# Spectral Boundary Value Problems and Elliptic Equations on Singular Manifolds

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# Spectral Boundary Value Problems and Elliptic Equations on Singular Manifolds

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**Abstract**

For elliptic operators on manifolds with boundary, we define *spectral boundary value problems*, which generalize the Atiyah–Patodi–Singer problem to the case of nonhomogeneous boundary conditions, operators of arbitrary order, and nonself-adjoint conormal symbols. The Fredholm property is proved and equivalence with certain elliptic equations on manifolds with conical singularities is established.

**Keywords**: APS problem, spectral resolution, nonhomogeneous boundary value problems, Fredholm property

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# Contents



# Introduction

The notion of a general boundary value problem (BVP) for an elliptic differential operator on a manifold with boundary was introduced in [11], where one can also find an extended discussion of motivations for considering this notion and its relations to previously known results. In [11], Calderón–Seeley projections (e.g., see [12, 13]) were used as one of important tools. Here we describe a class of general BVPs which will be referred to as *spectral boundary value problems*. In these problems, the boundary operator is the projection in the space of boundary data onto the "positive subspace" corresponding to the *conormal symbol* of the elliptic operator in question, i.e., the subspace corresponding to the spectral points of the conormal symbol[1] with positive imaginary parts.

For the case in which the conormal symbol has the form $p - i\hat{A}$, where $\hat{A}$ is a first-order operator on the boundary and is self-adjoint in the space $L^2$ with some

[1]The conormal symbol is obtained by choosing a normal direction globally in a collar neighborhood of the boundary, then freezing the coefficients of the operator at the boundary, and finally replacing the normal differentiation operator by the spectral parameter; this procedure is of course ambiguous in that it depends on the choice of the normal direction at the first step.

smooth density, *homogeneous* BVPs of this sort in the $L^2$ setting were studied by Atiyah, Patodi, and Singer in the famous papers [2], which gave an impetus for an extensive subsequent work (e.g., see Cheeger [3], Gilkey [7], Getzler [5], Melrose [9], etc.). In the present paper we essentially give a natural generalization of the APS problem to the case of *nonhomogeneous boundary conditions* and *higher-order operators, without any self-adjointness or normality requirements.*

The spectral BVPs turn out to be closely related to *equations on singular manifolds.* Specifically, if we attach a cone to the boundary and continue the elliptic operator in question (provided that the coefficients are independent of the normal coordinate near the boundary) to the cone in a natural way, then we obtain an elliptic operator on the resulting manifold with conical singularities. It turns out that the *spectral BVP for the original operator is equivalent to the corresponding equation in weighted Sobolev spaces for the latter operator* in the sense of natural isomorphisms between kernels and cokernels.

Perhaps one point that needs some further clarification is how one has to tackle with the ambiguity in the definition of the conormal symbol. In the present paper, we make no investigation of how the conormal symbol depends on the choice of the normal direction. However, the following should be pointed out.

- In any case, the boundary operator in the spectral BVP is a pseudodifferential operator whose principal symbol is independent of the choice of the conormal symbol (see Lemma 2 below).
- If the relationship with equations on singular manifolds is considered, the latter being the primary object, then the conormal symbol is defined invariantly [10].
- If the operators in question are geometric (as in [2]), then the choice of the normal direction is uniquely determined by the prescribed Riemannian metric.

The paper is organized as follows. Section 1 deals with spectral boundary value problems; we introduce the notion of these in Subsection 1.1, treat the model situation on the cylinder in Subsections 1.2 and 1.3, and prove the Fredholm property of the general spectral BVP in Subsection 1.4. Section 2 reveals the relationship between spectral BVPs and equations on singular manifolds and comprises two subsections, the first one stating two problems (one for a manifold with boundary and the other for the corresponding singular manifold) and the second one establishing the equivalence. In Section 3 we give two simple examples; here we consider equations on the model cylinder, which allows everything to be computed explicitly. Finally, the appendix contains the proof of all results of Section 1 pertaining to the model problem. The proofs are somewhat technical, and we have rendered it inappropriate to include it in the main text.

# 1 Spectral boundary value problems

## 1.1 Statement of the problem

Let $M$ be a compact $C^\infty$ manifold with $C^\infty$ boundary $X = \partial M$, and let

$$\hat{D} : H^s(M, E_1) \to H^{s-m}(M, E_2) \tag{1}$$

be an $m$th-order elliptic differential operator on $M$ acting in the Sobolev spaces of sections of some vector bundles $E_1$ and $E_2$ over $M$ (we consider only integer $s \geq m$). Let $U$ be a collar neighborhood of $X$ in $M$. We once and forever choose an isomorphism of $U$ onto the direct product $X \times [0, 1)$,

$$U \simeq X \times [0, 1), \tag{2}$$

so that each point $x \in X \subset U$ is represented by the pair $(x, 0)$. Then the operator $\hat{D}$ can be represented in $U$ in the form

$$\hat{D} = \sum_{j=0}^{m} \hat{D}_j(t) \left(-i\frac{\partial}{\partial t}\right)^j, \tag{3}$$

where $\hat{D}_j(t) \in \mathrm{Diff}^{m-j}(X)$ is a differential operator of order $m - j$ on $X$, smoothly depending on the parameter $t \in [0, 1)$. Since $\hat{D}$ is elliptic, it follows that $\hat{D}_m(t)$, which is a zero-order differential operator and hence a bundle homomorphism

$$\hat{D}_m(t) : E_1 \to E_2, \tag{4}$$

is an isomorphism. Hence, we can safely divide by $\hat{D}_m(t)$ on the left and assume, as far as our considerations are restricted to $U$, that $E_2 = E_1$ and $\hat{D}_m(t)$ is the identity operator:

$$\hat{D} = \left(-i\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} \hat{D}_j(t) \left(-i\frac{\partial}{\partial t}\right)^j, \tag{5}$$

with some new operators $\hat{D}_j(t)$. We freeze the coefficients of $\hat{D}$ at $t = 0$, thus obtaining the operator

$$\hat{D}_{(0)} = \left(-i\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} \hat{D}_j(0) \left(-i\frac{\partial}{\partial t}\right)^j. \tag{6}$$

The operator family

$$\hat{D}_{(0)}(p) = p^m + \sum_{j=0}^{m-1} \hat{D}_j(0) p^j \tag{7}$$

will be called the *conormal symbol* of the operator (5). *Our aim is to pose a BVP for the operator* (1) *with a boundary operator which is some sort of spectral projection corresponding to the family* (7). The operator (6) and the operator family (7) will be studied in Subsections 1.2 and 1.3, where we drop the subscript "(0)" to avoid clumsy notation. In particular, a BVP of the desired form is studied for the operator (6) on a half-infinite cylinder. In Subsection 1.4 we pose the spectral BVP for the general operator (1) and prove the Fredholm property.

## 1.2 The model operator

Let $X$ be a closed $C^\infty$ manifold, and let $E$ be a vector bundle over $X$. On the half-cylinder

$$C_+ = X \times \mathbf{R}_+$$

with base $X$, we consider an $m$th-order elliptic differential operator[2]

$$\hat{D} : H^s(C_+, E) \to H^{s-m}(C_+, E) \tag{8}$$

where

$$\hat{D} = \hat{D}\left(-i\frac{\partial}{\partial t}\right) = \left(-i\frac{\partial}{\partial t}\right)^m + \sum_{j=0}^{m-1} \hat{D}_j \left(-i\frac{\partial}{\partial t}\right)^j$$

with coefficients independent of $t$. Here $\hat{D}_j$ is a differential operator of order $\le m-j$ o $X$ acting in sections of $E$, and $s \ge m$ is assumed to be an integer.

Let $j_X^{m-1}$ be the operator that takes each function $u(x,t)$ on $C_+$ to its $(m-1)$st-order jet

$$j_X^{m-1} u = \left( u(x,0), -i\frac{\partial u}{\partial t}(x,0), \ldots, (-i)^{m-1}\frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) \right)$$

at the section $\{t=0\}$ (naturally identified with $X$). Then $j_X^{m-1}$ is continuous in the spaces

$$j_X^{m-1} : H^s(C_+, E) \to H_m^{s-1/2}(X, E) \equiv \bigoplus_{j=0}^{m-1} H^{s-1/2-j}(X, E). \tag{9}$$

We study the following two closely related questions for the equation

$$\hat{D}u = f \in H^{s-m}(C_+, E). \tag{10}$$

[2]For brevity, we write $H^s(C_+, E)$ instead of $H^s(C_+, \pi^* E)$, where $\pi : C_+ \to X$ is the natural projection. Sometimes we even abbreviate $H^s(C_+, E)$ and $H^s(X, E)$ to $H^s(C_+)$ and $H^s(X)$, respectively.

1) Suppose that we prescribe the value of $j_X^{m-1}$:

$$j_X^{m-1} u = \varphi \in H_m^{s-1/2}(X, E). \qquad (11)$$

What are necessary and sufficient conditions on the pair $(f, \varphi)$ for problem (10), (11) to be solvable, and is the solution unique?

2) How to equip (10) with a boundary condition so that the resulting problem will be uniquely solvable?

For the operator (8), let us consider the *conormal symbol*

$$\hat{D}(p) = p^m + \sum_{j=0}^{m} \hat{D}_j p^j : H^\nu(X, E) \to H^{\nu-m}(X, E), \quad p \in \mathbf{C}, \quad \nu \in \mathbf{R}. \qquad (12)$$

This is a family of differential operators on $X$ elliptic with parameter $p$ in the sense of Agranovich–Vishik [1] for $p \in \mathbf{R}$. Since $X$ is compact, it obviously follows that $\hat{D}(p)$ is elliptic with parameter in the double sector

$$\Lambda_\varepsilon = \{|\arg p| < \varepsilon\} \cup \{|\pi - \arg p| < \varepsilon\}$$

for some $\varepsilon > 0$.

Consequently (see [8]), $\hat{D}(p)$ is finite-meromorphically invertible, that is, $\hat{D}^{-1}(p)$ is a meromorphic operator function on the entire complex plane of the variable $p$, and the principal part of the Laurent series expansion of $\hat{D}^{-1}(p)$ at each pole is finite-dimensional. Moreover, there are at most finitely many poles of $\hat{D}^{-1}(p)$ in $\Lambda_\varepsilon$.

For simplicity, we impose the following condition on the conormal symbol.

**Condition 1** $\hat{D}(p)$ is invertible for all $p \in \mathbf{R}$.

Since our main interest is in exploring the relationship between BVPs and equations on manifolds with conical singularities, this condition is in fact not very restrictive: we can always ensure its validity by passing to an appropriate weight line $\{\operatorname{Im} p = \gamma\}$ in the conical problem.

It follows from Condition 1 in view of the preceding that $\hat{D}(p)$ is actually invertible in the strip

$$L_c = \{|\operatorname{Im} p| < c\}$$

for some $c > 0$. The set Spec $(\hat{D}(\cdot))$ of poles of $\hat{D}^{-1}(p)$ naturally splits into two parts,

$$\operatorname{Spec}(\hat{D}(\cdot)) = \operatorname{Spec}_+(\hat{D}(\cdot)) \cup \operatorname{Spec}_-(\hat{D}(\cdot)),$$

where

$$\operatorname{Spec}_\pm(\hat{D}(\cdot)) = \{p \in \operatorname{Spec}(\hat{D}(\cdot)) : \pm \operatorname{Im} p > 0\}.$$

The simplest way to study equation (10) is to reduce it to a first-order system in $\partial/\partial t$; as a by-product, we essentially reduce $\hat{D}^{-1}(p)$ to the resolvent of some operator. This fairly standard reduction can be carried out as follows. For any section $u(x,t)$ of the bundle $E$ over $C_+$, consider the vector function[3]

$$v = {}^t(v_0, \dots, v_{m-1}), \tag{13}$$

where

$$v_0 = u, \quad v_1 = -i\frac{\partial u}{\partial t}, \dots, \quad v_{m-1} = \left(-i\frac{\partial}{\partial t}\right)^{m-1} u. \tag{14}$$

Then equation (10) is equivalent to the system

$$\left(-i\frac{\partial}{\partial t} - \hat{A}\right) v = {}^t(0, , \dots, 0, f), \tag{15}$$

where $\hat{A}$ is the matrix differential operator

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -\hat{D}_0 & -\hat{D}_1 & 0 & \dots & -\hat{D}_{m-1} \end{pmatrix} \tag{16}$$

on $X$ with entries being differential operators acting in the space of sections of $E$.

The conormal symbol corresponding to the operator in (15) is $p - \hat{A}$, and its inverse is just the resolvent[4] of $\hat{A}$.

**Lemma 1** (i) *The operator $p - \hat{A}$ is invertible if and only if so is $\hat{D}(p)$ (in other words,* Spec $(\hat{A})$ = Spec $(\hat{D}(\cdot))$*).*

(ii) *One has*

$$(p - \hat{A})^{-1} = \hat{D}^{-1}(p) \cdot \hat{Q}(p), \tag{17}$$

*where $\hat{Q}(p) = \left|\left|\hat{Q}_{jk}(p)\right|\right|_{j,k=0}^{m-1}$ is a matrix of differential opertors polynomially depending on $p$.*

[3]The left superscript $t$ stands for the transpose of a matrix.

[4]We do not specify the function space in which this is considered; it is easy to see from the following that the resolvent of $\hat{A}$ is well defined in $L^2(X,E)$ and leaves $C^\infty(X,E)$ invariant.

More precisely, $\hat{Q}_{jk}(p)$ is a differential operator with parameter $p$ of order[5]

$$\operatorname{ord}\hat{Q}_{jk}(p) = m-1+j-k, \qquad j,k=0,\dots,m-1. \tag{18}$$

The proof of (ii) is straightforward; it also gives an explicit expression for $\hat{Q}(p)$ via $\hat{D}(p)$.

The "if" part of (i) readily follows from (17); to prove the "only if" part, it suffices to note that

$$\hat{D}^{-1}(p)f = \text{the first component of } (p-\hat{A})^{-1}\,{}^t(0,\dots,0,f) \tag{19}$$

by virtue of the specific structure of the matrix $\hat{A}$.

## 1.3 Projection operators and the model problem

We have seen that the spectrum of $\hat{A}$ splits into two parts, one constituted by spectral points with positive and the other by those with negative imaginary part. In this subsection we describe the corresponding spectral projections. Let $\Gamma_+$ and $\Gamma_-$ be the following contours in the plane of the complex variable $p$:

$$\Gamma_\pm = \{p = \pm[\tau + i(c-\varepsilon)],\ \tau\in\mathbf{R}\}, \tag{20}$$

where $\varepsilon > 0$ is sufficiently small and the sense of $\Gamma_\pm$ is determined by the condition that $\tau$ increases along $\Gamma_\pm$.

Set

$$\hat{P}_\pm = \frac{\hat{A}}{2\pi i}\int\limits_{\Gamma_\pm}(p-\hat{A})^{-1}\frac{dp}{p}. \tag{21}$$

**Lemma 2** (i) *For each $\sigma\in\mathbf{R}$, the integral* (21) *strongly converges in $H^\sigma_m(X,E)$ on the dense subset $H^{\sigma+1}_m(X,E)$.*

(ii) *The operators $\hat{P}_\pm$ extend to bounded operators in the spaces*

$$\hat{P}_\pm : H^\sigma_m(X,E)\to H^\sigma_m(X,E) \tag{22}$$

*for each $\sigma\in\mathbf{R}$.*

[5]To avoid misunderstanding, recall that this means that

$$\hat{Q}_{jk}(p) = \sum_{l=0}^{m-1+j-k} p^l\hat{Q}_{jkl},$$

where $\hat{Q}_{jkl}$ is a differential operator of order $m-1+j-k-l$.

(iii) *These operators are projections commuting with* $\hat{A}$,

$$(\hat{P}_\pm)^2 = \hat{P}_\pm, \qquad \hat{P}_\pm \hat{A} = \hat{A}\hat{P}_\pm, \tag{23}$$

*and one has*

$$\hat{P}_+ + \hat{P}_- = 1. \tag{24}$$

(iv) $\hat{P}_+$ *is a pseudodifferential operator whose principal symbol* $\sigma(\hat{P}_+)$ *(in the sense of Douglis–Nirenberg) is the projection on the subspace*

$$L_-(x,\xi) \subset E_x, \qquad x \in X, \qquad \xi \in T_x^*X\backslash\{0\}, \tag{25}$$

*of the initial data of exponentially decaying as* $t \to +\infty$ *solutions to the ordinary differential equation*

$$\sigma(\hat{D})\left(x,\xi,-i\frac{\partial}{\partial t}\right)\varphi(t) = 0 \tag{26}$$

*along the subspace*

$$L_+(x,\xi) \subset E_x$$

*of exponentially increasing solutions of* (26). *A similar assertion holds for* $\hat{P}_-$ *with accordingly interchanged signs* $+$ *and* $-$.

(v) *The operator function* $(p-\hat{A})^{-1}\hat{P}_+$ *extends to a holomorphic function for* $p \notin \mathrm{Spec}_+(\hat{A})$. *Likewise,* $(p-\hat{A})^{-1}\hat{P}_-$ *extends to a holomorphic function for* $p \notin \mathrm{Spec}_-(\hat{A})$.

The proof will be given in the appendix, where we also refine assertion (v) as to the behavior of the above operator functions.

We can now pose the spectral BVP for the operator $\hat{D}$. We study equation (10) with the help of the Fourier–Laplace transform

$$[\mathcal{F}u](p) \equiv \tilde{u}(p) = \int_0^\infty e^{-itp}u(t)\,dt \tag{27}$$

with the inverse

$$[\mathcal{F}^{-1}\tilde{u}](t) = \frac{1}{2\pi i}\int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{ipt}\,\tilde{u}(p)\,dp, \qquad t > 0, \tag{28}$$

where $\gamma > 0$. Consider the mappings

$$\begin{array}{rcl} \lambda : \mathbf{C} & \to & \mathbf{C}^m \\ z & \mapsto & (0,\ldots,0,z) \end{array} \tag{29}$$

and

$$\begin{array}{rcl} \pi_0 : \mathbf{C}^m & \to & \mathbf{C} \\ (z_0, \dots, z_{m-1}) & \mapsto & z_0. \end{array} \tag{30}$$

**Theorem 1** *The problem*

$$\begin{cases} \hat{D}u = f \in H^{s-m}(C_+), \\ j_X^{m-1} u = g \in H_m^{s-1/2}(X) \end{cases} \tag{31}$$

*is solvable for $u \in H^s(C_+)$ if and only if the consistency condition*

$$\hat{P}_- g + \frac{1}{2\pi} \int\limits_{\Gamma_-} (p - \hat{A})^{-1} \lambda(\tilde{f}\,(p))\, dp = 0 \tag{32}$$

*is satisfied. In that case, the solution is unique. It is given by*

$$u = \pi_0 \mathcal{F}^{-1}\{(p - \hat{A})^{-1}[\lambda(\tilde{f}\,(p)) - ig]\}. \tag{33}$$

**Theorem 2** *The problem*

$$\begin{cases} \hat{D}u = f \in H^{s-m}(C_+), \\ \hat{P}_+ j_X^{m-1} u = g \in \hat{P}_+ H_m^{s-1/2}(X) \end{cases} \tag{34}$$

*is uniquely solvable for $u \in H^s(C_+)$. The solution is given by*

$$u = \mathcal{R}[f, g] = \pi_0 \mathcal{F}^{-1}\{(p - \hat{A})^{-1}[\lambda(\tilde{f}\,(p)) - ig - \frac{1}{2\pi i} \int\limits_{\Gamma_-} (\eta - \hat{A})^{-1} \lambda(\tilde{f}\,(\eta))\, d\eta]\}, \tag{35}$$

*and the resloving operator $\mathcal{R}$ is continuous in the spaces*

$$\mathcal{R} : H^{s-m}(C_+) \oplus \hat{P}_+ H^{s-1/2}(X) \to H^s(C_+). \tag{36}$$

The proof of both theorems will be given in the appendix.

Problem (34) will be referred to as the *spectral boundary value problem* for the operator $\hat{D}$.

### 1.4 The general case

We now return to the notation of Subsection 1.1. Let

$$\hat{P}_{\pm}:\ H^{\sigma}_{m}(X,E_1)\to H^{\sigma}_{m}(X,E_1) \tag{37}$$

be the projection operators (21) constructed for the conormal symbol $\hat{D}_{(0)}(p)$ (cf. formula (7)). Consider the *spectral boundary value problem*

$$\left\{\begin{array}{l}\hat{D}u=f\in H^{s-m}(M,E_2),\\ \hat{P}_{+}j^{m-1}_{X}u=g\in \hat{P}_{+}H^{s-1/2}_{m}(X,E_1)\end{array}\right. \tag{38}$$

for $u\in H^{s}(M,E_1)$ (here $s\geq m$ is an integer).

**Theorem 3** *The spectral boundary value problem* (38) *possesses the Fredholm property.*

*Proof.* This immediately follows from Theorem 3 in [11], since Condition (GSL) (the generalized Shapiro–Lopatinskii condition) is satisfied by Lemma 2, (iv): the principal symbol $\sigma(\hat{P}_{+})$ is an isomorphism of $L_{-}(x,\xi)$ onto its range.

**Remark 1** We see that, to prove Theorem 3, no information is actually needed about the solvability of the model problem (34); we only use Lemma 2, (iv). However, this information is crucial in establishing the equivalence with equations on manifolds with conical singularities in the next section.

## 2 Elliptic equations on singular manifolds

### 2.1 Statement of the problem

In this section, we consider the connection between the above spectral boundary value problem (38) for an operator $\hat{D}$ on a manifold $M$ with boundary $X$ and some equation on a manifold $M^{\wedge}$ with conical singularities.

Let us begin with the geometric construction. Let $M$, as above, be a smooth manifold with compact smooth boundary $X=\partial M$. Consider the cylinder $C=X\times\mathbf{R}$ over $X$. We have

$$C=C_{+}\cup C_{-},\ \ C_{+}=\{(x,t)\in C\,|\,t\geq 0\},\ \ C_{-}=\{(x,t)\in C\,|\,t\leq 0\}.$$

We identify some collar neighborhood $U$ of the boundary $X$ in $M$ with the set $0 \le t < \varepsilon$ in $C_+$. Let $M^\wedge$ be the manifold obtained from $M$ by gluing the cylinder $C_-$ along the common boundary $X$,

$$M^\wedge = M \bigcup_X C_-.$$

With this construction, we can view $C \cap \{t < \varepsilon\}$ as a subset in $M^\wedge$. Then $M^\wedge$ is a manifold with cylindrical end. It can be viewed as a manifold with conical singularity (via the change of variables $r = e^t$ in a neighborhood of $\{t = -\infty\}$).

Let $\hat{D}$ be an elliptic differential operator on $M$. Suppose for simplicity that

(i) the coefficients of this operator are independent of $t$ in $U$ (that is, for $t < \varepsilon$);

(ii) the point $p = 0$ is not a spectral point of the conormal symbol $\hat{D}_{(0)}(p)$ corresponding to the operator $\hat{D}$.

Under assumption (i), we can extend the operator $\hat{D}$ to an elliptic operator $\hat{D}_1$ with smooth coefficients on the manifold $M^\wedge$ with conical singularities; to this end, it suffices to require that $\hat{D}_1 = \hat{D}$ on $M$ and the coefficients of $\hat{D}_1$ are independent of $t$ in $C \cap \{t < \varepsilon\}$. In the following, the extended operator $\hat{D}_1$ will be denoted by the same letter $\hat{D}$.

It is clear that $\hat{D}_1$ is elliptic as a Fuchs type operator on $M^\wedge$.

Consider the homogeneous spectral BVP

$$\begin{cases} \hat{D}u = f \in H^{s-m}(M), \\ \hat{P}_+ j_X^{m-1} u = 0, \quad u \in H^s(M). \end{cases} \tag{39}$$

We shall show that problem (39) is in some sense equivalent to the equation

$$\hat{D}v = f_1 \in H^{s-m,0}(M^\wedge), \qquad v \in H^{s,0}(M^\wedge) \tag{40}$$

on $M^\wedge$ in the weighted Sobolev spaces $H^{s,\gamma}(M^\wedge)$ (e.g., see [10]) with $\gamma = 0$. The equivalence is by no means straightforward since the extension of the right-hand side $f$ of problem (39) to the entire $M^\wedge$ is not unique. However, if we consider the narrower class of functions $f$ supported in $M \backslash X$, then these two problems become equivalent if we set $f_1 = f$ in $M$ and $f_1 = 0$ in $M^\wedge \backslash M$ (in that case we simply write $f_1 = f$). On the other hand, the solvability of (39) and (40) for general right-hand sides proves to be equivalent to that for right-hand sides in the narrower class.

In the following subsection we give the precise statements.

## 2.2 Equivalence theorems

**Theorem 4** *Suppose that conditions* (i) *and* (ii) *of Subsection* 2.1 *are satisfied. Then problem* (39) *is equivalent to equation* (40) *provided that $f_1 = f$ is supported in $M \backslash X$.*

*Proof.* Let $u$ be a solution of problem (39). Consider the following problem in $C_-$ (here $\hat{D}_{(0)}$ is obtained from $\hat{D}$ by freezing the coefficients at $t = 0$).

$$\begin{cases} \hat{D}_{(0)} w = 0 \text{ in } C_-, \\ \hat{P}_- j_X^{m-1} w = \hat{P}_- j_X^{m-1} u. \end{cases} \tag{41}$$

By Theorem 2, this problem has a unique solution $w \in H^s(C_-)$ (note that $\hat{P}_+$ and $\hat{P}_-$ interchange their roles if $C_+$ is replaced by $C_-$). Next, by Theorem 1 the function $w$ satisfies the compatibility condition

$$\hat{P}_+ j_X^{m-1} w = 0. \tag{42}$$

By combining this with the boundary conditions in (39) and (41), with regard to the fact that $\hat{P}_+ + \hat{P}_- = 1$, we see that

$$j_X^{m-1} u = j_X^{m-1} w. \tag{43}$$

Set

$$v = \begin{cases} u \text{ in } M, \\ w \text{ in } M^\wedge \setminus M. \end{cases} \tag{44}$$

It follows now from (43) that the derivatives $\partial^k v / \partial t^k$ for $k = 1, \ldots, m$ can be calculated by differentiating $u$ in $M$ and $w$ in $M^\wedge \setminus M$ (no $\delta$-like terms on $X$ occur in the differentiation). Consequently,

$$\hat{D} v = f \in H^{s-m,0}(M^\wedge), \tag{45}$$

and since $\hat{D}$ is elliptic of order $m$, we conclude that $v \in H^s_{\mathrm{loc}}(M^\wedge)$. This, combined with $w \in H^s(C_-)$ and $u \in H^s(M)$, implies

$$v \in H^{s,0}(M^\wedge). \tag{46}$$

Conversely, let $v$ be a solution of equation (40) with $f_1 = f$. Set

$$u = v|_M . \tag{47}$$

Then

$$\hat{P}_+ j_X^{m-1} u = \hat{P}_+ j_X^{m-1} v = 0 \tag{48}$$

by Theorem 1, since this is just the compatibility condition for the equation

$$\hat{D}_{(0)} v = 0, \tag{49}$$

which holds in $C_-$. The proof is complete.

**Theorem 5** *The following assertions hold:*

(a) *the kernel of problem* (39) *coincides with the kernel of equation* (40)*;*

(b) *the cokernel of problem* (39) *is naturaly isomorphic to the cokernel of equation* (40)*.*

*In particular, problem* (39) *and equation* (40) *have the same index.*

*Proof.* (a) immediately follows from Theorem 4. To prove (b), let us show that problem (39) with a general right-hand side can be equivalently reduced to the problem with right-hand side supported in $M\backslash X$, and likewise, the same is true of equation (40). To this end, consider the problem

$$\begin{cases} \hat{D}_{(0)}u = f \in H^{s-m}(C_+), \\ \hat{P}_+ j_X^{m-1}u = 0. \end{cases} \tag{50}$$

By Theorem 2, this problem is uniquely solvable; the solution is given by

$$u = \hat{R}_+ f \equiv \mathcal{R}[f,0] \in H^s(M). \tag{51}$$

Now let $e$ and $\psi$ be smooth functions on $C_+$ such that $\operatorname{supp} e \subset U$, $\operatorname{supp} \psi \subset U$, $e \equiv 1$ near $X$, and $\psi e \equiv e$.

Let us seek the solution of problem (39) in the form

$$u = u' + \psi \hat{R}_+(ef). \tag{52}$$

Then

$$\begin{aligned} \hat{D}u &= \hat{D}u' + [\hat{D},\psi]\hat{R}_+(ef) + \psi\hat{D}\hat{R}_+(ef) \\ &= \hat{D}u' + [\hat{D},\psi]\hat{R}_+(ef) + \psi\hat{D}_{(0)}\hat{R}_+(ef) \qquad (53) \\ &= \hat{D}u' + [\hat{D},\psi]\hat{R}_+(ef) + ef, \end{aligned}$$

(54)

whence for $u'$ we obtain the problem

$$\begin{cases} \hat{D}u' = f' \in H^{s-m}(M), \\ \hat{P}_+ j_X^{m-1}u' = 0, \end{cases} \tag{55}$$

where

$$f' = (1-e)f - [\hat{D},\psi]\hat{R}_+(ef), \tag{56}$$

so that

$$\operatorname{supp} f' \cap X = \emptyset. \tag{57}$$

Obviously, problem (39) is solvable if and only if so is problem (55) with the right-hand side (56). We claim that the cokernel of problem (39) is naturally isomorphic to the cokernel of problem (55) with right-hand sides ranging in the (nonclosed) subspace of $f' \in H^{s-m}(M)$ satisfying (57) (but not necessarily given by (56) for some $f \in H^{s-m}(M)$). Indeed, this is an easy consequence of the following obvious purely algebraic lemma.

**Lemma 3** *Let $K_0$ and $R$ be subspaces of a linear space $K$, and let there exist a linear mapping*

$$\mu : K \to K_0 \tag{58}$$

*such that $u(x) \in R$ if and only if $x \in R$. Then the quotient spaces $K_0/R \cap K_0$ and $K/R$ are naturally isomorphic provided at least one of them is finite-dimensional.*

*Proof of Lemma* 3. We have the natural injection

$$i : K_0/R \cap K_0 \to K/R, \tag{59}$$

and we only have to prove that the mapping (59) is surjective. Next, the mapping (58) induces an injection

$$\widetilde{\mu}\colon K/R \to K_0/R \cap K_0. \tag{60}$$

It follows from the existence of the two injections (59) and (60) that if one of the spaces $K/R$ and $K_0/R \cap K_0$ is finite-dimensional, then so is the other, and their dimensions coincide. But then $i$ given by (59) is the desired isomorphism. This completes the proof of the lemma.

Now let us return to the proof of Theorem 5. We can apply Lemma 3 with $K = H^{s-m}(M)$, $K_0 = \{f \in K \,|\, f \text{ satisfies } (57)\}$, $R$ = the range of the operator corresponding to (39), and the mapping $\mu$ given by (56), thus obtaining the isomorphism between the cokernels of problems (39) and (55), (57) (recall that problem (39) is Fredholm by Theorem 4). Likewise, using the resolving operator for the equation

$$\hat{D}_{(0)}u = f$$

on the infinite cylinder $C = C_+ \cup C_-$, we prove the isomorphism of cokernels of equation (40) with arbitrary right-hand sides and with those supported in $M\backslash X$. An application of Theorem 4 now completes the proof of Theorem 5.

**Remark 2** Condition (ii) in Subsection 2.1 is in fact unessential. Should it be violated, we can choose a weight line $\{\operatorname{Im} p = \gamma\}$ with $\gamma \neq 0$ which does not contain the spectral points of the conormal symbol, redefine $\hat{P}_+$ accordingly, and use the spaces $H^{s,\gamma}(M^\wedge)$ instead of $H^{s,0}(M^\wedge)$ in problem (40).

# 3 Examples

In this section, we present two examples of boundary value problems of the above type. To make explicit calculations possible, we shall consider operators on infinite cylinders, though everything can be considered on smooth compact manifolds with boundary as well.

In both examples, the operator in question has the form $\partial/\partial t + \hat{A}$, where $\hat{A}$ is a self-adjoint operator on $X$. *Homogeneous* boundary value problems for such operators were considered by Atiyah, Patodi, and Singer [2]. The projections $\hat{P}_{\pm}$ for such operators will be constructed, just as in the cited paper, by using eigenfuncion expansions rather than the Fourier–Laplace transform.

## 3.1 The Cauchy–Riemann operator

Let $C_+$ be an infinite (half-)cylinder over the one-dimensional circle $X = S^1$. We denote by $(t, \varphi)$ the corresponding (global) coordinates on $C_+$. Consider the operator

$$\hat{D} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial t} + i\frac{\partial}{\partial \varphi}\right),$$

where $z = t + i\varphi$; the latter relation determines a complex structure on $C_+$.

**Remark 3** The introduced operator is a representation near a connected component of the boundary of the operator $\overline{\partial}$ on a one-dimensional complex manifold with purely real boundary. The latter means that in a neighborhood of any point of the boundary there exists a holomorphic coordinate $z$ such that the equation of the boundary is $\operatorname{Im} z = 0$.

The corresponding operator pencil (conormal symbol) is

$$\hat{D}\left(p\right) = \frac{i}{2}\left(p + \frac{\partial}{\partial \varphi}\right).$$

Let us compute the spectrum of this operator pencil. Passing to the Fourier series representation

$$u\left(\varphi\right) = \sum_{k=-\infty}^{+\infty} u_k e^{ik\varphi},$$

we have

$$\hat{D}\left(p\right) u\left(\varphi\right) = \frac{i}{2} \sum_{k=-\infty}^{+\infty} \left(p + ik\right) u_k e^{ik\varphi},$$

and hence, the inverse operator is given by

$$\hat{D}^{-1}(p)\, u(\varphi) = -2i \sum_{k=-\infty}^{+\infty} \frac{u_k}{p+ik} e^{ik\varphi}.$$

The latter expression shows that

- the spectrum of the operator in question consists of the points
$$p_k = -ik, \quad k \in \mathbf{Z};$$
- each spectral point is simple;
- the eigenspace corresponding to the point $p_k$ is one-dimensional with the generator
$$u_k(\varphi) = e^{ik\varphi}.$$

Note that zero is a spectral point of $\hat{D}(p)$. Thus, we make a shift of the weight line and consider the problem in weighted Sobolev spaces $H^{s,\gamma}(C_+)$ with some $\gamma \in (0,1)$. Then the projections $\hat{P}_+$ and $\hat{P}_-$ are given by

$$\begin{aligned} \hat{P}_+ u(\varphi) &= \sum_{k=0}^{+\infty} u_k e^{ik\varphi}, \\ \hat{P}_- u(\varphi) &= \sum_{k=-\infty}^{-1} u_k e^{ik\varphi}. \end{aligned}$$

Now it follows from Theorem 2 that the boundary value problem

$$\begin{cases} \dfrac{\partial}{\partial \bar{z}} u = f, \\ \hat{P}_+ u\Big|_{t=0} = g, \end{cases}$$

is uniquely solvable in these spaces.

## 3.2 The Euler operator

Let $X$ be a compact oriented Riemannian manifold without boundary, and let

$$C_+ = \mathbf{R}_+ \times X$$

be an infinite half-cylinder over $X$ with the product metric

$$g = dt^2 + g_X,$$

where $g_X$ is a metric on $X$.

On $C_+$ we consider the Euler operator[6]

$$\hat{D} = (d+\delta)_{\mathrm{ev}} \ : \ \Lambda^{\mathrm{ev}}(C_+) \to \Lambda^{\mathrm{odd}}(C_+), \tag{61}$$

where $\delta$ is the adjoint of $d$ with respect to the metric $g$, and $\Lambda^{\mathrm{ev}}(C_+)$, (resp., $\Lambda^{\mathrm{odd}}(C_+)$) is the bundle of forms of even (resp., odd) degree on $C_+$.

By virtue of the decompositions

$$\omega = \omega_0 + dt \wedge \omega_1, \ \omega \in \Lambda^k(C_+), \omega_0 \in \Lambda^k(X), \omega_1 \in \Lambda^{k-1}(X),$$

where the forms $\omega_0$ and $\omega_1$ do not contain the differential $dt$, we obtain the isomorphisms

$$\Lambda^{\mathrm{ev}}(C_+) \approx \pi^*(\Lambda(X)) \approx \pi^*(\Lambda^{\mathrm{ev}}(X)) \oplus \pi^*\left(\Lambda^{\mathrm{odd}}(X)\right). \tag{62}$$

Here

$$\pi \ : \ C_+ \to X$$

is the natural projection; we shall omit the operator $\pi^*$ in the sequel.

Simple computations show that the operator (61) can be rewritten in matrix form

$$\hat{D} = \begin{pmatrix} (d_X+\delta_X)_{\mathrm{ev}} & -\partial/\partial t \\ \partial/\partial t & -(d_X+\delta_X)_{\mathrm{odd}} \end{pmatrix} : \begin{pmatrix} \Lambda^{\mathrm{ev}}(X) \\ \Lambda^{\mathrm{odd}}(X) \end{pmatrix} \to \begin{pmatrix} \Lambda^{\mathrm{odd}}(X) \\ \Lambda^{\mathrm{ev}}(X) \end{pmatrix}$$

with respect to the decomposition (62). Interchanging the components in the range of the latter operator and changing the sign in the odd component, we arrive at the representation of the Euler operator in the form

$$\hat{D} = \begin{pmatrix} \partial/\partial t & (d_X+\delta_X)_{\mathrm{odd}} \\ (d_X+\delta_X)_{\mathrm{ev}} & \partial/\partial t \end{pmatrix} : \begin{pmatrix} \Lambda^{\mathrm{ev}}(X) \\ \Lambda^{\mathrm{odd}}(X) \end{pmatrix} \to \begin{pmatrix} \Lambda^{\mathrm{ev}}(X) \\ \Lambda^{\mathrm{odd}}(X) \end{pmatrix}.$$

The corresponding conormal symbol is

$$\hat{D}(p) = \begin{pmatrix} ip & (d_X+\delta_X)_{\mathrm{odd}} \\ (d_X+\delta_X)_{\mathrm{ev}} & ip \end{pmatrix}.$$

[6]Concerning the boundary value problems for the Euler operator, see [6, 4].

To investigate the spectrum of this operator pencil, we first note that

$$\hat{D}^{-1}(p) = \left(\Delta_X - p^2\right)^{-1} \begin{pmatrix} ip & -(d_X + \delta_X)_{\mathrm{odd}} \\ -(d_X + \delta_X)_{\mathrm{ev}} & ip \end{pmatrix},$$

and hence, the spectrum of the pencil $\hat{D}(p)$ is given by the relation

$$p_k^{\pm} = \pm i\sqrt{\lambda_k},$$

where the $\lambda_k$ are the (nonnegative) eigenvalues of the Beltrami–Laplace operator $\Delta_X$:

$$\Delta_X u_k = -\lambda_k u_k$$

for some nonvanishing form $u_k$. Furthermore, the spaces $\Lambda^{\mathrm{ev}}(X)$ and $\Lambda^{\mathrm{odd}}(X)$ are invariant with respect to the operator $\Delta_X$, and we denote by

$$\left(\varphi_k^{\mathrm{ev}}, \lambda_k^{\mathrm{ev}}\right) \text{ and } \left(\varphi_k^{\mathrm{odd}}, \lambda_k^{\mathrm{odd}}\right)$$

the corresponding complete sequences of eigenvectors and eigenvalues. It is easy to see that

$$\lambda_k^{\mathrm{ev}} = \lambda_k^{\mathrm{odd}} = \lambda_k,$$

and that the operator $(d_X + \delta_X)_{\mathrm{ev}}$ determines an isomorphism between the corresponding eigenspaces for $\lambda_k \neq 0$. Hence, we can suppose that

$$(d_X + \delta_X)_{\mathrm{ev}}\, \varphi_k^{\mathrm{ev}} = \varphi_k^{\mathrm{odd}},\ (d_X + \delta_X)_{\mathrm{odd}}\, \varphi_k^{\mathrm{odd}} = \lambda_k \varphi_k^{\mathrm{ev}}$$

for $\lambda_k \neq 0$ and

$$(d_X + \delta_X)_{\mathrm{ev}}\, \varphi_k^{\mathrm{ev}} = 0,\ (d_X + \delta_X)_{\mathrm{odd}}\, \varphi_k^{\mathrm{odd}} = 0 \text{ for } \lambda_k = 0.$$

Now we see that the eigenvectors corresponding to the eigenvalues $p_k^+$ in the upper half-plane are

$$\begin{pmatrix} \varphi_k^{\mathrm{ev}} \\ \lambda_k^{-1/2} \varphi_k^{\mathrm{odd}} \end{pmatrix}. \tag{63}$$

Likewise, the vectors

$$\begin{pmatrix} \varphi_k^{\mathrm{ev}} \\ -\lambda_k^{-1/2} \varphi_k^{\mathrm{odd}} \end{pmatrix}$$

are the eigenvectors corresponding to the eigenvalues $p_k^-$ in the lower half-plane. The basis of the kernel of the operator $\hat{D}$, i.e., the set of eigenfunctions corresponding to the eigenvalue $p_k^\pm = 0$ is

$$\begin{pmatrix} \varphi_k^{\mathrm{ev}} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \varphi_k^{\mathrm{odd}} \end{pmatrix}. \tag{64}$$

The above analysis shows that the spectral boundary value problem for the Euler operator $\hat{D}$ of the form

$$\begin{cases} \hat{D}u = f, \\ \hat{P}_+ u\Big|_{t=0} = g, \end{cases}$$

where $\hat{P}_+$ is the projection on the linear span $L_+$ of the set of vectors (63), (64), is uniquely solvable in the weighted Sobolev spaces $H^{s,\gamma}(C_+)$ with $\gamma \in (0, \varepsilon)$, where $\varepsilon$ is the square root of the minimal nonzero eigenvalue of $\Delta_X$.

# 4 Appendix. Computations on the half-infinite cylinder

We now prove Lemma 2 and Theorems 1 and 2 of Section 1.

In the sequel, we shall use the following definitions and assertions, which are adapted from [14], Chapter 2.

Let $\Lambda$ be a subset of the complex plane and $X$ a smooth compact manifold.

**Definition 1** The space $L^m(X, \Lambda)$ consists of pseudodifferential operators $\hat{A}(p)$ on $X$ depending on the parameter $p \in \Lambda$ and satisfying the following condition: in any coordinate neighborhood $U$ on $X$ the operator $\hat{A}(p)$ can be represented by a pseudodifferential operator with symbol $A(x, \xi, p)$ satisfying the estimates

$$\left| \frac{\partial^{\alpha+\beta} A}{\partial x^\alpha \partial \xi^\beta}(x, \xi, p) \right| \le C_{\alpha\beta}(1 + |\xi| + |p|)^{m-|\beta|}, \qquad |\alpha| + |\beta| = 0, 1, 2, \ldots, \tag{65}$$

plus an operator on $X$ with smooth kernel $K(x, y, p)$ satisfying (in local coordinates) the estimates

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} K(x, y, p) \le C_{\alpha\beta N}(1 + |p|)^{-N}, \qquad |\alpha|, |\beta|, N = 0, 1, 2, \ldots . \tag{66}$$

The subspace $CL^m(X,\Lambda)$ of classical pseudodifferential operators consists of operators whose symbols admit asymptotic expansions

$$A(x,\xi,p) \sim \sum_{j=0}^{\infty} A_{m-j}(x,\xi,p), \qquad |\xi|+|p| \to \infty, \tag{67}$$

where

$$A_{m-j}(x,\lambda\xi,\lambda p) = \lambda^{m-j} A_{k-j}(x,\xi,p) \tag{68}$$

provided that $p, \lambda p \in \Lambda$. The corresponding symbol spaces will be denoted by $S^m(X,\Lambda)$ and $CS^m(X,\Lambda)$.

The principal symbol of a classical $\psi DO$ with parameter is, of course, invariantly defined on $T^*X \times \Lambda$.

**Theorem 6** (a) *Let $\hat{A}(p) \in L^m(X,\Lambda)$, and let $l \geq m$. Then the following estimates hold:*

$$\left|\left|\hat{A}(p)\right|\right|_{H^s(X)\to H^{s-l}(X)} \leq C_{s,l}(1+|p|)^m, \qquad l \geq 0, \tag{69}$$

$$\left|\left|\hat{A}(p)\right|\right|_{H^s(X)\to H^{s-l}(X)} \leq C_{s,l}(1+|p|)^{-(l-m)}, \qquad l \leq 0. \tag{70}$$

(b) *Let $\hat{A}(p) \in CL^m(X,\Lambda)$ be elliptic with parameter $p \in \Lambda$ (that is, the principal symbol $A_m(x,\xi,p)$ satisfies*

$$A_m(x,\xi,p) \neq 0 \tag{71}$$

*for $(x,\xi) \in T^*X$, $p \in \Lambda$, $|\xi|+|p| \neq 0$). Then $\hat{A}(p)$ is invertible in*

$$\Lambda_R = \{p \in \Lambda \mid |p| > R\} \tag{72}$$

*for sufficiently large $R$, and for any closed subset $\widetilde{\Lambda} \subset \Lambda$ such that $\hat{A}(p)$ is invertible for $p \in \widetilde{\Lambda}$, one has*

$$\hat{A}(p)^{-1} \in CL^{-m}(X,\widetilde{\Lambda}). \tag{73}$$

*Moreover, for any parametrix $\hat{B}(p)$ of the operator $\hat{A}(p)$ one has*

$$\hat{A}(p)^{-1} - \hat{B}(p) \in L^{-\infty}(X,\widetilde{\Lambda}) \tag{74}$$

*(that is, the difference (74) is an operator with smooth kernel $K(x,y,p)$ satisfying the estimates (66)).*

Now let us prove the assertions of Subsection 1.3.

First, we shall carry out the proof for the case in which $\hat{D}$ is a first-order operator ($m = 1$), so that the reduced system coincides with the original equation.

Thus, the situation is as follows. We are given a first-order differential operator

$$\hat{A}: C^\infty(X, E) \to C^\infty(X, E) \tag{75}$$

acting in the space of sections of a vector bundle $E$ over a compact manifold $X$ and possessing the following properties:

(i) The family $p - \hat{A}$ is elliptic with parameter in the sense of Agranovich–Vishik in the set

$$\Lambda_\varepsilon = \{\, |\arg p| < \varepsilon\} \cup \{\, |\arg p - \pi| < \varepsilon\}; \tag{76}$$

(ii)

$$\operatorname{Spec}(\hat{A}) \cap \mathbf{R} = \emptyset. \tag{77}$$

**Remark 4** It follows from condition (i) that $\hat{A}$ itself is elliptic. It is then well known that for each $s$ the minimal and the maximal operators generated by $\hat{A}$ in the Sobolev space $H^s(X, E)$ coincide (they will also be denoted by $\hat{A}$), and the invertibility of $p - \hat{A}$ in $C^\infty$ is equivalent to the invertibility in any $H^s$, so we do not indicate the space in which Spec $(\hat{A})$ is taken in (77).

We have already indicated in Section 1 that the spectrum of $\hat{A}$ is discrete, and there is a strip

$$L_c = \{-c \le \operatorname{Im} p \le c\} \tag{78}$$

in which there are no spectral points of $\hat{A}$.

Let $\Gamma_\pm$ be the following contours in the complex plane of the variable $p$:

$$\Gamma_\pm = \{p = \pm\tau \pm ic/2,\ \tau \in \mathbf{R}\} \tag{79}$$

(the sense of $\Gamma_\pm$ corresponds to increasing $\tau$). We set

$$\hat{P}_\pm = \frac{\hat{A}}{2\pi i} \int\limits_{\Gamma_\pm} (p - \hat{A})^{-1} \frac{dp}{p}. \tag{80}$$

**Theorem 7** (i) *For each $\sigma \in \mathbf{R}$, the integral* (80) *strongly converges in $H^\sigma(X, E)$ on the dense subset $H^{\sigma+1}(X, E)$.*

(ii) *The operators $\hat{P}_\pm$ are bounded in the spaces*

$$\hat{P}_\pm : H^\sigma(X, E) \to H^\sigma(X, E) \tag{81}$$

*for each* $\sigma \in \mathbf{R}$.

(iii) *These operators are projections,*

$$\hat{P}_\pm^2 = \hat{P}_\pm, \tag{82}$$

*commute with* $\hat{A}$, *i.e.,*

$$\hat{P}_\pm \hat{A} = \hat{A}\hat{P}_\pm, \tag{83}$$

*and moreover,*

$$\hat{P}_+ + \hat{P}_- = 1. \tag{84}$$

(iv) $\hat{P}_\pm$ *are pseudodifferential operators of order zero. For each* $(x,\xi) \in T^*X$, $\xi \neq 0$, *the principal symbol of* $\hat{P}_+$ *(resp.,* $\hat{P}_-$*) is the projection on the subspace* $L_-(x,\xi)$ *(respectively,* $L_+(x,\xi)$*) of* $E_x$ *formed by the initial data of the solutions* $\varphi(t)$ *of the ordinary differential equation*

$$-i\frac{\partial \varphi}{\partial t} - \sigma(\hat{A})(x,\xi)\varphi = 0 \tag{85}$$

*that are exponentially decaying as* $t \to \infty$ *(resp., as* $t \to -\infty$*) along the subspace* $L_-(x,\xi)$ *(resp.,* $L_+(x,\xi)$*).*

(v) *The operator function* $(p-\hat{A})^{-1}\hat{P}_+$ *is holomorphic for*

$$p \notin \operatorname{Spec}_+(\hat{A}) = \operatorname{Spec}(\hat{A}) \cap \{\operatorname{Im} p > 0\}. \tag{86}$$

*Proof.* The operator family $p-\hat{A}$ belongs to $L^1(X,L_c)$ and is elliptic and invertible there. By Theorem 6, (b) we have $(p-\hat{A})^{-1} \in L^{-1}(X,L_c)$, so that by Theorem 6, (a) we have

$$\left|\left|(p-\hat{A})^{-1}\right|\right|_{H^{\sigma+1}(X)\to H^{\sigma+1}(X)} \leq C_\sigma(1+|p|)^{-1}. \tag{87}$$

Now (i) follows immediately.

Let

$$\sigma(\hat{A}) = A_0(x,\xi) = A_0(x)\xi \tag{88}$$

be the principal symbol of $\hat{A}$. The principal symbol of $(p-\hat{A})^{-1}$ (treated as an operator with parameter) is then $(p-A_0(x)\xi)^{-1}$. To prove that $\hat{P}_+$ is a classical pseudodifferential operator, it sufficies to prove that in any system of local coordinates the integral giving the complete symbol of $\hat{P}_+$ is a classical symbol of order 0. In fact, it sufficies to consider the principal symbol alone, since the convergence of the subsequent terms is an easy exercise. More precisely, in any coordinate neighborhood on $X$ we have

$$\begin{aligned}(p-\hat{A})^{-1} &= \{(p-A_0(x)\xi)^{-1}\varphi(|\xi|)\}^{\wedge} \\ &\quad +\{(p-A_0(x)\xi)^{-1}(1-\varphi(|\xi|))\}^{\wedge}+\hat{B}(p), \end{aligned} \tag{89}$$

where $\varphi(|\xi|)$ is an excision function,

$$\varphi(|\xi|)=\begin{cases} 0 & \text{for} \quad |\xi|\leq 1, \\ 1 & \text{for} \quad |\xi|\geq 2, \end{cases} \tag{90}$$

and $\hat{B}(p)\in L^{-2}(X,L_c)$. It is easy to see that only the first term contributes to the principal symbol of $\hat{P}_{\pm}$. Now we can perform the symbolic calculation:

$$\begin{aligned}\sigma(\hat{P}_+)(x,\xi) &= \frac{A_0(x)\xi}{2\pi i}\int\limits_{\Gamma_+}(p-A_0(x)\xi)^{-1}\frac{dp}{p} \\ &= \frac{A_0(x)\omega}{2\pi i}\int\limits_{\Gamma_+/|\xi|}(\eta-A_0(x)\omega)^{-1}\frac{d\eta}{\eta}, \end{aligned} \tag{91}$$

where $\omega=\xi/|\xi|$. The last expression can be transformed to

$$\sigma(\hat{P}_+)(x,\xi)=\frac{A_0(x)\xi}{2\pi i}\int\limits_{\gamma_+}(\eta-A_0(x)\omega)^{-1}\frac{d\eta}{\eta}, \tag{92}$$

where $\gamma_+$ is a closed contour, independent of $\xi$, that lies in the upper half-plane and surrounds all eigenvalues of $A_0(x)\omega$ with positive imaginary part. It is now clear that $\hat{P}_+$ is a $\psi DO$ of order 0, whence (ii) follows.

The proof of (iv) now is reduced to the assertion that $\sigma(\hat{P}_+)$ is the projection onto $L_-(x,\xi)$ along $L_+(x,\xi)$, which is quite standard so that we omit the calculation. The proof of (ii) and (iv) for $\hat{P}_-$ goes in a similar manner.

Let us prove (iii). Equation (83) is obvious. To prove (84) note that

$$\hat{P}_+ + \hat{P}_- = \frac{\hat{A}}{2\pi i}\int\limits_{\Gamma_0}(p-\hat{A})^{-1}\frac{dp}{p}, \tag{93}$$

where $\Gamma_0$ is a small circle surrounding the origin clockwise. By Cauchy's residue theorem, the integral equals $2\pi i\hat{A}^{-1}$, which proves (84). Next, let us calculate $(\hat{P}_+)^2$

(the calculation for $\hat{P}_-$ is similar). Let $\Gamma_+ = \Gamma_+ + i\varepsilon$, where $\varepsilon > 0$ is sufficiently small. We have, by the resolvent identity,

$$\begin{aligned}
(\hat{P}_+)^2 &= -\left(\frac{1}{2\pi}\right)^2 \int\limits_{\Gamma'_+} \left\{ \int\limits_{\Gamma_+} \frac{\hat{A}^2}{\xi\mu}(\xi - \hat{A})^{-1}(\mu - \hat{A})^{-1}\, d\mu \right\} d\xi \\
&= -\left(\frac{1}{2\pi}\right)^2 \int\limits_{\Gamma'_+} \left\{ \int\limits_{\Gamma_+} \frac{\hat{A}^2}{\mu\xi(\mu - \xi)}[(\xi - \hat{A})^{-1} - (\mu - \hat{A})^{-1}]\, d\mu \right\} d\xi \\
&= -\left(\frac{\hat{A}}{2\pi}\right)^2 \int\limits_{\Gamma'_+} (\xi - \hat{A})^{-1} \left\{ \int\limits_{\Gamma_+} \frac{d\mu}{\mu(\mu - \xi)} \right\} \frac{d\xi}{\xi} \\
&\quad -\left(\frac{\hat{A}}{2\pi}\right)^2 \int\limits_{\Gamma'_+} \left\{ \int\limits_{\Gamma_+} (\mu - \hat{A})^{-1} \frac{d\mu}{\mu(\mu - \xi)} \right\} \frac{d\xi}{\xi} \\
&= I_1 + I_2.
\end{aligned}$$

Now

$$I_2 = -\left(\frac{\hat{A}}{2\pi}\right)^2 (\mu - \hat{A})^{-1} \left\{ \int\limits_{\Gamma'_+} \frac{d\xi}{\xi(\mu - \xi)} \right\} \frac{d\mu}{\mu} = 0,$$

since the inner integral is zero. On the other hand we have

$$\begin{aligned}
I_1 &= -\left(\frac{\hat{A}}{2\pi}\right)^2 \cdot 2\pi i \int\limits_{\Gamma'_+} (\xi - \hat{A})^{-1} \frac{d\xi}{\xi^2} \\
&= \frac{\hat{A}}{2\pi i} \left[ \int\limits_{\Gamma_+} (\xi - \hat{A})^{-1} \frac{d\xi}{\xi} + \int\limits_{\Gamma_+} \frac{d\xi}{\xi^2} \right] = \hat{P}_+
\end{aligned}$$

as desired.

Let us now prove (v). Suppose that $\mu$ lies in the lower half-plane; then

$$\begin{aligned}
(\mu - \hat{A})^{-1}\hat{P}_+ &= \frac{\hat{A}}{2\pi i} \int\limits_{\Gamma_+} (\xi - \hat{A})^{-1}(\mu - \hat{A})^{-1} \frac{d\xi}{\xi} \\
&= \frac{\hat{A}}{2\pi i} \int\limits_{\Gamma_+} (\xi - \hat{A})^{-1} \frac{d\xi}{\xi(\mu - \xi)} - \frac{\hat{A}(\mu - \hat{A})^{-1}}{2\pi i} \int\limits_{\Gamma_+} \frac{d\xi}{\xi(\mu - \xi)}. \qquad (94)
\end{aligned}$$

The second integral in (94) is zero, and the first one is holomorphic for $\mu$ lying in the lower half-plane. Theorem 7 is thereby proved.

**Remark 5** It follows from Theorem 6 that

$$(\mu - \hat{A})^{-1}\hat{P}_+ \in L^{-1}(X, L_c \cup \{\operatorname{Im} p < 0\}), \tag{95}$$

so that $(\mu - \hat{A})^{-1}\hat{P}_+$ satisfies the corresponding estimates in the lower half-plane.

Now let us prove Theorems 1 and 2. It suffices to prove that the mapping (35) is the continuous two-sided inverse for the mapping

$$(\hat{D}, \hat{P}_+ \circ j_X) : \, H^s(C_+) \to H^{s-1}(C_+) \oplus \hat{P}_+ H^{s-1/2}(X) \tag{96}$$

(recall that we assume $m = 1$, so that we have $j_X \equiv j_X^0$ instead of $j_X^{m-1}$, and the mappings $\pi_0$ and $\lambda$ in the formula for $\mathcal{R}[f, g]$ have to be omitted).

We shall only prove the continuity of the mapping (35); the fact that it is a two-sided inverse for the mapping (96) can be proved by routine computations with the help of the Laplace transform, which we leave to the reader. Next, it sufficies to prove the continuity of $\mathcal{R}$ for $s = 1$; then we can apply a trivial induction over $s$.

Thus, we have

$$\mathcal{R}[f, g] = \mathcal{F}^{-1}\{(p - \hat{A})^{-1}[\tilde{f}\,(p) - ig - \frac{1}{2\pi i}\int_{\Gamma_-}(\nu - \hat{A})^{-1}\,\tilde{f}\,(\eta)\,d\eta]\}, \tag{97}$$

and we have to prove that $\mathcal{R}$ is continuous as an operator

$$\mathcal{R} : \, L^2(C_+) \oplus \hat{P}_+ H^{1/2}(X) \to H^1(C_+). \tag{98}$$

**Lemma 4** *The mapping*

$$U_0 : \, f \mapsto \int_{\Gamma_-}(\nu - \hat{A})^{-1}\,\tilde{f}\,(\eta)\,d\eta, \tag{99}$$

*is continuous from $L^2(C_+)$ to $H^{1/2}(X)$.*

*Proof.* We have

$$U_0 = i^* \circ \left(-i\frac{\partial}{\partial t} - \hat{A}\right)^{-1} \circ j, \tag{100}$$

where

$$j : \, L^2(C_+) \to L^2(C) \tag{101}$$

is the natural isometric embedding, the operator

$$\left(-i\frac{\partial}{\partial t} - \hat{A}\right)^{-1} : L^2(C) \to H^1(C) \tag{102}$$

is continuous (since $-i\partial/\partial t - \hat{A}$ is elliptic and invertible), and the restriction operator

$$i^* : H^1(C) \to H^{1/2}(X) \tag{103}$$

is continuous by the trace theorem.

**Lemma 5** *The mappings*

$$U_1 : f \mapsto \mathcal{F}^{-1}\{(p-\hat{A})^{-1} \tilde{f}(p)\}, \tag{104}$$

$$U_2 : f \mapsto \hat{A}\mathcal{F}^{-1}\{(p-\hat{A})^{-1} \tilde{f}(p)\} \tag{105}$$

*are continuous from* $L^2(C_+)$ *to* $L^2(C)$.

*Proof.* This is obvious, since

$$\left(-i\frac{\partial}{\partial t} - \hat{A}\right)^{-1} \quad \text{and} \quad \hat{A}\left(-i\frac{\partial}{\partial t} - \hat{A}\right)^{-1}$$

are pseudodifferential operators of nonpositive order.

**Lemma 6** *The mappings*

$$U_3 : g \mapsto \mathcal{F}^{-1}\{(p-\hat{A})^{-1}g\}, \tag{106}$$

$$U_4 : g \mapsto \hat{A} \circ \mathcal{F}^{-1}\{(p-\hat{A})^{-1}g\} \tag{107}$$

*are continuous from* $H^{1/2}(X)$ *to* $L^2(C_+)$.

*Proof.* Let us prove the second assertion (the proof of the first one is much easier). Since $\hat{A}$ is continuous as an operator

$$\hat{A} : H^{1/2}(X) \to H^{-1/2}(X),$$

it suffices to prove that the operator

$$j : f(x) \to jf(x,t) = \mathcal{F}^{-1}_{p\to t}\{(p-\hat{A})^{-1}f\}$$

is continuous in the spaces

$$j:\, H^{-1/2}(X) \to L^2(C).$$

Without loss of generality, we argue in terms of local coordinates on $X$ (to reduce everything to this case, we can use a partition of unity). Then, modulo smoothing operators on $X$ with kernel rapidly decreasing as $|p| \to \infty$, $(p-\hat{A})^{-1}$ is a pseudodifferential operator,

$$(p-\hat{A})^{-1} \simeq B\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}, p\right)$$

with symbol $B(x,\xi,p)$ compactly supported in $x$ and satisfying the estimates

$$\left|\frac{\partial^{\alpha+\beta}B(x,\xi,p)}{\partial x^\alpha \partial \xi^\beta}\right| \le C(1+|\xi|+|p|)^{-1-|\beta|}, \qquad |\alpha|+|\beta| = 0,1,2,\ldots.$$

Now, by Parseval's indentity

$$\begin{aligned} I &= \left\|\mathcal{F}^{-1}_{p\to t}\left\{B\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}, p\right) f(x)\right\}\right\|^2_{L^2(C)} \\ &= \int_{-\infty}^{\infty} \left\|B\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}, p\right) f(x)\right\|^2_{L^2(X)} dp \\ &= \int_{-\infty}^{\infty} dp \int d\xi \left|\int \tilde{f}\,(\eta)\, d\eta\right|^2, \end{aligned}$$

where $\tilde{f}\,(\eta)$ is the Fourier transform of $f$ with respect to $x$ and

$$\tilde{B}\,(\xi-\eta,\eta,p) = \left(\frac{1}{2\pi}\right)^{n/2} \int e^{ix(\xi-\eta)} B(x,\xi,p)\, dx. \tag{108}$$

Note that $\tilde{B}\,(\xi-\eta,\eta,p)$ satisfies the estimates

$$|\,\tilde{B}\,(\xi-\eta,\eta,p)| \le C\cdot(1+|\xi-\eta|)^{-N}(1+|p|+|\xi|)^{-1},$$

$N$ is arbitrary, which can immediately be proved by integration by parts in (108).

Now

$$
\begin{aligned}
|I| &= \left| \int_{-\infty}^{\infty} dp \int d\eta \int d\eta' \{ \tilde{B}\,(\xi-\eta,\eta,p)\ \tilde{f}\,(\eta)\ \tilde{B}^*(\xi-\eta',\eta',p)\overline{\tilde{f}\,(\eta')}\} \right| \\
&\le \int\!\!\int\!\!\int\!\!\int (1+|\xi-\eta|)^{-N}(1+|p|+|\xi|)^{-2} \qquad (109) \\
&\quad \times |\ \tilde{f}\,(\eta)|\,|\ \tilde{f}\,(\eta')|\, dp\, d\xi\, d\eta\, d\eta'.
\end{aligned}
$$

We have, for large $|\xi|$,

$$
\begin{aligned}
\int (1+|p|+|\xi|)^{-2} dp &= \int \left( \frac{1}{|\xi|} + \left| \frac{p}{\xi} \right| + 1 \right)^{-2} |\xi|^{-2}\, |\xi|\, d\left( \frac{p}{|\xi|} \right) \\
&\le |\xi|^{-1} \int \frac{d\xi}{(1+|\xi|)^2} = C \cdot |\xi|^{-1},
\end{aligned}
$$

and for small $|\xi|$ the integral is bounded. Thus the estimate (109) can be continued as

$$
|I| \le C \int\!\!\int\!\!\int (1+|\xi-\eta|)^{-N}(1+|\xi-\eta'|)^{-N}(1+|\xi|)^{-1}|\ \tilde{f}\,(\eta)|\,|\ \tilde{f}\,(\eta')|\, d\xi\, d\eta\, d\eta'.
$$

By Peetre's inequality, we have

$$
(1+|\xi|)^{-1/2} \le (1+|\eta|)^{-1/2}(1+|\xi-\eta|)^{1/2},
$$

which implies for $N_1 = N - 1/2$ the estimate

$$
\begin{aligned}
|I| &\le C \int\!\!\int\!\!\int (1+|\xi-\eta|)^{-N_1}(1+|\xi-\eta'|)^{-N_1}|(1+|\eta|)^{-1/2}\ \tilde{f}\,(\eta)| \\
&\qquad \cdot |(1+|\eta'|)^{-1/2}\ \tilde{f}\,(\eta')|\, d\xi\, d\eta\, d\eta'.
\end{aligned}
$$

Moreover,

$$
\int (1+|\xi-\eta|)^{-N_1}(1+|\xi-\eta'|)^{-N_1} d\xi
$$

$$\leq \int (1+|\xi-\eta|)^{-N_1}(1+|\xi-\eta'|)^{-N_2}d\xi \qquad \{\text{where} \quad N_2 = N_1 - \left[\frac{n}{2}\right] - 1\}$$

$$= \int (1+|\xi|)^{-N_1}(1+|\xi+\eta-\eta'|)^{-N_2}d\xi$$

$$\leq \int (1+|\xi|)^{N_2-N_1}d\xi(1+|\eta-\eta'|)^{-N_2} \qquad \{\text{by Peetre's inequality}\}$$

$$\leq C(1+|\eta-\eta'|)^{-N_2},$$

and hence

$$|I| \leq C \int\int (1+|\eta-\eta'|)^{-N_2}\varphi(\eta)\varphi(\eta')d\eta\, d\eta' \equiv I_1, \tag{110}$$

where

$$\varphi(\eta) = (1+|\eta|)^{-1/2}\left|\tilde{f}\,(\eta)\right| \in L_2$$

and

$$||\varphi(\eta)||_{L^2} \sim ||f||_{H^{-1/2}(X)}.$$

The right-hand side of (110) is the inner product $(K\varphi,\varphi)_{L_2}$, where $K$ is the integral operator with kernel

$$K(\eta-\eta') = C(1+|\eta-\eta'|)^{-N_2}.$$

By using Parseval's identity and properties of convolution, we obtain

$$I_1 = (\tilde{K}\,(\tau)\,\tilde{\varphi}\,(\tau), \tilde{\varphi}\,(\tau))_{L_2},$$

where the tilde stands for the Fourier transform.

Since

$$\tilde{K}\,(\tau) = \left(\frac{1}{2\pi}\right)^{n/2}\int \frac{e^{i\xi\tau}}{(1+|\xi|)^{N_2}}d\tau$$

is uniformly bounded (provided $N_2$ is sufficiently large), we obtain $|I_1| \leq C \cdot ||\varphi||^2_{L_2}$, whence the desired property of the operator $j$ follows. The proof is complete.

Now let

$$u = \mathcal{R}[f,g].$$

By combining Lemmas 4, 5 and 6, we can readily estimate

$$||u||_{L^2(C_+)} \leq C(||f||_{L^2(C_+)} + ||g||_{H^{1/2}(X)}) \tag{111}$$

and

$$\left|\left|\hat{A}u\right|\right|_{L^2(C_+)} \leq C(||f||_{L^2(C_+)} + ||g||_{H^{1/2}(X)}). \tag{112}$$

Since

$$\frac{\partial u}{\partial t} = i(\hat{A}u + f),$$

we see that (with some other constant $C$)

$$\left|\left|\frac{\partial u}{\partial t}\right|\right|_{L^2(C_+)} \leq C(||f||_{L^2(C_+)} + ||g||_{H^{1/2}(X)}); \tag{113}$$

recall that $\hat{A}$ is elliptic on $X$, so that finally we have

$$||u||_{H^1(C_+)} \leq C(||f||_{L^2(C_+)} + ||g||_{H^{1/2}(X)}),$$

as desired.

Now let us indicate the modifications needed in the proof for the case in which $\hat{D}$ is an operator of arbitrary order $m > 1$. The main difference is in that now $\hat{A}$ will be an operator of the first order in the sense of Douglis–Nirenberg in the spaces $H^s_m(X)$ , and the only essential novelty in the proof is that $\hat{Q}(p)$ in the formula

$$(p - \hat{A})^{-1} = \hat{D}^{-1}(p)\hat{Q}(p) \tag{114}$$

is a polynomial of order $\leq m - 1$ in $p$, which ensures the convergence of the integral (21) and the corresponding integral for the complete symbol in appropriate spaces. The subsequent proof goes, *mutatis mutandis,* along the same lines. We omit these calculations, which are extremaly awkward.

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