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Preprint No. 97/35
November 1997

Potsdam 1997

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November 19, 1997

Abstract

We construct a theory of general boundary value problems for differential operators whose symbols do not necessarily satisfy the Atiyah–Bott condition [3] of vanishing of the corresponding obstruction. A condition for these problems to be Fredholm is introduced and the corresponding finiteness theorems are proved.

Keywords: elliptic boundary value problems, Atiyah–Bott condition, Calderón projections, Cauchy–Riemann operator, Euler operator

1991 AMS classification: 35J, 35P, 35S

*Supported by Deutsche Forschungsgemeinschaft and RFBR grant No. 97-01-00703.

Contents

Introduction

The theory of boundary value problems in Sobolev spaces for elliptic differential equations is at present well known (e.g., see [1, 18]). The main theorem concerning these problems states that under some algebraic conditions (the *Shapiro–Lopatinskii conditions*) this problem is Fredholm. One of the important features of this theory is that not any elliptic operator on a manifold with boundary admits boundary conditions of the above type. It was found out ([3], see also [22, 23]) that the obstruction to the existence of (pseudo)differential Fredholm boundary value problems in Sobolev spaces is of topological character, and hence a given elliptic operator admits a Fredholm boundary value problem only if the corresponding obstruction vanishes.

Unfortunately, this obstruction does not vanish for some important geometric operators like the Hirzebruch (signature) or Dirac operators. In particular, this leads to the fact that the general formula for the index of elliptic operators on manifolds with boundary (e.g., see [10, 22]) does not apply to these operators, which are important in topology and Riemannian geometry.

An attempt to find a formula for the signature in the case of manifolds with boundary has led Atiyah, Patodi, and Singer [2] to the consideration of a boundary value problem for the Hirzebruch (and Dirac) operator in the space L_2. More precisely, these operators are treated as unbounded operators in L_2 with domains determined by homogeneous boundary conditions of a special form. In this setting, these operators are Fredholm, and for example, the index computation for the Hirzebruch operator on a manifold with boundary results in an expression for the signature of the manifold in terms of its L-genus and an additional term called the *η-invariant* [2].

However, the two cases are apparently quite different: while for an elliptic differential operator A with boundary conditions of Shapiro–Lopatinskii type we can either consider the boundary problem itself or treat A as an unbounded operator in L^2 corresponding to the homogeneous boundary conditions, only the latter possibility is available if boundary conditions of Shapiro–Lopatinskii type do not exist for A. Hence the following question is quite natural: *Is there a general theory of boundary value problems which includes the classical (Shapiro–Lopatinskii) problems but also permits one to pose Fredholm nonhomogeneous boundary value problems for elliptic operators for which classical boundary value problems fail to exist?* In the present paper, we describe such a theory. Most of the ingredients needed there are in fact contained in Seeley's papers [24, 25]. However, for operators violating the Shapiro–Lopatinskii condition he only considered homogeneous boundary value problems in L^2, of which the problems considered in [2] are a very special case.

Let us outline our main idea. Simple examples given by the Cauchy–Riemann, Bitsadze [5], and other equations show that although they do not possess Fredholm boundary value problems in Sobolev spaces, such problems do exist if the right-hand sides in the boundary conditions belong to finer spaces (for example, for the Cauchy–Riemann equations these are the Hardy spaces; e.g., see [6, 21]). In fact, these spaces are (closed) subspaces of some Sobolev spaces, which permits one to suggest that to define a Fredholm boundary value problem one must in the general case use *subspaces* of Sobolev spaces. In the present paper, we implement this scheme. More precisely, the $(m-1)$st-order jets at the boundary of solutions of a homogeneous mth-order elliptic equation always form a subspace of the Sobolev space of sections of the corresponding bundle over the boundary, which readily gives a trivial example of a boundary value problem of the above type. In classical boundary value problems, the boundary operator can be viewed as an isomorphic (or almost isomorphic, i.e. Fredholm) mapping of this subspace onto the Sobolev space of sections of some other bundle over the boundary. In nonclassical (general) boundary value problems, the mapping is onto a subspace that may be *infinite-codimensional.* From the topological viewpoint, the obstruction to posing a classical (Shapiro–Lopatinskii) boundary value problem is equivalent to the nonexistence of an isomorphism of a certain vector bundle over $T_0^*X = T^*X\backslash\{0\}$, where X is the boundary, to the pullback of a vector bundle over X. From the analytical viewpoint, the obstruction is the nonexistence of a *pseudodifferential almost isomorphism* between a certain subspace of the Sobolev space of boundary jets and the Sobolev space of sections of a vector bundle over X. It is easily recognized that the latter condition is the "quantized" version of the former.

The structure of the paper is as follows. It consists of three sections. The first section comprised the main results. Specifically, the definition of a general boundary value problem is introduced and discussed in Subsection 1.1; a criterion for the Fredholm property to hold is established in Subsection 1.2; a pseudodifferential statement of general boundary value problems is described and the corresponding finiteness theorem is proved in Subsection 1.3. Finally, in Subsection 1.4 we discuss the Shapiro–Lopatinskii conditions.

The reasoning in Section 1 is based on the use of the Calderón–Selley boundary projection operator [7, 24, 25], whose construction involves the inverse of an elliptic operator on the double of the original manifold. This is a little disadvantage, because it it intuitively clear that everything concerning the boundary conditions must be determined by the behavior of the operator in question near the boundary (or even at the boundary) rather that on the entire manifold (not to speak of the rather ambiguous continuation to the double). That is why we have included Section 2, where the finiteness theorem of Subsection 1.3 is proved be constructing a parametrix of the problem in quite a "classical" manner (we freeze the coefficients at an arbitrary point of the boundary, pass to the Fourier transform with respect to the tangential variables, and study the resulting ordinary differential equation). We do some preliminary work in Subsection 2.1, examine the model problem with frozen coefficients in the half-space in Subsection 2.2, and construct the global parametrix in Subsection 2.3.

Section 3 contains two simple and familiar examples, in one of which there are no classical boundary value problems (the Cauchy–Riemann operator, Subsection 3.1), whereas the other possesses those (the Euler operator, Subsection 3.2).

Acknowledgement. The authors are deeply grateful to Dr. Vladimir Nazaikinskii, who has thoroughly read the manuscript, made a lot of valuable remarks, and suggested quite a few improvements.

1 Main Results

1.1 Definition of general boundary value problems

Let M be a compact C^∞ manifold with smooth boundary $\partial M = X$, and let

$$\hat{D}\colon C^\infty(M,E) \to C^\infty(M,F), \tag{1}$$

where E and F are vector bundles over M, be an mth-order elliptic differential operator on M. We shall define the abstract notion of a *general boundary value problem* (BVP) for the operator (1) (which includes classical BVPs as a special case), introduce a specific construction of general BVPs, and show that with this construction one can always achieve a BVP that is Fredholm in relevant function spaces. As a by-product, we obtain the well-known condition for the existence of classical boundary value problems satisfying the Shapiro–Lopatinskii condition.

As usual in the theory of elliptic operators, we consider the operator (1) in Sobolev spaces,

$$\hat{D}\colon H^s(M,E) \to H^{s-m}(M,F), \tag{2}$$

where $s > m - 1/2$ is an integer. The boundary conditions will be imposed on the $(m-1)$st-order jet $j_X^{m-1}(u)$ of the solution $u \in H^s(M,E)$ at the boundary; to treat them conveniently, we take a collar neighborhood U of X in M and identify it with the product $X \times [0,1)$ (for example, this can be done by choosing a Riemannian metric on M, whence $(x,t) \in X \times [0,1)$ can be identified with the point at a distance t from X on the geodesic issuing from $x \in X$ in the inward normal direction). By the trace theorem, we then have a continuous mapping

$$j_X^{m-1} : H^s(M,E) \to \mathcal{H}_m^{s-1/2}(X,E) \equiv \bigoplus_{j=0}^{m-1} H^{s-1/2-j}(X, i^*E),$$

which takes each $u \in H^s(M,E)$ to the $(m-1)$st-order jet

$$j_X^{m-1}(u) = \left(u|_X\,, -\,i\frac{\partial u}{\partial t}\bigg|_X, \ldots, \left[\left(-i\frac{\partial}{\partial t}\right)^m u\right]\bigg|_X \right)$$

at the boundary (here i^*E is the pullback of E under the embedding

$$i : X \hookrightarrow M,$$

and the restriction $\varphi|_X$ is defined by

$$\varphi|_X = \lim_{\tau\to 0+} \varphi|_{t=\tau}\,,$$

the limit being taken in the corresponding Sobolev space on X). For brevity, in the following we sometimes write E instead of i^*E.

Definition 1 A *general boundary value problem* for the operator (2) is a problem of the form

$$\begin{aligned} \hat{D}\, u = f \in H^{s-m}(M, F), \\ \hat{B}\,(j_X^{m-1}(u)) = g \in \mathcal{L} \end{aligned} \tag{3}$$

for the unknown function $u \in H^s(M, E)$, where $\mathcal{L}$ is some Banach space and

$$\hat{B}\colon \mathcal{H}_m^{s-1/2}(X, E) \to \mathcal{L} \tag{4}$$

is a continuous linear operator.

In other words, a general BVP is an operator of the form

$$(\hat{D}, \hat{B} \circ j_X^{m-1}) :\ H^s(M, E) \to H^{s-m}(X, E) \oplus \mathcal{L} \tag{5}$$

with $\hat{D}$ and $\hat{B}$ as in (2) and (4).

Remark 1 We must draw a distinction between the boundary operator $\hat{B} \circ j_X^{m-1}$ in (3) and (5) and the "general boundary operators" (e.g., see Sternin [27, 28], where they were considered in the framework of relative elliptic theory). The latter have the form $i^*\, \hat{b}$, where $\hat{b} \equiv j_X^0\, \hat{b}$ is a pseudodifferential operator on M rather than on X. On the one hand, our definition is more restrictive in that $\hat{B} \circ j_X^{m-1}$ is necessarily a *differential* operator of order $\leq m-1$ in the direction normal to the boundary (this requirement sounds quite natural for boundary value problems, as opposed to Sobolev problems). On the other hand, the codomain of $i^* \circ b$ is always a Sobolev space, whereas $\hat{B} \circ j_X^{m-1}$ is allowed to act into an arbitrary Banach space.

Remark 2 A *classical* BVP is a specific case of (3) in which $\mathcal{L}$ is the Sobolev space of sections of some vector bundle over X and $\hat{B}$ is a (pseudo)differential operator.

The main reason for introducing the notion of a general BVP is that for a given $\hat{D}$ we can always find a problem (3) with the Fredholm property (which is *not* the case with classical BVPs). As we shall see shortly, this readily follows from the results of Seeley [24, 25], who however did not make the final step—for operators $\hat{D}$ such that classical BVPs with the Shapiro–Lopatinskii condition fail to exist, he only considered problems with homogeneous boundary conditions ($g = 0$) and with $\hat{B}$ a pseudodifferential operator. We point out that Seeley's work essentially uses and develops the ideas due to Calderón ([7]; see also [8]), who was the first to introduce projection operators of this type in order to study boundary value problems. Close results are due to Boutet de Monvel [9] and Hörmander [17]. Calderón's projections found various applications in differential equations and mathematical physics (e.g., see [1, 15, 16, 17, 19, 20]).

1.2 The finiteness theorem (abstract case)

Let us show how Seeley's reasoning can be adapted to our aims. First, we give an intuitive argument to clarify the idea, and then fill in the missing details. By a fairly simple technique, $\hat{D}$ can be extended to an elliptic differential operator on the double $2M$ of the manifold M (note that $2M$ is a closed compact manifold), and we can define a continuous operator extending any $f \in H^{s-m}(M, F)$ to $2M$ with smoothness $s - m$ preserved. Since $\hat{D}$ is elliptic, it is now pretty clear that (modulo a finite-dimensional defect, which can be neglected as far as the Fredholm property is concerned) we can use a right almost inverse of $\hat{D}$ to reduce problem (3) to a problem of the same form with $f = 0$ (and, of course, with different g):

$$\begin{array}{ll} \hat{D}\, u = 0, & \\ \hat{B}\, (j_X^{m-1}(u)) = g, & \end{array} \qquad u \in H^s(M, E).$$

Now let

$$N(\hat{D}, s) = \{u \in H^s(M, E) \mid \ \hat{D}\, u = 0\}$$

be the kernel of the operator (2). We see that the point is to describe the linear manifold

$$R_0(\hat{D}, s) = j_X^{m-1}(N(\hat{D}, s)) \subset \mathcal{H}_m^{s-1/2}(X, E),$$

that is, the space of boundary data for the solutions in $H^s(M, E)$ of the homogeneous equation. If $R_0(\hat{D}, s)$ is a subspace (i.e. is closed), then we can hope that any operator $\hat{B}$ (see (4)) such that

$$\hat{B}\Big|_{R_0(\hat{D}, s)} : R_0(\hat{D}, s) \to \mathcal{L} \tag{6}$$

is an isomorphism or at least a Fredholm operator gives rise to a Fredholm BVP (3). In particular, the simplest choice is as follows:

$$\mathcal{L} = R_0(\hat{D}, s), \quad \text{and} \quad \hat{B} \quad \text{is a continuous projection onto } \mathcal{L}.$$

Now we proceed to rigorous exposition. Let

$$N_0(\hat{D}) = \{u \in C^\infty(M) \mid \hat{D}\, u = 0,\ \ j_X^{m-1}(u) = 0\}. \tag{7}$$

This is a finite-dimensional space.

Seeley proved the following assertion.

Theorem 1 ([24, 25]) *There exists an operator*

$$\hat{S}\colon\ (C^\infty(X,E))^m \to C^\infty(M,E)$$

such that

i) *for any s, $\hat{S}$ extends to a continuous mapping*

$$\hat{S}\colon\ \mathcal{H}_m^{s-1/2}(X,E) \to N(\hat{D},s) \subset H^s(M,E);$$

ii) *$N(\hat{D},s)$ is the direct sum of $N_0(\hat{D})$ and $\hat{S}\,(\mathcal{H}_m^{s-1/2}(X,E))$, that is,*

$$N(\hat{D},s) = N_0(\hat{D}) \oplus \hat{S}\,(\mathcal{H}_m^{s-1/2}(X,E)); \tag{8}$$

iii) *the operator*

$$\hat{P}^{\,+} = j_X^{m-1} \circ \hat{S}\colon\ \mathcal{H}_m^{s-1/2}(X,E) \to \mathcal{H}_m^{s-1/2}(X,E)$$

is a continuous projection onto $R_0(\hat{D},s)$; moreover, $\hat{P}^{\,+}$ is a pseudodifferential operator whose principal symbol $\sigma(\hat{P}^{\,+})(x,\xi)$ is a projection onto the space $L^-(x,\xi)$ of initial data of the solutions of the ordinary differential equation

$$\sigma(\hat{D})\left(x,0,\xi,-i\frac{\partial}{\partial t}\right)\varphi(t) = 0 \tag{9}$$

such that $\varphi(t) \to 0$ as $t \to +\infty$.

Corollary 1 $R_0(\hat{D},s) = \operatorname{Im}\ \hat{P}^{\,+}$ is closed.

Next, for each s Seeley constructed a bounded operator

$$\hat{C}\colon\ H^{s-m}(M,E) \to H^s(M,E)$$

such that

$$\hat{D}\hat{C}\, f = f \tag{10}$$

whenever $f \in H^{s-m}(M, E)$ is orthogonal[1] to the finite-dimensional space $N_0(\hat{D}^{\,*})$. In other words, $\hat{C}$ is a right inverse of $\hat{D}$ modulo finite-dimensional operators.

Now we can state and prove our first theorem concerning general BVPs.

Theorem 2 *The general boundary value problem* (3) (*or, which is the same, the operator* (5)) *is Fredholm if and only if the operator* (6)*, i.e., the restriction*

$$\hat{B}\Big|_{\mathrm{Im}\,\hat{P}^{+}} : \mathrm{Im}\ \hat{P}^{\,+} \to \mathcal{L},$$

has the Fredholm property.

Proof. First, we reduce the assertion to the case in which the right-hand side f is zero.

Lemma 1 *Problem* (3) *is Fredholm if and only if so is the problem*

$$\begin{aligned} &\hat{D}\, u = 0, \\ &\hat{B}\, (j_X^{m-1}(u)) = g \in \mathcal{L}. \end{aligned} \tag{11}$$

In other words, the operator (5) *is Fredholm if and only if so is the operator*

$$\hat{B} \circ j_X^{m-1} : N(\hat{D}, s) \to \mathcal{L}. \tag{12}$$

Proof of Lemma 1. Obviously, the kernels of the operators (5) and (12) coincide. Let us study the cokernels. We claim that the cokernel of the operator (5) is isomorphic to that of the operator (12) plus (direct sum) $N_0(\hat{D}^{\,*})$. Indeed, let f be orthogonal to $N_0(\hat{D}^{\,*})$. Then, by (10), the substitution

$$u = \hat{C}\, f + \tilde{u}$$

reduces problem (3) to problem (11) for $\tilde{u}$ with g replaced by

$$\tilde{g} = g - \hat{B}\, j_X^{m-1}\, \hat{C}\, f.$$

[1]We assume that some measure is chosen on M and some Hermitian metrics in the bundles under consideration are fixed.

Since dim $N_0(\hat{D}^*) < \infty$, the assertion follows readily.

Now we have the decomposition

$$\hat{B} \circ j_X^{m-1} : N(\hat{D}, s) \xrightarrow{j_X^{m-1}} R_0(\hat{D}, s) = \mathrm{Im}\ \hat{P}^+ \xrightarrow{\hat{B}} \mathcal{L},$$

and the assertion of Theorem 2 can readily be obtained from Lemma 1 and the following statement.

Lemma 2 *The operator*

$$j_X^{m-1} : N(\hat{D}, s) \to R_0(\hat{D}, s) \tag{13}$$

is Fredholm.

Proof. By the definition of $R_0(\hat{D}, s)$, the operator (13) is an epimorphism. Next, by virtue of (7) the kernel of the operator (13) is just $N_0(\hat{D})$, which is finite-dimensional. This completes the proof of Lemma 2 and Theorem 2.

1.3 The finiteness theorem (pseudodifferential case)

In applications, it is often important to describe the space $\mathcal{L}$ and the boundary operator $\hat{B}$ in explicit terms. The form of the "simplest" Fredholm BVP in which $\hat{B}=\hat{P}^+$ and $\mathcal{L} = \mathrm{Im}\ \hat{P}^+$ suggests such a description: $\hat{B}$ must be a pseudodifferential operator acting in sections of vector bundles on X,

$$\hat{B}\colon \mathcal{H}_m^{s-1/2}(X, E) \to H^\sigma(X, G),$$

and the subspace $\mathcal{L} \subset H^\sigma(X, G)$ must be described as the image of some pseudodifferential operator

$$\hat{P}\colon H^\sigma(X, G) \to H^\sigma(X, G)$$

(for simplicity, we assume that $\hat{P}$ is a pseudodifferential operator of order zero). Moreover, we assume that the principal symbol $\sigma(\hat{P})(x,\xi)$ is a projection operator in $(\pi^* G)_{(x,\xi)}$, $(x, \xi) \in T_0^* X$, where

$$\pi : T_0^* X \to X$$

is the natural projection, the range of $\hat{P}$ is closed, and Im $\hat{B} \subset$ Im $\hat{P}$. We endow Im $\hat{P}$ with the Hilbert space structure inherited from $H^\sigma(X, G)$.

Consider the general boundary value problem ($\hat{D}$ is an elliptic operator)

$$
\begin{aligned}
&\hat{D}\, u = f \in H^{s-m}(M,F),\\
&\hat{B}\, (j_X^{m-1} u) = g \in \operatorname{Im}\ \hat{P} \subset H^{\sigma}(X,G)
\end{aligned}
\tag{14}
$$

for the unknown function $u \in H^s(M,E)$.

Theorem 3 *Suppose that the following condition is satisfied:*

(GSL) *For any $(x,\xi) \in T_0^*X$, the principal symbol $\sigma(\hat{B})(x,\xi)$ of the operator $\hat{B}$ induces an isomorphism between the spaces*[2] *$L^-(x,\xi)$ and* Im *$\sigma(\hat{P})(x,\xi)$.*

Then problem (14) *is Fredholm. In other words, the operator*

$$
(\hat{D}, \hat{B} \circ j_X^{m-1}) : \ H^s(M,E) \to H^{s-m}(M,F) \oplus \operatorname{Im}\ \hat{P}
$$

has the Fredholm property.

We shall refer to condition (GSL) as the *coerciveness condition*, or the *generalized Shapiro–Lopatinskii condition.* For the case in which $\hat{P}= 1$, we arrive at the usual Shapiro–Lopatinskii condition (e.g., see [1]). This will be discussed in Subsection 1.4. The advantage of the general condition is that a boundary value problem satisfying this condition can be posed for an arbitrary elliptic operator $\hat{D}$ (it suffices to take[3] $\hat{B}=\hat{P}=\hat{P}^{\,+}$).

Proof of Theorem 3. By Theorem 2, it suffices to prove that

$$
\hat{B} \colon \operatorname{Im}\ \hat{P}^{\,+} \to \operatorname{Im}\ \hat{P}
\tag{15}
$$

is a Fredholm operator. First, let us make a technical remark. The operator $\hat{P}^{\,+}$ acts in the space

$$
\mathcal{H}_m^{s-1/2}(X,E) = \bigoplus_{j=0}^{m-1} H^{s-1/2-j}(X,E)
\tag{16}
$$

and is of order 0 in this space; hence the orders of matrix entries of $\hat{P}^{\,+}$ vary according to the orders of the direct summands in (16), and the principal symbol of

[2] Recall that $L^-(x,\xi)$ is constructed from the principal symbol of the operator $\hat{D}$ as the space of solutions to Eq. (9) decaying as $t \to +\infty$.

[3] More precisely, order reduction is needed so that $\hat{P}$ be a zero-order operator.

$\hat{P}^+$ that we speak about is defined in the sense of Douglis–Nirenberg [13]. To make things more convenient, let us take an invertible first-order elliptic pseudodifferential operator $\hat{\Lambda}$ in $C^\infty(X,E)$ and use the isomorphism

$$\hat{U} = \operatorname{diag}(\hat{\Lambda}^{m-1}, \hat{\Lambda}^{m-2}, \dots, \hat{\Lambda}^{0}) : \mathcal{H}_m^{s-1/2}(X,E) \to (H^{s-m+1/2}(X,E))^m$$

to reduce the orders so as to avoid using principal symbols in the sense of Douglis–Nirenberg. Thus, we replace

$$\hat{P}^+ \quad \text{by} \quad \hat{U}\hat{P}^+\hat{U}^{-1} \quad \text{and} \quad \hat{B} \quad \text{by} \quad \hat{B}\hat{U}^{-1},$$

denoting the newly obtained operators by the same letters.

Now $\hat{P}^+$ is of order 0, $\hat{B}$ is of order $s-m+1/2-\sigma$, and the principal symbol $\sigma(\hat{B})$ of $\hat{B}$ is an isomorphism between the ranges of $\sigma(\hat{P}^+)$ and $\sigma(\hat{P})$. Momentarily, let us write A instead of $\sigma(\hat{A})$ for the principal symbol of any pseudodifferential operator $\hat{A}$.

Since any short exact sequence of vector bundles splits, it is an easy exercise in linear algebra to find symbols

$$R_1, R_2 \in \operatorname{Hom}(\pi^*G, \pi^*E^m), \quad \pi : T_0^*X \to X,$$

homogeneous of order $\sigma+m-s-1/2$ such that

$$\begin{aligned} P^+R_i &= R_i, \quad i=1,2, \\ R_1B &= P^+, \quad BR_2 = P. \end{aligned}$$

Set

$$\hat{\mathcal{R}}_i = \hat{P}^+ \hat{R}_i, \quad i=1,2.$$

Then

$$\hat{\mathcal{R}}_i(\operatorname{Im} \hat{P}) \subset \operatorname{Im} \hat{P}^+$$

and

$$\hat{\mathcal{R}}_1 \hat{B} = \hat{P}^+ + \hat{Q}_1, \tag{17}$$

$$\hat{B}\hat{\mathcal{R}}_2 = \hat{P} + \hat{Q}_2, \tag{18}$$

where the $\hat{Q}_{1,2}$ are pseudodifferential operators of order -1 on X (hence compact operators), and moreover,

$$\begin{aligned} \hat{Q}_1 \operatorname{Im}(\hat{P}^+) &\subset \operatorname{Im}(\hat{P}^+), \\ \hat{Q}_2 \operatorname{Im}(\hat{P}) &\subset \operatorname{Im}(\hat{P}) \end{aligned}$$

(the latter inclusion is due to the fact that Im $\hat{B} \subset$ Im $\hat{P}$). Now restricting (17) and (18) to Im $\hat{P}^+$ and Im $\hat{P}$, respectively, we obtain

$$\begin{aligned} &\hat{\mathcal{R}}_1 \hat{B} = 1_{\mathrm{Im}\hat{P}^+} + \hat{Q}_1, \\ &\hat{B}\hat{\mathcal{R}}_2 = 1_{\mathrm{Im}\hat{P}} + \hat{Q}_2 + (\hat{P} - 1)\Big|_{\mathrm{Im}\hat{P}}. \end{aligned} \tag{19}$$

Lemma 3 *The operator* $(\hat{P} - 1)\Big|_{\mathrm{Im}\hat{P}}$ *is compact.*

Proof. Let S be the unit sphere in Im $\hat{P}$. Consider the bounded operator

$$\tilde{P}\colon H^\sigma(X,G)/\mathrm{Ker}\ \hat{P} \to \mathrm{Im}\ \hat{P}$$

induced by $\hat{P}$. This operator is one-to-one, and since Im $\hat{P}$ is closed, it follows from Banach's open mapping theorem that $\tilde{P}^{-1}$ is bounded, and so $\tilde{P}^{-1}(S)$ is a bounded set in $H^\sigma(X,G)/\mathrm{Ker}\ \hat{P}$. Consequently, there exists a bounded set $\tilde{S} \subset H^\sigma(X,G)$ such that $S = \hat{P}(\tilde{S})$. Now

$$(\hat{P} - 1)(S) = (\hat{P}^2 - \hat{P})(\tilde{S})$$

is a relatively compact subset of $H^\sigma(X,G)$, since $\hat{P}^2 - \hat{P}$ is an operator of order -1 (recall that $P^2 = P$). Lemma 3 is thereby proved.

Now it follows from (19) that $\hat{\mathcal{R}}_1$ and $\hat{\mathcal{R}}_2$ are, respectively, left and right regularizers of $\hat{B}$ in the spaces (15). Thus, the operator (15) is Fredholm, which completes the proof of Theorem 3.

1.4 The Shapiro–Lopatinskii condition

If $\hat{P}$ is the identity operator, $\hat{P} = 1$, then problem (14) turns into the classical boundary value problem for the elliptic operator $\hat{D}$ with boundary conditions specified by the operator $\hat{B}$ (the right-hand side g in the boundary conditions is allowed to range over the entire Sobolev space $H^\sigma(X,G)$). Note that the principal symbol $P(x,\xi)$ of the operator $\hat{P}$, which acts in the spaces

$$P(x,\xi) : (\pi^*G)_{(x,\xi)} \equiv G_x \to (\pi^*G)_{(x,\xi)},$$

in this case is the identity operator,

$$\operatorname{Im} P(x,\xi) = (\pi^* G)_{(x,\xi)},$$

so that condition (GSL) is reduced to the requirement that the symbol B defines an isomorphism

$$B : L^- \to \pi^* G, \tag{20}$$

of bundles over X, where, of course,

$$L^- \to T_0^* X$$

is the bundle with fiber $L^-(x,\xi)$ at any $(x,\xi) \in T_0^* X$, and $\pi^* : T_0^* M \to M$ is the natural projection.

This is just the usual Shapiro–Lopatinskii condition.

We see that *classical boundary value problems satisfying the Shapiro–Lopatinskii condition exist if and only if L^- is isomorphic to the pullback under the natural projection of some bundle over X.*

The obstruction to the existence of such an isomorphism can be re presented as follows (sf. [3, 22]). It suffices to deal with the cosphere bundle S^*X instead of T_0^*X, since the former is a retract of the latter (in plain words, it suffices to extablish the existence of an isomorphism (20) on S^*X and then extend it by homogeneity). For each $(x,\xi) \in S^*X$, consider the ordinary differential operator

$$\hat{D}\,(x,\xi) \equiv \sigma(\hat{D})\left(x, 0, \xi, -i\frac{\partial}{\partial t}\right) : H^m(\mathbf{R}_+) \to L^2(\mathbf{R}_+)$$

with constant coefficients (cf. (9)). Since the coefficient of $(\partial/\partial t)^m$ is nonzero, it follows that $\{\hat{D}\,(x,\xi)\}$ is a conditions family of Fredholm operators parametrized by $(x,\xi) \in S^*X$, and consequently, the K-theoretic index

$$\operatorname{index}\{\hat{D}\,(x,\xi)\} \in K(S^*X)$$

is well defined. Note that

$$\operatorname{index}\{\hat{D}\,(x,\xi)\} = [L^-],$$

where $[L^-]$ is the class of the bundle $L^-|_{S^*X}$ in $K(S^*X)$, since one has the isomorphisms

$$\operatorname{Ker}\ \hat{D}\,(x,\xi) = L^-(x,\xi)$$

($L^-(x,\xi)$ is the space of initial data of exponentially decaying solutions of (9) hence of those solutions which belong to $L^2(\mathbf{R}_+)$) and

$$\operatorname{Coker} \hat{D}(x,\xi) = \{0\}$$

(recall that $\hat{D}(x,\xi)$ is a *differential* operator). Now for the existence of an isomorphism (20) it is necessary that

$$\operatorname{index}\{\hat{D}(x,\xi)\} \in \pi^* K(X),$$

where $\pi : S^*X \to X$ is the canonical projection.

Summarizing, not every elliptic operator admits a classical boundary condition of Shapiro–Lopatinskii type, and the obstuction to the existence of such problems is of topological nature [3].

2 Construction of the parametrix

The proof of Theorem 3 given in Subsection 1.3 is quite abstract in that it is based on the Calderón–Seeley projection; here we give a different proof of this theorem by explicity constructing a parametrix for problem (14).

The reader should be aware that the notation we use here (see Subsection 2.1) slightly differs from that adopted in Section 1. The main difference is that we use $\partial/\partial t$ instead of $-i\partial/\partial t$ so as to avoid an excessive amount of factors $\pm i$ in all the formulas.

2.1 Notation and preliminary considerations

Let M be a smooth manifold with boundary $X = \partial M$, and let E_i, $i = 1, 2$, be complex vector bundles over M. Next, let

$$\hat{D} \colon H^s(M, E_1) \to H^{s-m}(M, E_2)$$

be an elliptic pseudodifferential operator with principal symbol D.

In a neighborhood of $X = \partial M$ we introduce special coordinates (x,t) as in Section 1. The dual variables will be denoted by (ξ, p). In this neighborhood, the operator and the symbol have, respectively, the form

$$\hat{D} = D\left(x, t, -i\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) + \text{lower-order terms}, \tag{21}$$

$$D = D(x,t,\xi,p), \qquad \xi \in T^*_x X, \quad p \in \mathbf{C}. \tag{22}$$

Let

$$p_j = p_j(x, t, \xi)$$

be the points at which the symbol (22) is not invertible. The p_j are obviously the roots of the polynomial equation

$$\det D(x, t, \xi, p) = 0.$$

To describe them more conveniently as eigenvalues of some matrix, we use the following, quite standard trick. We have[4]

$$D = D(x, t, \xi, p) = p^m + A_{m-1}p^{m-1} + \ldots + A_1 p + A_0, \quad A_j = A_j(x, t, \xi).$$

Consider the matrix operator

$$\mathcal{A} = \begin{pmatrix} p & -1 & 0 & \ldots & 0 \\ 0 & p & -1 & \ldots & 0 \\ 0 & 0 & p & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_0 & A_1 & A_2 & \ldots & p + A_{m-1} \end{pmatrix}.$$

It is a block matrix each of whose blocks is an endomorphism of E (more precisely, $\pi^* E$, where $\pi : T_0^* X \to X$).

Then the equation

$$D(x, t, \xi, p)\, u = 0$$

is equivalent to

$$\mathcal{A}(x, t, p, \xi)\ \vec{u} = 0,$$

where

$$\vec{u} = \begin{pmatrix} u \\ pu \\ \ldots \\ p^{m-1}u \end{pmatrix}.$$

Consequently, $p_j = p_j(x, \xi, t)$ are the eigenvalues of the endomorphism

$$A = \begin{pmatrix} 0 & +1 & 0 & \ldots & 0 \\ 0 & 0 & +1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{m-1} \end{pmatrix} : (\pi^* E)^m \to (\pi^* E)^m. \qquad (23)$$

[4]By virtue of the ellipticity, the coefficient $D(x, t, 0, 1)$ of $(\partial/\partial t)^m$ is an invertible homomorphism $E_1 \to E_2$. Hence, we can assume that $E_1 = E_2$ and $D(x, t, 0, 1) = \mathbf{1}$ (the identity homomorphism).

Let $\varphi_j(x,t,\xi)$ be the corresponding eigenvectors and associated eigenvectors of this homomorphism.

Note that Re $p_j(x,t,\xi) \neq 0$ for $\xi \neq 0$ by virtue of the ellipticity, so that the functions $p_j(x,t,\xi)$ (and, accordingly, the eigenfunctions $\varphi_j(x,t,\xi)$) split into two subsets

$$\{p_j^+,\, j=1,\ldots,k_+\}, \quad \operatorname{Re} p_j^+ > 0,$$

$$\{p_j^-,\, j=1,\ldots,k_-\}, \quad \operatorname{Re} p_j^- < 0.$$

For simplicity, we assume that all eigenvalues of the endomorphism (23) are simple, so that there are no associated eigenvectors.

For each triple (x,t,ξ), $\xi \neq 0$, by

$$L^- = L^-(x,t,\xi)$$

we denote the sum of eigenspaces of (23) corresponding to the eigenvalues

$$\{p_j^-(x,t,\xi),\, j=1,\ldots,k_-\}.$$

Similarly, we introduce the spaces

$$L^+ = L^+(x,t,\xi).$$

Obviously, for sufficiently small $t < \varepsilon$ we have the direct sum expansion

$$L^+(x,t,\xi) \oplus L^-(x,t,\xi) = \pi^* E^m_{(x,\xi)}.$$

Let $P^-(x,t,\xi)$ be the projection onto $L^-(x,t,\xi)$ along $L^+(x,t,\xi)$, and let $P^+(x,t,\xi)$ be the projection onto $L^+(x,t,\xi)$ along $L^-(x,t,\xi)$. Obviously,

i) $P^\pm(x,t,\xi)$ are matrices smoothly depending on (x,t,ξ), $\xi \neq 0$;

ii) $P^+(x,t,\xi) + P^-(x,t,\xi) = 1$;

iii) $P^\pm(x,t,\xi)$ are zero-order homogeneous in ξ.

2.2 The equation in the half-space

In the half-space $\mathbf{R}^n_+$, consider the operator

$$\hat{D}_{x_0} = D\left(x_0, 0, -i\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) : H^s(\mathbf{R}^n_+, E) \to H^{s-m}(\mathbf{R}^n_+, E)$$

obtained from (21) by freezing the coefficients at a point $(x_0, 0)$ of the boundary X.

Let us supplement the operator $\hat{D}_{x_0}$ with boundary conditions so as to obtain a Fredholm problem.

Remark 3 According to Section 1, we consider boundary operators of the form

$$\hat{B} \circ j^{m-1},$$

where $\hat{B}$ is a matrix pseudodifferential operator on X acting on sections of E^m, and j^{m-1} is the $(m-1)$st-order jet of a function u with respect to t at $t=0$:

$$u \mapsto \left(u(x,0), \frac{\partial u}{\partial t}(x,0), \ldots, \frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) \right).$$

Obviously, the problem

$$\begin{cases} \hat{D}_{x_0} u = f, \\ \hat{B}_{x_0} \circ j^{m-1} u = g, \end{cases}$$

where

$$\hat{B}_{x_0} \colon \bigoplus_{j=0}^{m-1} H^{s-j-1/2}(X,E) \to H^{\sigma}(X,F)$$

for some bundle F over X, is equivalent to the problem

$$\begin{cases} \overset{\wedge}{\mathcal{A}}_{x_0} v = \psi, \\ \hat{B}_{x_0} v|_{t=0} = g, \end{cases}$$

where

$$v = j^{m-1} u, \quad \psi = (0, \ldots, 0, f)$$

are m-component vectors.

Let us study the kernel of the operator $\overset{\wedge}{\mathcal{A}}_{x_0}$. By performing the Fourier transform in x, we obtain the equation

$$\mathcal{A}\left(x_0, 0, \xi, \frac{\partial}{\partial t} \right) \tilde{v} = 0.$$

Consequently, the elements of the kernel have the form[5]

$$\tilde{v}_j^{\pm}(\xi) = C_j^{\pm}(\xi) e^{p_j^{\pm} t} \varphi_j^{\pm}(\xi),$$

where the $C_j^{\pm}(\xi)$ are arbitrary functions of ξ and the $\varphi_j^{\pm}(\xi)$ are the corresponding eigenfunctions. Since the spectrum is simple, it follows that these functions depend

[5]In the following formulas, in the coefficients of the operator and in the related $p_j^{\pm}$ and $\varphi_j^{\pm}$ we everywhere assume $t=0$.

on the parameter $\xi \neq 0$ regularly. Since the solutions of the original equation must belong to the Sobolev space, the solutions with superscript $+$ are excluded automatically,

$$C_j^+(\xi) = 0,$$

and the system of solutions of the homogeneous equation has the form

$$\left\{\tilde{v}_j^-(\xi,t) = C_j^-(\xi)e^{p_j^- t}\varphi_j^-(\xi), \quad j = 1,\dots,k_-\right\}.$$

Obviously, the boundary conditions must be chosen so as to determine the constants $C_j^-(\xi)$ uniquely. Let some boundary conditions

$$\hat{B}_{x_0}\, v|_{t=0} = g$$

be given, where $\hat{B}_{x_0}$ is a pseudodifferential operator with constant coefficients. The Fourier transform algebraizes these conditions:

$$B_{x_0}(\xi)\,\tilde{v}\,(\xi,0) = \tilde{g}\,(\xi). \tag{24}$$

Note that the general solution of the nonhomogeneous equation

$$\mathcal{A}\left(x_0, 0, \xi, \frac{\partial}{\partial t}\right)\tilde{v} = \tilde{\psi}$$

has the form

$$\tilde{v} = \tilde{v}^{\,*} + \sum_{j=1}^{k_-} C_j^-(\xi)e^{p_j^-(\xi)t}\varphi_j^-(\xi),$$

where $\tilde{v}^{\,*}$ is some particular solution, and consequently, the initial data

$$\tilde{v}\,|_{t=0} = \tilde{v}^{\,*}|_{t=0} + \sum_{j=1}^{k_-} C_j^-(\xi)\varphi_j^-(\xi)$$

form a coset modulo the subspace $L^-(x_0,\xi)$. Thus, the boundary condition (24) acquires the form

$$B_{x_0}(\xi)\sum_{j=1}^{k_-} C_j^-(\xi)\varphi_j^-(\xi) = \tilde{g}\,(\xi) - B_{x_0}(\xi)\,\tilde{v}^{\,*}|_{t=0}.$$

It follows that for $C_j^-(\xi)$ to be determined uniquely, we must require that

i) the homomorphism $B_{x_0}(\xi)$ be a monomorphism on $L^-(x_0,\xi)$;

ii) the data $\tilde{g}(\xi)$ lie in the range of this homomorphism, which is a subspace of the fibre of F.

It is natural to describe this subspace as the image of some projection $P(\xi)$,

$$\tilde{g}(\xi) = P(\xi)g_1(\xi)$$

for arbitrary $g_1(\xi)$. For conditions (24) to be well-posed, we must require that the range of $B_{x_0}(\xi)$ be contained in the range of $P(\xi)$.

Proposition 1 *One has the equivalence*

$$\operatorname{Im} B_{x_0}(\xi) \subset \operatorname{Im} P(\xi) \Leftrightarrow B_{x_0}(\xi) = P(\xi)B^{(1)}_{x_0}(\xi)$$

for some homomorphism $B^{(1)}_{x_0}(\xi)$.

Proof. Suppose that

$$\operatorname{Im} B_{x_0}(\xi) \subset \operatorname{Im} P(\xi).$$

Then

$$P(\xi)B_{x_0}(\xi) = B_{x_0}(\xi),$$

since $P(\xi)$ is the identity operator on the range.

Conversely, if

$$B_{x_0}(\xi) = P(\xi)B^{(1)}_{x_0}(\xi),$$

then obviously

$$\operatorname{Im} B_{x_0}(\xi) \subset \operatorname{Im} P(\xi).$$

Now, in order that the problem with conditions (24) have no cokernel in the boundary conditions, we must require the operator $B_{x_0}(\xi)$ to be an *isomorphism* of the spaces $L^-(x_0,\xi)$ and Im $P(\xi)$. Under this condition, the problem

$$\begin{cases} \mathcal{A}\left(x_0,0,\xi,\frac{\partial}{\partial t}\right) \tilde{v}(t,\xi) = \tilde{\psi}(t,\xi), \\ B_{x_0}(\xi)\,\tilde{v}(\xi,0) = \tilde{g}(\xi), \quad \tilde{g}(\xi) \in \operatorname{Im} P(\xi) \end{cases} \tag{25}$$

is uniquely solvable. Let us find the solution.

1) We have

$$\left[\frac{\partial}{\partial t} - A(x_0,0,\xi)\right] \tilde{v}(t,\xi) = \tilde{\psi}(t,\xi).$$

Let us expand $\overset{\sim}{\psi}(t,\xi)$ in the basis

$$\{\varphi_j^+(\xi), j = 1, \dots, k_+;\ \varphi_j^-(\xi), j = 1, \dots, k_-\}.$$

Then the corresponding components satisfy the equation

$$\left[\frac{\partial}{\partial t} - p_j^\pm(\xi)\right] \tilde{v}{}_j^\pm(t,\xi) = \overset{\sim}{\psi}{}_j^\pm(t,\xi),$$

whose solution has the form

$$\tilde{v}{}_j^\pm(t,\xi) = C_j^\pm e^{p_j^\pm(\xi)t} + \int^t e^{p_j^\pm(\xi)(t-\tau)} \overset{\sim}{\psi}{}_j^\pm(\tau,\xi)\, d\tau.$$

For the sign "+", we have $C_j^+(\xi) = 0$ and the integration is from $+\infty$ to t, so that the solution can be represented in the form

$$\tilde{v}{}_j^+(t,\xi) = -\int_t^{+\infty} e^{p_j^+(\xi)(t-\tau)} \overset{\sim}{\psi}{}_j^+(\tau,\xi)\, d\tau.$$

For the sign "–", the functions $C_j^-(\xi)$ are arbitrary, and hence the integration is from 0 to t:

$$\tilde{v}{}_j^-(t,\xi) = C_j^-(\xi) e^{p_j^-(\xi)t} + \int_0^t e^{p_j^-(\xi)(t-\tau)} \overset{\sim}{\psi}{}_j^-(\tau,\xi)\, d\tau.$$

Finally, the general solution of the equation has the form

$$\begin{aligned}
\tilde{v}(t,\xi) &= \sum_{j=1}^{k_-} C_j^-(\xi) e^{p_j^-(\xi)t}\varphi_j(\xi) + \int_0^t \sum_{j=1}^{k_-} e^{p_j^-(\xi)(t-\tau)} \overset{\sim}{\psi}{}_j^-(\tau,\xi)\varphi_j(\xi)\, d\tau \\
&- \int_t^{+\infty} \sum_{j=1}^{k_+} e^{p_j^+(\xi)(t-\tau)} \overset{\sim}{\psi}{}_j^+(\tau,\xi)\varphi_j(\xi)\, d\tau.
\end{aligned}$$

Using the projecttions $P^+(\xi)$ and $P^-(\xi)$, we can expand the operator A into the components

$$A = P^+A + P^-A = A^+ + A^-.$$

In these terms, we have

$$\widetilde{v}\,(t,\xi) = e^{A^-(\xi)t}\,\vec{C}\,(\xi) + \int\limits_0^t e^{A^-(\xi)(t-\tau)}P^-(\xi)\,\widetilde{\psi}\,(\tau,\xi)\,d\tau - \int\limits_t^{+\infty} e^{A^+(\xi)(t-\tau)}P^+(\xi)\,\widetilde{\psi}\,(\tau,\xi)\,d\tau,$$

where

$$\vec{C}\,(\xi) = \sum_{j=1}^{k_-} C_j^-(\xi)\varphi_j(\xi)$$

is an arbitrary element of the space $L^-(x_0,\xi)$.

2) Let us now satisfy the boundary conditions. We have

$$\widetilde{v}\,(0,\xi) = \vec{C}\,(\xi) - \int\limits_0^{+\infty} e^{-A^+(\xi)\tau}P^+(\xi)\,\widetilde{\psi}\,(\tau,\xi)\,d\tau,$$

and the boundary conditions acquire the form

$$B_{x_0}(\xi)\,\vec{C} = \widetilde{g}\,(\xi) + B_{x_0}(\xi)\int\limits_0^{+\infty} e^{-A^+(\xi)\tau}P^+(\xi)\,\widetilde{\psi}\,(\tau,\xi)\,d\tau.$$

Since $B_{x_0}(\xi)$ is an isomorphism of $L^-(x_0,\xi)$ onto $\mathrm{Im}\ P(\xi) = \mathrm{Im}\,B_{x_0}(\xi)$, it follows that the inverse

$$B_{x_0}^{-1}(\xi):\ \mathrm{Im}\,B_{x_0}(\xi) \to L^-(x_0,\xi)$$

exists. We can extend the latter homomorphism to the entire F by setting

$$B_{x_0}^{(-1)}(\xi) = B_{x_0}^{-1}(\xi)P(\xi).$$

Obviously, $B_{x_0}^{(-1)}(\xi)$ is a homomorphism of F into E, and moreover,

i) $B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi) = 1$ on $L^-(x_0,\xi)$;

ii) $B_{x_0}(\xi)B_{x_0}^{(-1)}(\xi) = P(\xi)$;

iii) the range of $B_{x_0}^{(-1)}(\xi)$ coincides with $L^-(x_0,\xi)$.

Now we have

$$\vec{C}(\xi) = B_{x_0}^{(-1)}(\xi)\,\tilde{g}(\xi) + B_{x_0}^{(-1)}(\xi) B_{x_0}(\xi) \int\limits_0^{+\infty} e^{-A^+(\xi)\tau}\,\tilde{\psi}(\tau,\xi)\,d\tau$$

and the solution of the problem acquires the form

$$\begin{aligned}
\tilde{v}(t,\xi) &= e^{A^-(\xi)t} B_{x_0}^{(-1)}(\xi) \left[\tilde{g}(\xi) + B_{x_0}(\xi) \int\limits_0^{+\infty} e^{-A^+(\xi)\tau} P^+(\xi)\,\tilde{\psi}(\tau,\xi)\,d\tau \right] \\
&\quad + \int\limits_0^t e^{A^-(\xi)(t-\tau)} P^-(\xi)\,\tilde{\psi}(\tau,\xi)\,d\tau - \int\limits_t^{+\infty} e^{A^+(\xi)(t-\tau)} P^+(\xi)\,\vec{\psi}(\tau,\xi)\,d\tau \\
&\equiv \mathcal{R}[\tilde{\psi}, \tilde{g}].
\end{aligned}$$

Let us prove that $\mathcal{R}$ is the exact resolving operator of problem (25).

First, we show that $\mathcal{R}$ is a right inverse.

1) The substitution into the equation gives

$$\begin{aligned}
\left[\frac{\partial}{\partial t} - A(\xi)\right] \mathcal{R}[\tilde{\psi}, \tilde{g}] &= e^{A^-(\xi)t}(A^-(\xi) - A(\xi)) B_{x_0}^{(-1)}(\xi) \\
&\times \left[\tilde{g}(\xi) + B_{x_0}(\xi) \int\limits_0^{\infty} e^{-A^+(\xi)\tau}\,\tilde{\psi}(\tau,\xi)\,d\tau \right] + P^-(\xi)\,\tilde{\psi}(\tau,\xi) \\
&+ \int\limits_0^t e^{-A^+(\xi)(t-\tau)}(A^-(\xi) - A(\xi)) P^-(\xi)\,\tilde{\psi}(\tau,\xi)\,d\tau + P^+(\xi)\,\tilde{\psi}(t,\xi) \\
&- \int\limits_t^{+\infty} e^{A^+(\xi)(t-\tau)}(A^+(\xi) - A(\xi)) P^+(\xi)\,\tilde{\psi}(\tau,\xi)\,d\tau = \tilde{\psi}(t,\xi),
\end{aligned}$$

since

$$\begin{aligned}
&(A^-(\xi) - A(\xi)) B_{x_0}^{(-1)}(\xi) = 0 \qquad (\text{recall that } \operatorname{Im} B_{x_0}^{(-1)}(\xi) \subset L^-(x_0,\xi)), \\
&(A^-(\xi) - A(\xi)) P^-(\xi) = 0, \\
&(A^+(\xi) - A(\xi)) P^+(\xi) = 0.
\end{aligned}$$

2) The boundary conditions are satisfied. Indeed, we have

$$B_{x_0}(\xi)\mathcal{R}(\tilde{\psi},\tilde{g})|_{t=0} = B_{x_0}(\xi)B_{x_0}^{(-1)}(\xi)\,\tilde{g}\,(\xi)$$

$$+B_{x_0}(\xi)B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)\int\limits_0^{+\infty} e^{-A^+(\xi)\tau}P^+(\xi)\,\tilde{\psi}\,(\tau,\xi)\,d\tau$$

$$-B_{x_0}(\xi)\int\limits_0^{+\infty} e^{-A^+(\xi)\tau}P^+(\xi)\,\tilde{\psi}\,(\tau,\xi)\,d\tau = P(\xi)\,\tilde{g}\,(\xi) =\tilde{g}\,(\xi),$$

since $B_{x_0}(\xi)B_{x_0}^{(-1)}(\xi) = P(\xi)$, $\tilde{g}\,(\xi) \in \operatorname{Im} P(\xi)$, and

$$B_{x_0}(\xi)B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi) = B_{x_0}(\xi)$$

by virtue of the inclusion $\operatorname{Im} B_{x_0}(\xi) \subset \operatorname{Im} P(\xi)$.

Now let us prove that $\mathcal{R}$ is a left inverse. We have

$$\mathcal{R}\left[\left(\frac{\partial}{\partial t} - A(\xi)\right)\tilde{v}, B_{x_0}(\xi)\,\tilde{v}\,|_{t=0}\right] = e^{A^-(\xi)t}B_{x_0}^{(-1)}(\xi)(B_{x_0}\,\tilde{v}\,|_{t=0}$$

$$+e^{-A^-(\xi)t}B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)\int\limits_0^{+\infty} e^{A^+(\xi)\tau}P^+(\xi)\left(\frac{\partial\,\tilde{v}}{\partial\tau} - A(\xi)\,\tilde{v}\,(\xi,\tau)\right)\,d\tau$$

$$+\int\limits_0^{t} e^{A^-(\xi)(t-\tau)}P^-(\xi)\left(\frac{\partial\,\tilde{v}}{\partial\tau} - A(\xi)\,\tilde{v}\,(\xi,\tau)\right)\,d\tau$$

$$-\int\limits_t^{+\infty} e^{A^+(\xi)(t-\tau)}P^+(\xi)\left(\frac{\partial\,\tilde{v}}{\partial\tau} - A(\xi)\,\tilde{v}\,(\xi,\tau)\right)\,d\tau.$$

Integration by parts in the term with $\partial\,\tilde{v}\,/\partial\tau$ in all three integrals yields

$$\mathcal{R}\left[\left(\frac{\partial}{\partial t} - A(\xi)\right)\tilde{v}, B_{x_0}(\xi)\,\tilde{v}\,|_{t=0}\right] = e^{A^-(\xi)t}B_{x_0}^{(-1)}(\xi)(B_{x_0}(\xi)\,\tilde{v}\,|_{t=0})$$

$$-e^{-A^-(\xi)t}B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+\,\tilde{v}\,|_{t=0}) + e^{-A^-(\xi)t}B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)\int\limits_0^{\infty} e^{-A^+(\xi)\tau}$$

$$\times P^+(\xi)(A^+(\xi)-A(\xi))\,\tilde{v}\,(\xi,\tau)\,d\tau+P^-(\xi)\,\tilde{v}\,(t,\xi)-e^{A^-(\xi)t}P^-(\xi)\,\tilde{v}\,|_{t=0}$$

$$+\int_0^t e^{A^-(\xi)(t-\tau)}P^-(\xi)(A^-(\xi)-A(\xi))\,\tilde{v}\,(\xi,\tau)\,d\tau-P^+(\xi)\,\tilde{v}\,(\xi,\tau)$$

$$-\int_t^{+\infty} e^{A^+(\xi)(t-\tau)}P^+(\xi)(A^+(\xi)-A(\xi))\,\tilde{v}\,(\xi,\tau)\,d\tau$$

$$=e^{A^-(\xi)t}[B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)-B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)-P^-(\xi)]\,\tilde{v}\,|_{t=0}+\,\tilde{v}\,(t,\xi).$$

Note that we have

$$\begin{aligned}&[B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)-B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)-P^-(\xi)]P^+(\xi)\\&\qquad=B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)-B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)=0\end{aligned}$$

and

$$\begin{aligned}&[B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)-B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)-P^-(\xi)]P^-(\xi)\\&\qquad=B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^-(\xi)-P^-(\xi)=0,\end{aligned}$$

since $B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)=1$ on $L^-(x_0,\xi)$. By summing these equations, we obtain

$$[B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)-B_{x_0}^{(-1)}(\xi)B_{x_0}(\xi)P^+(\xi)-P^-(\xi)]=0,$$

and consequently,

$$\mathcal{R}\left[\left(\frac{\partial}{\partial t}-A(\xi)\right)\tilde{v},\;B_{x_0}(\xi)\,\tilde{v}\,|_{t=0}\right]=\tilde{v}\,(t,\xi),$$

as desired.

It follows that the problem

$$\begin{cases}\mathcal{A}\left(x_0,0,-i\dfrac{\partial}{\partial x},\dfrac{\partial}{\partial t}\right)v(t,x)=\psi(t,x),\\ B_{x_0}\left(-i\dfrac{\partial}{\partial x}\right)v(0,x)=g(x)\end{cases}\tag{26}$$

has the regularizer

$$\mathcal{R}_0[\psi,g]=\mathcal{F}_{x\to\xi}\mathcal{R}[\mathcal{F}_{x\to\xi}\psi,\mathcal{F}_{\xi\to x}g],$$

where $\mathcal{F}$ stands for the Fourier transform, and the regularizer of the model problem

$$\left\{\begin{array}{l} D\left(x_0, 0, -i\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(t,x) = f(t,x), \\ \left[B_{x_0}\left(-i\frac{\partial}{\partial x}\right) j^{m-1} u\right](x) = g(x) \end{array}\right. \tag{27}$$

under the above assumptions is given in the cited spaces by[6]

$$\mathcal{R}_1[f,g] = \mathcal{R}_0[\alpha(f), g],$$

where $\alpha(f) = (0, \dots, 0, f)$.

Remark 4 If we do not assume that $g(x) \in \operatorname{Im} P\left(-i\frac{\partial}{\partial x}\right)$, then, modulo smoothing operators, one has

$$\mathcal{R}_1\left[\hat{D}_{x_0} u, B_{x_0}\left(-i\frac{\partial}{\partial x}\right) j^{m-1}\right] = u$$

and

$$\left\{\begin{array}{l} \hat{D}_{x_0} \mathcal{R}_1[f,g] = f, \\ B_{x_0} j^{m-1} \mathcal{R}_1[f,g] = P\left(-i\frac{\partial}{\partial x}\right) g. \end{array}\right.$$

Note that the estimates of the regularizers are standard, and we omit them altogether.

2.3 The general situation

Consider the boundary boundary value problem

$$\left\{\begin{array}{l} \hat{D} u = f, \\ \hat{B} j_X^{m-1} u = g, \end{array}\right. \tag{28}$$

with $u \in H^s(M, E_1)$, $f \in H^{s-m}(M, E_2)$, and $g \in H^\sigma(X, F)$, where the E_i are bundles over M and F is a bundle over X. We assume that the conditions of Theorem 3 are satisfied. Namely, $\hat{D}$ is an mth-order elliptic operator, and $\hat{B}$ is an operator in sections of bundles whose order is compatible with the indices of Sobolev spaces, so that

$$\hat{B}\colon \bigoplus_{j=0}^{m-1} H^{s-j-1/2}(X, i^*E) \to H^\sigma(X, F)$$

[6]We omit the standard cutoff functions in a neighborhood of $\xi = 0$. Note that it is due to these functions that the exact resolving operator for problem (25) becomes only a regularizer of problem (27) after the Fourier transform.

is a bounded operator and $s - m + 1/2 > 0$. We also assume that the right-hand side g in problem (28) belongs to the range of a pseudo-differential operator

$$\stackrel{\wedge}{P}\colon\ H^\sigma(X,F) \to H^\sigma(X,F)$$

of order zero with closed range, whose principal symbol $P(x,\xi)$ is a projection in the fibres of π^*F for any (x,ξ), $\xi \neq 0$, and that the range of $\stackrel{\wedge}{B}$ is contained in the range of $\stackrel{\wedge}{P}$. Finally, we assume that condition (GSL) is satisfied.

Now we are in a position to construct a parametrix for problem (28)

To this end, for any $x_0 \in X$ we consider the model problem (27). Let U_{x_0} be a sufficiently small neighborhood of x_0 in M (the size of the neighborhood will be specified later). By $\stackrel{\wedge}{D}_U$, $\stackrel{\wedge}{B}_U$, and $\stackrel{\wedge}{P}_U$ we denote pseudodifferential operators coinciding on functions with support in U with $\stackrel{\wedge}{D}$, $\stackrel{\wedge}{B}$, and $\stackrel{\wedge}{P}$, respectively, and satisfying the following condition: the symbols D_U, B_U, and P_U differ from D_{x_0}, B_{x_0}, and P_{x_0} on the unit sphere at most by $\varepsilon > 0$ (obviously, for any $\varepsilon > 0$ there exists a neighborhood U in which $\stackrel{\wedge}{D}_U$, $\stackrel{\wedge}{B}_U$, and $\stackrel{\wedge}{P}_U$ can be constructed). Let $\mathcal{R}_1^{(x_0)}$ be the regularizer of the model problem at x_0. Assuming f and g to be supported in U, we have

$$(\stackrel{\wedge}{D}, \stackrel{\wedge}{B}\, j^{m-1}) \circ \psi_U \mathcal{R}_1^{(x_0)}[f,g]^t = \begin{pmatrix} \stackrel{\wedge}{D}\ \psi_U \mathcal{R}_1^{(x_0)}[f,g] \\ \stackrel{\wedge}{B}\ j^{m-1}\psi_U \mathcal{R}_1^{(x_0)}[f,g] \end{pmatrix},$$

where ψ_U is a function with support in U such that $\psi_U = 1$ on supp f and supp g and $0 \le \psi_U \le 1$. Next, we have

$$\begin{aligned} \stackrel{\wedge}{D}\ \psi_U \mathcal{R}_1^{(x_0)}[f,g] &= \stackrel{\wedge}{D}_U\ \psi_U \mathcal{R}_1^{(x_0)}[f,g] = (\stackrel{\wedge}{D}_U - \stackrel{\wedge}{D}_{x_0})\psi_U \mathcal{R}_1^{(x_0)}[f,g] \\ &+ [\stackrel{\wedge}{D}_{x_0}, \psi_U]\mathcal{R}_1^{(x_0)}[f,g] + \psi_U\ \stackrel{\wedge}{D}_{x_0}\ \mathcal{R}_1^{(x_0)}[f,g] \\ &= (\stackrel{\wedge}{D}_U - \stackrel{\wedge}{D}_{x_0})\psi_U \mathcal{R}_1^{(x_0)}[f,g] + [\stackrel{\wedge}{D}_{x_0}, \psi_U]\mathcal{R}_1^{(x_0)}[f,g] + f. \end{aligned}$$

The principal symbol of $\stackrel{\wedge}{D}_U - \stackrel{\wedge}{D}_{x_0}$ does not exceed ε. It follows that there exists a pseudodifferential operator $T_U^{(1)}$ with norm less than 2ε and a smoothing operator Q_U such that

$$\stackrel{\wedge}{D}_U - \stackrel{\wedge}{D}_{x_0} = T_U + Q_U.$$

Consequently,

$$\stackrel{\wedge}{D}\ \psi_U \mathcal{R}_1^{(x_0)}[f,g] = f + T_U^{(1)}(f,g) + Q_U^{(1)}(f,g),$$

where $\left|\left|T_U^{(1)}\right|\right| \leq 2\varepsilon \left|\left|\mathcal{R}_1^{(x_0)}\right|\right|$ and $Q_U^{(1)}$ is a smoothing operator.

Similarly, we have

$$\begin{aligned}
&\hat{B}\, j^{m-1}\psi_U \mathcal{R}_1^{(x_0)}[f,g] = \hat{B}_U \circ j^{m-1}\mathcal{R}_1^{(x_0)}[f,g] \\
&= (\hat{B}_U - \hat{B}_{x_0}) \circ j^{m-1}\psi_U \mathcal{R}_1^{(x_0)}[f,g] + [B_{x_0}, \psi_U] \circ j^{m-1} \circ \mathcal{R}_1^{(x_0)}[f,g] \\
&\quad + \psi_U \vec{B}_{x_0} \circ j^{m-1}\mathcal{R}_1^{(x_0)}[f,g] \\
&= (\hat{B}_U - \hat{B}_{x_0}) \circ j^{m-1}\psi_U \mathcal{R}_1^{(x_0)}[f,g] + [B_{x_0}, \psi_U] \circ j^{m-1}\mathcal{R}_1^{(x_0)}[f,g] + \psi_U \vec{P}_{x_0} g \\
&= (\hat{B}_U - \hat{B}_{x_0}) \circ j^{m-1}\psi_U \mathcal{R}_1^{(x_0)}[f,g] + [B_{x_0}, \psi_U] \circ j^{m-1}\mathcal{R}_1^{(x_0)}[f,g] \\
&\quad + [\psi_U, \vec{P}_{x_0}]g + (\hat{P}_U - \hat{P}_{x_0})g + \hat{P}_g \,.
\end{aligned}$$

Hence, there exists operators $T_U^{(2)}$ and $Q_U^{(2)}$ such that

$$\hat{B}\, j^{m-1} \circ \psi_U \mathcal{R}_1^{(x_0)}[f,g] = \hat{P}\, g + T_U^{(2)}(f,g) + Q_U^{(2)}(f,g),$$

and moreover,

$$\left|\left|T_U^{(2)}\right|\right| \leq 2\varepsilon(||\mathcal{R}_1|| + 1)$$

and $Q_U^{(2)}$ is a smoothing operator.

Finally, we have the relation

$$(\hat{D}, \hat{B} \circ j^{m-1})\psi_U \mathcal{R}_1^{(x_0)}[f,g] = (f, \hat{P}\, g) + T_U(f,g) + (f,g),$$

where Q_U is a smoothing operator and T_U satisfies the estimate

$$||T_U|| \leq 2\varepsilon(2\,||\mathcal{R}_1|| + 1).$$

By choosing ε so small that

$$2\varepsilon(2\,||\mathcal{R}_1|| + 1) < 1/2, \tag{29}$$

we ensure that $1 + T_U$ is invertible and

$$(\hat{D}, \hat{B} \circ j^{m-1})\psi_U \mathcal{R}_1^{(x_0)}(1 + T_U)^{-1}(f,g) = (f, \hat{P}\, g) + Q_U(f,g).$$

Note that although the operator T_U depends on the choice of ψ_U, the size of the neighborhood in which inequality (29) is satisfied is independent of ψ_U.

From the cover $\{U_{x_0}\}$ we now choose a finite subcover $\{U_{x_j}, j = 1, \dots, N\}$. Let $V \subset M \backslash X$ be an open set supplementing this subcover to a cover of M. Let $\hat{R}$ be a pseudodifferential regularizer of $\hat{D}$ on $M \backslash X$. Finally, let $\{e_j, e\}$ be a partition of unity subordinate to the cover, and let ψ_i, ψ be the corresponding cutoff functions.

Standard computations show that the operator

$$\mathcal{R}_{gl}(f, g) = \sum_{j=1}^{N} \psi_j \mathcal{R}_1^{(x_j)} (1 + T_U^{(j)})^{-1} (f, g) + \psi \hat{R} f$$

is a regularizer of problem (28) in the sense that

$$(\hat{D}, \hat{B} \circ j^{m-1}) \mathcal{R}_{gl} = (1, \hat{P}) + \hat{Q},$$

where $\hat{Q}$ is a smoothing operator. Obviously, by considering $\mathcal{R}_{gl}$ on the subspace $\operatorname{Im} \hat{P} \subset H^{\sigma}(X, F)$ (in the boundary component), we obtain a regularizer of problem (28). The left regularizer can be constructed in a similar way. The proof is complete.

3 Examples

3.1 The Cauchy-Riemann operator

Consider the operator $\partial/\partial\bar{z}$ on a complex manifold M of (complex) dimension 1. Obviously, this operator is elliptic. Suppose that the boundary X of M is purely real, i.e., there exist coordinates $z = x + iy$ in a neighborhood of the boundary such that the equation of X is $\{y = 0\}$. In these coordinates we have

$$\hat{D} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{i}{2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial x}\right).$$

The symbol of this operator is

$$\hat{D}(p, \xi) = \frac{i}{2}(p + \xi)$$

(the variable y plays the role of t in the general construction). We take E to be the one-dimensional trivial bundle over X. For each point $x_0 \in X$, we obviously have $p_1(\xi) = -\xi$. Furthermore

$$\begin{array}{lll} L^+(\xi) = \mathbf{R}, & L^-(\xi) = \{0\} & \text{for } \xi < 0; \\ L^-(\xi) = \mathbf{R}, & L^+(\xi) = \{0\} & \text{for } \xi > 0. \end{array}$$

Consequently,

$$P^+(\xi) = \theta(-\xi), \quad P^-(\xi) = \theta(\xi),$$

where θ is the Heaviside function.

Since the dimension of $L^-(\xi)$ is not the same for $\xi < 0$ and $\xi > 0$, it follows that $L^-(\xi)$ is not isomorphic to the pullback of any bundle on X.

Now let

$$\begin{cases} \hat{D}\, u = f, \\ \hat{B}\, u|_{y=0} = g \end{cases} \tag{30}$$

be a boundary value problem for the operator $\hat{D}$. Obviously, the bundle F used in the boundary conditions must be one-dimensional if we want the generalized Shapiro–Lopatinskii condition to be satisfied. Next, in this case $B(x, \xi)$ is a scalar function. The generalized Shapiro–Lopatinskii condition gives isomorphisms

$$\begin{array}{ll} B(x,\xi) : \mathbf{R} \to \mathbf{R}, & \text{for } \xi > 0 \\ B(x,\xi) : \{0\} \to \mathbf{R}, & \text{for } \xi < 0 \end{array}$$

on some subspaces. Clearly, the condition is satisfied if $B(x, \xi) \neq 0$ for $\xi > 0$, $B(x, \xi) = 0$ for $\xi < 0$, and the projection $P(x, \xi)$ coincides with $P^-(\xi)$. Under these conditions, problem (30) with

$$u \in H^s(M), \; f \in H^{s-1}(M), \; g \in \operatorname{Im} \hat{P}^{\,-}(\xi) \subset H^{s-1/2}(M),$$

is Fredholm. Note that the range of the operator $\hat{P}^{\,-}(\xi)$ is called the Hardy space [6, 21].

3.2 The Euler operator

Consider an even-dimensional Reimannian manifold M with boundary[7] $\partial M = X$. Let

$$d + \delta : \Lambda^{\mathrm{ev}}(M) \to \Lambda^{\mathrm{odd}}(M)$$

be the Euler operator (e.g., see [4, 14, 11, 12]) on M. In the coordinates (t, x) near X, this operator can be rewritten in the form

$$\begin{pmatrix} \dfrac{\partial}{\partial t} & (d_X + \delta_X)_{\mathrm{odd}} \\ (d_X + \delta_X)_{\mathrm{ev}} & \dfrac{\partial}{\partial t} \end{pmatrix} : \begin{pmatrix} \Lambda^{\mathrm{ev}}(X) \\ \Lambda^{\mathrm{odd}}(X) \end{pmatrix} \to \begin{pmatrix} \Lambda^{\mathrm{ev}}(X) \\ \Lambda^{\mathrm{odd}}(X) \end{pmatrix}, \tag{31}$$

[7] It is assumed that near the boundary the metric is the direct product of a metric on X by the standard metric dt^2 on $\mathbf{R}^1$.

where d_X and δ_X are, respectively, the exterior differential on X and its metric adjoint. To calculate the symbol of the Euler operator, note that

i) the symbol of d_X is the exterior multiplication by $i\xi dx$;

ii) the symbol of δ_X is the interior multiplication by $-iV_\xi$, where V_ξ is the vector corresponding to ξdx with respect to the metric g_X.

Furthermore, we need the relation

iii)

$$(\xi dx \wedge -V_\xi\rfloor)^2\omega = -((\xi dx\wedge)(V_\xi\rfloor) + (V_\xi\rfloor)(\xi dx\rfloor)\omega = -\xi^2\omega,$$

where $\wedge$ and $\rfloor$ are the operators of exterior and interior multiplication, respectively (e.g., [26]).

It is convenient to prove i) — iii) in the coordinates in which the metric g of the boundary is $(dx^1)^2 + \ldots + (dx^n)^2$ over a given (fixed) point $x_0 \in X$. For $\omega \in \Lambda^k(X)$, we have

$$\begin{aligned}
\omega &= \sum_{j_1<\ldots<j_k} \omega_{j_1\ldots j_k}(x)dx^{j_1}\wedge\ldots\wedge dx^{j_k}, \\
d\omega &= \sum_{j_1<\ldots<j_k}\left(\sum_{s=1}^{n-1}\frac{\partial\omega_{j_1\ldots j_k}(x)}{\partial x^s}dx^s\right)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_k} \\
&= \sum_{j_1<\ldots<j_k}\left(\sum_{s=1}^{n-1} i\left(-i\frac{\partial}{\partial x}\right)\omega_{j_1\ldots j_k}(x)dx^j\right)\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_k},
\end{aligned}$$

which proves i).

Next, we have

$$\delta\omega = -*d*\omega = -*d\sum_{j_1<\ldots<j_k}(-1)^{\sigma(j,l)}\omega_{j_1\ldots j_k}(x)dx^{l_1}\wedge\ldots\wedge dx^{l_{n-k-1}},$$

where $l = (l_1,\ldots,l_{n-k-1}) = l(j)$ is the tuple of indices complementary to $j = (j_1,\ldots,j_k)$ and $\sigma(j,l)$ is the number of transpositions in the permutation taking (j,l) to $(1,\ldots,n-1)$.

Consequently,

$$\begin{aligned}
\delta\omega &= -*\sum_{j_1<\ldots<j_k}(-1)^{\sigma(j,l)}\sum_{s=1}^{n-1}\frac{\partial\omega_{j_1\ldots j_k}}{\partial x^s}dx^s\wedge dx^{l_1}\wedge\ldots\wedge dx^{l_{n-k-1}} \\
&= -\sum_{j_1<\ldots<j_k}\sum_{s=1}^{n-1}(-1)^{\sigma(j,l)}\frac{\partial\omega_{j_1\ldots j_k}}{\partial x^s}(-1)^{\sigma(\{s\},l,l')}dx^{l'_1}\wedge\ldots\wedge dx^{l'_{k-1}},
\end{aligned}$$

where $(l'_1,\ldots,l'_{k-1})$ is the index tuple complementary to $(s,l_1,\ldots,l_{n-k-1})$. Let us consider each term of the sum. If $s\notin(j_1,\ldots,j_k)$, then $s\in(l_1,\ldots,l_{n-k-1})$ and consequently, the exterior product $dx^s \wedge dx^{l_1} \wedge \ldots \wedge dx^{l_{n-k-1}}$ is zero. If $s \in (j_1, \ldots, j_k)$, then $(l'_1, \ldots, l'_{k-1}) = (j_1, \ldots, j_k) \backslash \{s\}$. Next, it is easy to verify that the sign of each term is $(-1)^s$ (n is even!), and we have

$$\begin{aligned}
\delta\omega &= -\sum_{j_1 < \ldots < j_k} \sum_{s=1}^{n-1} \frac{\partial \omega_{j_1 \ldots j_k}}{\partial x^s} \frac{\partial}{\partial x^s} \rfloor dx^{j_1} \wedge \ldots \wedge dx^{j_k} \\
&= \sum_{j_1 < \ldots < j_k} \sum_{s=1}^{n-1} (-i) \left(-i \frac{\partial}{\partial x^s} \right) \omega_{j_1 \ldots j_k} \frac{\partial}{\partial x^s} \rfloor dx^{j_1} \wedge \ldots \wedge dx^{j'_k}.
\end{aligned}$$

Since $V_\xi = \partial / \partial x$ in our coordinates, this proves ii).

Finally, for any form ω we have

$$\omega = dx^j \wedge \omega' + \omega'',$$

where ω' and ω'' do not contain x^j. Consequently,

$$\begin{aligned}
&\left[(\xi_j dx^j \wedge) \left(\xi_j \frac{\partial}{\partial x^j} \rfloor \right) + \left(\xi_j \frac{\partial}{\partial x^j} \rfloor \right) (\xi_j dx^j \wedge) \right] \omega \\
&= \xi_j dx^j \wedge \xi_j \omega' + \left(\xi_j \frac{\partial}{\partial x^j} \right) \rfloor \xi_j dx^j \wedge \omega'' = \xi_j^2 dx^j \wedge \omega' + \xi_j^2 \omega'' = \xi_j^2 \omega.
\end{aligned}$$

By summing this over $j = 1, \ldots, n-1$, we obtain iii).

Now the symbol of the operator (31) is

$$D(p, \xi) = \begin{pmatrix} p & i(\xi dx \wedge - V_\xi \rfloor) \\ i(\xi dx \wedge - V_\xi \rfloor) & p \end{pmatrix}.$$

Let us find the spectral points and the corresponding subspaces of this operator family.

We must have

$$\begin{pmatrix} p & i(\xi dx \wedge - V_\xi \rfloor) \\ i(\xi dx \wedge - V_\xi \rfloor) & p \end{pmatrix} \begin{pmatrix} \omega^{\mathrm{ev}} \\ \omega^{\mathrm{odd}} \end{pmatrix} = 0, \tag{32}$$

where (note that it follows from general considerations that $p \neq 0$ for $\xi \neq 0$)

$$p\omega^{\mathrm{ev}} + i(\xi dx \wedge - V_\xi \rfloor) \omega^{\mathrm{odd}} = 0,$$

that is,

$$\omega^{\mathrm{ev}} = -\frac{i}{p} (\xi dx \wedge - V_\xi \rfloor) \omega^{\mathrm{odd}}$$

and

$$\begin{aligned}0 &= i(\xi dx \wedge -V_\xi \rfloor)\omega^{\mathrm{ev}} + p\omega^{\mathrm{odd}} = \frac{1}{p}(\xi dx \wedge -V_\xi \rfloor)^2\omega^{\mathrm{ev}} + p\omega^{\mathrm{odd}} \\ &= \frac{1}{p}(p^2 - |\xi|^2)\omega^{\mathrm{odd}},\end{aligned}$$

whence it follows that

$$p(\xi) = p_\pm(\xi) = \pm\,|\xi|\,.$$

Conversely, if $p = \pm\,|\xi|$,

$$\omega^{\mathrm{ev}} = \mp\frac{i}{|\xi|}(\xi dx \wedge -V_\xi \rfloor)\omega^{\mathrm{odd}}, \tag{33}$$

and $\omega^{\mathrm{odd}} \in \Lambda^{\mathrm{odd}}(X)$ is artibrary, then equation (32) holds. Hence the spectrum of $D(p,\xi)$ consists of the two points

$$p = \pm\,|\xi|$$

for any $\xi \neq 0$, and the corresponding eigenspace is described by (33), where ω^{odd} ranges over the entire space $\Lambda^{\mathrm{odd}}(X)$. Similarly, we can prove that the eigenspace is described by the formula

$$\omega^{\mathrm{odd}} = \mp\frac{i}{|\xi|}(\xi dx \wedge -V_\xi \rfloor)\omega^{\mathrm{ev}}, \tag{34}$$

where ω^{ev} ranges over $\Lambda^{\mathrm{ev}}(X)$.

Equations (33) and (34) show that the projections

$$P_{\mathrm{ev}}:\ \Lambda(X) = \Lambda^{\mathrm{ev}}(X) \oplus \Lambda^{\mathrm{odd}}(X) \to \Lambda^{\mathrm{ev}}(X)$$

and

$$P_{\mathrm{odd}}:\ \Lambda(X) = \Lambda^{\mathrm{ev}}(X) \oplus \Lambda^{\mathrm{odd}}(X) \to \Lambda^{\mathrm{odd}}(X)$$

are isomorphisms of the space $L_-(\xi)$ corresponding to the eigenvalue $p_-(\xi) = -\,|\xi|$ onto $\Lambda^{\mathrm{ev}}(X)$ and $\Lambda^{\mathrm{odd}}(X)$, respectively. Consequently, for the Euler operator we have the classical (Shapiro–Lopatinskii) Fredholm boundary value problems

$$\begin{cases} \hat{D}\, u = f, \\ P_{\mathrm{ev}} u|_X = g_1 \end{cases} \quad \text{and} \quad \begin{cases} \hat{D}\, u = f, \\ P_{\mathrm{odd}} u|_X = g_2, \end{cases}$$

where

$$\hat{D} = d + \delta:\ H^s(M, \Lambda^{\mathrm{ev}}(M)) \to H^{s-1}(M, \Lambda^{\mathrm{odd}}(M))$$

is the Euler operator, and g_1 and g_2 belong to

$$H^{s-1/2}(X, \Lambda^{\mathrm{ev}}(X))$$

and

$$H^{s-1/2}(X, \Lambda^{\mathrm{odd}}(X)),$$

respectively.

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