On the Index Formula for Singular Surfaces

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Abstract

In the preceding paper we proved an explicit index formula for elliptic pseudodifferential operators on a two-dimensional manifold with conical points. Apart from the Atiyah-Singer integral, it contains two additional terms, one of the two being the 'eta' invariant defined by the conormal symbol. In this paper we clarify the meaning of the additional terms for differential operators.

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Contents

Introduction

In [FST97] we proved the following index formula for elliptic pseudodifferential operators on a two-dimensional manifold with a conical point:

$$\begin{aligned}\operatorname{ind} A = &\int_{S^*M} \mathrm{AS}\,(A) - \frac{1}{2}\,\eta(A_c)\\ &-\frac{1}{4\pi^2}\int_{\mathbb{S}^1\times\mathbb{R}} \left(\frac{1}{2}\operatorname{tr}\left(a_0^{-1}\frac{\partial a_0}{\partial \xi}a_0^{-1}\frac{\partial a_0}{\partial x}\right) - i\operatorname{tr} a_0^{-1}a_1\right)\Bigg|_{\tau=-1}^{\tau=1} d\xi dx, \quad (0.1)\end{aligned}$$

M being the manifold in question whose cross-section close to the conical point is identified with the unit circle $\mathbb{S}^1$.

The index is evaluated for A acting on weighted Sobolev spaces on M as $H^{s,\gamma}(M,E^0) \to H^{s-m,\gamma}(M,E^1)$, where E^0 and E^1 are C^∞ vector bundles over the smooth part of M which behave properly under approaching the conical point.

The first term on the right-hand side of this formula is the Atiyah-Singer integral derived from the principal interior symbol a_0 of A and the curvature forms Ω^0 and Ω^1 of the bundles E^0 and E^1, respectively. We have

$$\mathrm{AS}\,(A) = \frac{1}{4\pi^2}\left(\frac{1}{6}\operatorname{tr}\,(a_0^{-1}\partial a_0)^3 - \frac{1}{2}\operatorname{tr}\left(\Omega^0 a_0^{-1}\partial a_0 + \Omega^1 \partial a_0 a_0^{-1}\right)\right).$$

The weight exponent γ enters only the second term on the right side of (0.1) which is known as the 'eta' invariant of the conormal symbol A_c of A at the conical point. More precisely,

$$\eta(A_c) = -\frac{1}{\pi i}\,\overline{\mathrm{Tr}}\left(A_c^{-1}(\tau+i\gamma)A_c'(\tau+i\gamma) - i\gamma\,\frac{d}{d\tau}(A_c^{-1}(\tau+i\gamma)A_c'(\tau+i\gamma))\right),$$

$\overline{\mathrm{Tr}}$ being a regularised trace (cf. Melrose [Mel95]).

Both these terms occur in the Atiyah-Patodi-Singer formula for the index of Dirac operators (cf. [APS75]). In contrast to this latter formula, (0.1) contains an additional third term which does not vanish even for the Cauchy-Riemann operator on the plane. This summand is in excess determined by the conormal symbol of A because the symbol components a_0 and a_1 entering into the expression are evaluated at the conical point.

Of course, formula (0.1) is still true for manifolds with several conical points. A slight change we have to do is that the 'eta' invariant and the additional terms should be summed up over all conical points of M.

The aim of this paper is to give an explicit description of the contribution of a conical point for elliptic *differential* operators. This description is given in terms of the monodromy matrix $M(\tau)$ for an ordinary differential equation defined by the conormal symbol $A_c(\tau)$. More precisely, we introduce a phase function

$$\varphi(\tau) = \frac{1}{2} \log\det\Big(M(\tau) + M^{-1}(\tau) - 2\Big)$$

which is an analytic function of τ with logarithmic ramification points. Then the 'eta' invariant term in (0.1) may be characterised as the variation of $\varphi(\tau)/2\pi i$ along a suitable contour defined by the weight line Γ. For first-order systems, the whole contribution of the conical point may be described as the variation of $\varphi(\tau)/2\pi i$ along yet another contour (Theorem 5.3). We would like to emphasise that this variation is integer or half-integer, hence so is also the interior contribution. To our mind, this property is a special feature of the two-dimensional case.

Our method is based on the asymptotical analysis of solutions and the monodromy matrix for a system of differential equations with a large parameter. Although there exists vast literature on this topic, we have not found the desired facts and were forced to prove them. The proof uses the ideas of Faddeev and Takhtajan [FT87] for the non-linear Schrödinger equation.

If $\varphi(\tau)$ possesses some natural symmetry properties, then the contribution of the conical point has very simple nature (Theorem 7.1). We show some sufficient conditions for this symmetry.

1 Ellipticity and splitting

We start by describing special coordinates and bundle trivialisations near a conical point. Recall that the neighbourhood of a conical point is treated as a cylindrical end with coordinates $t \in \mathbb{R}_+$, $x \in \mathbb{R} \bmod (2\pi)$. Since any complex vector bundle over a circle is trivial, we may assume that $E^0 \cong E^1 \cong \mathbb{C}^r$ over the cylindrical end and, for given trivialisations, the connection one-forms Γ^0, Γ^1 are equal to 0. There is a freedom in choosing global frames of the bundles E^0, E^1 over a circle, we use it to simplify the conormal symbol of an elliptic operator.

For a first-order differential elliptic operator its conormal symbol has the form

$$A_c(\tau) = A_1(x)\tau - iA_2(x)\frac{\partial}{\partial x} + B(x), \tag{1.1}$$

where $A_1(x)$, $A_2(x)$, $B(x)$ are, for given trivialisations of the bundles E^0 and E^1, $(r \times r)$-matrices of smooth functions on the circle. Thus, the principal interior symbol restricted to the boundary is

$$a_0 = A_1(x)\tau + A_2(x)\xi \tag{1.2}$$

and the lower-order term is given by a matrix-valued function

$$a_1 = B(x).$$

The interior ellipticity means that the matrix (1.2) is non-degenerate for any real $(\tau,\xi) \neq (0,0)$; in particular, $A_1(x)$ and $A_2(x)$ are non-degenerate matrices.

Using $A_1(x)$ as a transition matrix, we introduce a new frame of the bundle E^1, so that (1.1) becomes

$$A_c(\tau) = \tau - iA_1^{-1}(x)A_2(x)\frac{\partial}{\partial x} + A_1^{-1}(x)B(x).$$

The next consequence of the ellipticity is that the matrix $A_1^{-1}(x)A_2(x)$ has no real eigenvalues, so at any $x \in \mathbb{S}^1$ its spectrum consists of two disjoint parts belonging to the upper and lower half-planes, respectively. The corresponding spectral projectors are given by the Cauchy integrals [1]

$$P_\pm(x) = \frac{1}{2\pi i}\int_{c_\pm}\left(\xi - A_2^{-1}(x)A_1(x)\right)^{-1} d\xi \qquad (1.3)$$

where $c_\pm$ are closed contours in the upper and lower half-planes, respectively, consisting of a large semicircle and its diameter. These projectors depend smoothly on $x \in \mathbb{S}^1$, thus defining a decomposition of the trivial bundle $\mathbb{C}^r$ into a direct sum of two subbundles of dimensions $r_\pm$. Like any complex bundle over a circle these subbundles are trivial. It follows that we may choose a frame in $\mathbb{C}^r$ with a transition matrix $C(x)$, so that

$$\begin{aligned} P_+(x) &= C(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} C^{-1}(x), \\ P_-(x) &= C(x)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} C^{-1}(x) \end{aligned}$$

and

$$A_1^{-1}(x)A_2(x) = iC(x)\begin{pmatrix} a_+(x) & 0 \\ 0 & a_-(x) \end{pmatrix} C^{-1}(x),$$

where $a_\pm(x)$ are $(r_\pm \times r_\pm)$-matrices having the spectra in the right (left) half-plane. Passing to new frames in E^0, E^1 with the same transition matrix $C(x)$, we reduce the conormal symbol to the form

$$A_c(\tau) = \tau + \begin{pmatrix} a_+ & 0 \\ 0 & a_- \end{pmatrix}\frac{\partial}{\partial x} + C^{-1}A_1^{-1}BC + C^{-1}C' \qquad (1.4)$$

which will be referred to as the *canonical form*. To simplify notation, we will write it simply as

$$A_c(\tau) = \tau + A(x)\frac{\partial}{\partial x} + B(x), \qquad (1.5)$$

with $A(x)$ having a block-diagonal structure as in (1.4).

We investigate first the additional integral terms in (0.1).

[1] We use spectral projectors for the inverse matrix by reasons to be clear later on. Of course, these are the same as for the matrix itself.

Lemma 1.1 *For the canonical form, we have*

$$\int_{\mathbb{S}^1\times\mathbb{R}} \operatorname{tr}\ a_0^{-1}\frac{\partial a_0}{\partial \xi}a_0^{-1}\frac{\partial a_0}{\partial x}\Big|_{\tau=-1}^{\tau=1} d\xi dx = 0. \tag{1.6}$$

Proof. For $A_c(x)$ given by (1.5),

$$a_0 = \tau + iA(x)\xi,$$

so that

$$\begin{aligned} a_0^{-1}\frac{\partial a_0}{\partial \xi}a_0^{-1}\frac{\partial a_0}{\partial x} &= -\frac{\partial a_0^{-1}}{\partial \xi}\frac{\partial a_0}{\partial x} \\ &= \frac{\partial}{\partial \xi}\left(\xi - iA^{-1}(x)\tau\right)^{-1} A^{-1}(x)\frac{\partial A(x)}{\partial x}\xi. \end{aligned}$$

The integral over ξ in (1.6) is

$$\int_{-\infty}^{\infty} \xi\frac{\partial}{\partial \xi}\left((\xi - iA^{-1}(x))^{-1} - (\xi + iA^{-1}(x))^{-1}\right) d\xi.$$

It is absolutely convergent and the real axis may be replaced by the contour c_+ (or c_- as well). Integrating by parts and using (1.3), we obtain

$$\int_{c_+} \left((\xi + iA^{-1}(x))^{-1} - (\xi - iA^{-1}(x))^{-1}\right) d\xi = 2\pi i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that the whole integral (1.6) becomes

$$\int_{\mathbb{S}^1} \operatorname{tr} a_+^{-1} da_+ - \operatorname{tr} a_-^{-1} da_-$$

up to a constant factor. This expression is homotopy-invariant and thus equals 0 since $a_+(x)$ may be contracted to $+1$ and $a_-(x)$ to -1, proving the lemma.

□

Let us calculate the remaining integral in (0.1).

Lemma 1.2 *For the canonical form* (1.5)*, we have*

$$\frac{i}{4\pi^2}\int_{\mathbb{S}^1\times\mathbb{R}} \operatorname{tr}\ a_0^{-1}a_1\Big|_{\tau=-1}^{\tau=1} d\xi dx = \frac{i}{2\pi}\int_{\mathbb{S}^1} \operatorname{tr} \begin{pmatrix} a_+^{-1} & 0 \\ 0 & a_-^{-1} \end{pmatrix} B(x)dx. \tag{1.7}$$

Proof. Consider

$$a_0^{-1}a_1\Big|_{\tau=-1}^{\tau=1} = -i\left((\xi - iA^{-1}(x))^{-1} - (\xi + iA^{-1}(x))^{-1}\right) A^{-1}(x)B(x).$$

Integrating over ξ just as in the previous lemma, we arrive to (1.7).

□

2 The Green function and the monodromy matrix

The conormal symbol $A_c(\tau)$ given by (1.4), (1.5), is an ordinary differential operator with 2π-periodic matrix-valued coefficients. We will study the properties of this operator and its inverse when the spectral parameter τ varies in the complex plane $\mathbb{C}$. In particular, we are interested in asymptotic properties for $\Re\tau \to \pm\infty$ while $\Im\tau$ remains bounded.

Let $U(x,\tau)$ be a matrix-valued function satisfying the following differential equation and initial condition:

$$\begin{aligned} A_c(\tau)U &= 0, \\ U|_{x=0} &= 1. \end{aligned} \tag{2.1}$$

We call U a *fundamental solution.*

The *monodromy matrix* is, by definition,

$$M(\tau) = U(2\pi, \tau).$$

It is known from the theory of ordinary differential equations that $U(x,\tau)$ (and hence $M(\tau)$) is an entire function of the spectral parameter τ. It is invertible for each $x \in \mathbb{R}$ and $\tau \in \mathbb{C}$ and its determinant is given by the Liouville formula

$$\det U(x,\tau) = \exp\left(-\int_0^x \operatorname{tr} A^{-1}(y)\,(\tau + B(y))\,dy\right). \tag{2.2}$$

Sometimes one considers more general fundamental solutions $U(x,y,\tau)$ satisfying the same equation and the initial condition at $x = y$,

$$U|_{x=y} = 1.$$

The corresponding monodromy matrix is then

$$M(\tau, y) = U(y + 2\pi, y, \tau).$$

This more general case may be easily reduced to the previous one by using

$$U(x,y,\tau) = U(x,\tau)\,U^{-1}(y,\tau)$$

and

$$M(y,\tau) = U(y,\tau)\,M(\tau)\,U^{-1}(y,\tau).$$

In particular, the eigenvalues of the monodromy matrix are independent of the initial point y.

We will need the following expression for the logarithmic derivative of the monodromy matrix.

Lemma 2.1 *We have*

$$M^{-1}(\tau)\,\frac{\partial}{\partial\tau}M(\tau) = -\int_0^{2\pi} U^{-1}(x,\tau)A^{-1}(x)U(x,\tau)\,dx. \tag{2.3}$$

Proof. Set

$$U_\tau(x) = \frac{\partial}{\partial\tau}U(x,\tau).$$

Derivating (2.1) in τ, we obtain an equation and an initial condition for $U_\tau(x)$, namely

$$\begin{aligned} A_c(\tau)U_\tau(x) + U(x,\tau) &= 0,\\ U_\tau(0) &= 0. \end{aligned}$$

We look for a $U_\tau(x)$ of the form $U(x,\tau)V(x,\tau)$ (variation of constants). Substituting this into the above system gives

$$A(x)U(x,\tau)\frac{\partial}{\partial x}V(x,\tau) + U(x,\tau) = 0.$$

Hence

$$V(x,\tau) = -\int_0^x U^{-1}(y,\tau)A^{-1}(y)U(y,\tau)\,dy$$

and

$$U_\tau(x) = -U(x,\tau)\int_0^x U^{-1}(y,\tau)A^{-1}(y)U(y,\tau)\,dy.$$

Taking $x = 2\pi$, we arrive at (2.3), as desired.

□

Here is the starting point of our investigations.

Theorem 2.2 *The following formula holds:*

$$\begin{aligned} \operatorname{Tr}\frac{\partial}{\partial\tau}A_c^{-1}(\tau) &= \frac{1}{2}\,\frac{\partial^2}{\partial\tau^2}\log\det\left(M(\tau)+M^{-1}(\tau)-2\right)\\ &= \frac{\partial^2}{\partial\tau^2}\log\det\left(M^{\frac{1}{2}}(\tau)-M^{-\frac{1}{2}}(\tau)\right). \end{aligned} \tag{2.4}$$

Remark 2.3 From the monodromy theorem of complex analysis it follows that $M^{\frac{1}{2}}(\tau)$ is also an entire function.

Proof. The operator $A_c^{-1}(\tau)$, when considered on periodic functions on $[0,2\pi]$, is an integral operator whose kernel $G(x,y,\tau)$ (the *Green function*) is a periodic solution of the equation

$$A_c(\tau)G(x,y,\tau) = \delta(x-y).$$

We treat x as an argument while $y\in[0,2\pi]$ is considered as a parameter, δ being the Dirac δ-function. This equation means that G satisfies the homogeneous equation on $[0,y)$ and $(y,2\pi]$, whence

$$\begin{aligned} G(x,y,\tau) &= U(x,\tau)C_-, &&\text{for } x\in[0,y),\\ G(x,y,\tau) &= U(x,\tau)C_+, &&\text{for } x\in(y,2\pi], \end{aligned}$$

the matrices $C_\pm$ being independent of x. To produce the δ-function, these matrices should satisfy the relation

$$C_+ - C_- = U^{-1}(y,\tau)A^{-1}(y)$$

while periodicity yields

$$\begin{aligned} C_- &= U(x,2\pi)C_+ \\ &= M(\tau)C_+. \end{aligned}$$

Solving this system, we get a usual expression for the Green function, namely

$$G(x,y,\tau) = \begin{cases} U(x,\tau)(1-M(\tau))^{-1}U^{-1}(y,\tau)A^{-1}(y), & x \in [0,y), \\ U(x,\tau)M(\tau)(1-M(\tau))^{-1}U^{-1}(y,\tau)A^{-1}(y), & x \in (y,2\pi], \end{cases}$$

or equivalently

$$\begin{aligned} G(x,y,\tau) &= \frac{1}{2}U(x,\tau)(1+M(\tau))(1-M(\tau))^{-1}U^{-1}(y,\tau)A^{-1}(y) \\ &\quad + \frac{1}{2}\,\mathrm{sgn}\,(x-y)U(x,\tau)U^{-1}(y,\tau)A^{-1}(y). \end{aligned}$$

From the latter formula it follows that $(\partial/\partial\tau)G(x,y,\tau)$ has no jump at $x=y$. Moreover,

$$\left.\frac{\partial}{\partial\tau}G(x,y,\tau)\right|_{x=y} = \frac{\partial}{\partial\tau}\,\frac{1}{2}\,U(y,\tau)(1+M(\tau))(1-M(\tau))^{-1}U^{-1}(y,\tau)A^{-1}(y).$$

We now take the matrix trace, integrate over $y \in [0,2\pi]$ and make use of Lemma 2.1, thus obtaining

$$\begin{aligned} &\int_0^{2\pi} \mathrm{tr}\,\frac{\partial G}{\partial\tau}(y,y,\tau) \\ &= -\frac{1}{2}\,\frac{\partial}{\partial\tau}\left(\mathrm{tr}\,(1+M(\tau))(1-M(\tau))^{-1}M^{-1}(\tau)\frac{\partial}{\partial\tau}M(\tau)\right) \\ &= \frac{\partial}{\partial\tau}\left(\mathrm{tr}\,(M(\tau)-1)^{-1}\frac{\partial}{\partial\tau}M(\tau) - \frac{1}{2}\,\mathrm{tr}\,M^{-1}(\tau)\frac{\partial}{\partial\tau}M(\tau)\right), \end{aligned}$$

which is precisely (2.4).

□

3 Lyapunov estimates

In this section we consider the so-called *stable case* of equation (2.1). It means that the equation may be rewritten in the form

$$\frac{\partial U}{\partial x} = \lambda_-(x,\tau)U, \qquad (3.1)$$

where

$$\lambda_-(x,\tau) = a_-(x)\tau + b(x) \tag{3.2}$$

is a linear function in τ and the matrix $a_-(x)$ has its spectrum in the left half-plane (such matrices will be called *stable*). We will consider (3.1) for $\Re\tau \to +\infty$, $|\Im\tau| \le C$. Thus, without loss of generality we may regard τ as a large positive number by including $a_-(x)\Im\tau$ into $b(x)$. The whole matrix λ_- will be stable for τ large enough. Moreover, we may add to (3.2) a finite number of terms with negative powers of τ without spoiling the stability property for large τ.

The main result of this section consists in the following estimate for the fundamental solution $U(x,y,\tau) = U(x,\tau)U^{-1}(y,\tau)$.

Theorem 3.1 *Let $a_-(x)$ be a stable matrix. Then there exist constants C, $d > 0$ such that, for $\tau \gg 1$,*

$$\|U(x,y,\tau)\| \le C \ \exp(-d\,(x-y)\,\tau) \tag{3.3}$$

provided $x \ge y$, where $\|\cdot\|$ means any matrix norm.

Remark 3.2 In the case of constant coefficients a_- and b, estimate (3.3) is obvious, because the solutions can be expressed in terms of exponential functions. For variable coefficients it is not, however, so obvious (recall stable and unstable zones for the Schrödinger equation).

The following necessary and sufficient condition of stability is due to Lyapunov (see e.g. [Gan86]).

Lemma 3.3 *A complex matrix a_- is stable if and only if there exists a Hermitian positive definite matrix X such that*

$$a_-^* X + X a_- = -1. \tag{3.4}$$

Proof. If a_- is stable, so is a_-^*. Hence both $\exp a_- t$ and $\exp a_-^* t$ are exponentially decaying as $t \to +\infty$. The matrix X may be defined by an explicit expression, namely

$$X = \int_0^\infty \exp(a_-^* t)\exp(a_- t)dt. \tag{3.5}$$

Indeed,

$$\begin{aligned} a_-^* X + X a_- &= \int_0^\infty \frac{\partial}{\partial t}\Big(\exp(a_-^* t)\exp(a_- t)\Big)\,dt \\ &= -1. \end{aligned}$$

Conversely, from (3.4) it follows, for an eigenvector e of a_- with an eigenvalue λ, that

$$\begin{aligned} (e,e) &= -(Xa_-e, e) - (Xe, a_-e) \\ &= -(\lambda + \bar\lambda)(Xe, e). \end{aligned}$$

Hence $\Re\lambda < 0$, as desired.

$\square$

Proof of Theorem 3.1. If $a_-(x)$ is a smooth periodic function in x, then (3.5) shows that $X(x)$ is also a smooth periodic function. In particular, there are bounds independent of x, for

$$0 < C_1 \leq X(x) \leq C_2 \tag{3.6}$$

in the sense of quadratic forms. Denoting the usual norm in $\mathbb{C}^r$ by $\|e\| = \sqrt{(e,e)}$, we define a new norm

$$\|e\|_X = \sqrt{(Xe,e)}$$

which is equivalent to the usual one. Then, inequalities (3.6) give a precise form of the equivalence relations

$$C_1\,\|e\|^2 \leq \|e\|_X^2 \leq C_2\,\|e\|^2. \tag{3.7}$$

For a fundamental solution $U(x,y,\tau)$, we consider the function

$$\begin{aligned} f(x) &= \|Ue\|_X^2 \\ &= (U^*(x,y,\tau)X(x)U(x,y,\tau)e,\, e)\,. \end{aligned}$$

Differentiating and using (3.4), we get

$$\begin{aligned} \frac{\partial f}{\partial x} &= \left(U^*\left(\lambda_-^* X + X\lambda_- + X'\right)Ue,\, e\right) \\ &= -\tau\,(Ue,Ue) + ((b^*X + Xb + X')\,Ue,\, Ue)\,. \end{aligned}$$

The matrix $b^*X + Xb + X'$ is Hermitian and, for τ large enough, we have

$$-\frac{\tau}{2} \leq b^*X + Xb + X' \leq \frac{\tau}{2}$$

in the sense of quadratic forms. By (3.7), the norm $\|Ue\|^2$ may be replaced by $\|Ue\|_X^2$, hence

$$\frac{\partial f}{\partial x} \leq -d\,\tau\, f(x)$$

with some positive constant d. Dividing by $f(x)$ and integrating from y to x, with $x \geq y$, we obtain

$$\log\frac{f(x)}{f(y)} \leq -d\,\tau\,(x-y)$$

which means that

$$\|U(x,y,\tau)e\|_{X(x)}^2 \leq \exp(-d\,(x-y)\,\tau)\;\|e\|_{X(y)}^2.$$

Since the norms $\|\cdot\|_{X(x)}$ are equivalent to any fixed norm $\|\cdot\|$, we come to (3.3), which completes the proof.

□

This theorem has some obvious modifications. For example, an estimate

$$\|U(x,\tau)U^{-1}(y,\tau)\| \le C\ \exp(-d\,(x-y)\,\tau) \tag{3.8}$$

holds if $\tau \to -\infty$ and $x \le y$. Next, we may replace a stable matrix a_- by a matrix a_+ with a spectrum in the right half-plane. In this case we have

$$\|U(x,\tau)U^{-1}(y,\tau)\| \le C\ \exp(d\,(x-y)\,\tau) \tag{3.9}$$

for $\tau \to +\infty$ and $x \le y$ or $\tau \to -\infty$ and $x \ge y$, with some C, $d > 0$.

4 Asymptotics of solutions

In this section we consider the general case of equation (2.1) with a splitted matrix $A(x)$. So, we write it in the form

$$\frac{\partial U}{\partial x} = (\Lambda(x,\tau) + B(x))\,U \tag{4.1}$$

where

$$\begin{aligned} \Lambda(x,\tau) &= \begin{pmatrix} \lambda_+(x,\tau) & 0 \\ 0 & \lambda_-(x,\tau) \end{pmatrix} \\ &= \begin{pmatrix} a_+(x)\tau + b_{11}(x) & 0 \\ 0 & a_-(x)\tau + b_{22}(x) \end{pmatrix} \end{aligned} \tag{4.2}$$

is a block-diagonal part and

$$B(x) = \begin{pmatrix} 0 & b_{12}(x) \\ b_{21}(x) & 0 \end{pmatrix}$$

is an antidiagonal part of the coefficients. We assume that both $a_-(x)$ and $-a_+(x)$ are stable matrices.

Let us look for a solution of (4.1) in the form (cf. (4.5) in [FT87, Ch. 1])

$$U(x,\tau) = (1 + W(x,\tau))Z(x,\tau), \tag{4.3}$$

where Z is a block-diagonal matrix and W is an antidiagonal matrix. Substituting (4.3) into (4.1) and separating diagonal and antidiagonal parts, we obtain

$$\begin{aligned} \frac{\partial W}{\partial x}Z + W\frac{\partial Z}{\partial x} &= \Lambda W Z + BZ, \\ \frac{\partial Z}{\partial x} &= (\Lambda + BW)\,Z. \end{aligned} \tag{4.4}$$

Eliminating Z, we arrive at a matrix Riccati equation for W

$$\frac{\partial W}{\partial x} = \Lambda W - W\Lambda + B - WBW. \tag{4.5}$$

Were W a solution of (4.5), the second equation in (4.4) would give us an equation for Z with a block-diagonal coefficient $\Lambda + BW$.

To find W, we observe that equation (4.5) is equivalent to two separate equations for w_{12} and w_{21},

$$\frac{\partial w_{12}}{\partial x} = \lambda_+ w_{12} - w_{12}\lambda_- + b_{12} - w_{12}b_{21}w_{12}, \tag{4.6}$$

$$\frac{\partial w_{21}}{\partial x} = \lambda_- w_{21} - w_{21}\lambda_+ + b_{21} - w_{21}b_{12}w_{21}. \tag{4.7}$$

Assuming $\lambda_\pm$ to be of the form (4.2), let us consider τ positive and large enough. We will look for solutions to (4.6) and (4.7) on the closed interval $x \in [0, 2\pi]$ with initial conditions

$$w_{12}(2\pi) = 0, \tag{4.8}$$

$$w_{21}(0) = 0. \tag{4.9}$$

Lemma 4.1 *The solutions of* (4.6), (4.8) *and* (4.7), (4.9) *exist for* τ *large enough and satisfy the estimates*

$$\begin{aligned} \|w_{12}(x,\tau)\| &= O\left(\frac{1}{\tau}\right), \\ \|w_{21}(x,\tau)\| &= O\left(\frac{1}{\tau}\right) \end{aligned} \tag{4.10}$$

uniformly in $x \in [0, 2\pi]$.

Proof. Let us consider the case of w_{12}, the reasoning for w_{21} is similar. First we reduce (4.6), (4.8) to an equivalent integral equation. To this end, let us treat $f = b_{12} - w_{12}b_{21}w_{12}$ as a known function and apply the variation of constants to the equation

$$w_{12}' = \lambda_+ w_{12} - w_{12}\lambda_- + f.$$

In other words, we look for a solution of the form

$$w_{12}(x) = U_+(x)V(x)U_-^{-1}(x), \tag{4.11}$$

where $U_\pm(x,\tau)$ are fundamental solutions to the Cauchy problems

$$\begin{aligned} \frac{\partial U_\pm}{\partial x} &= \lambda_\pm U_\pm, \\ U_\pm|_{x=0} &= 1. \end{aligned}$$

Substituting, we obtain

$$\frac{\partial V}{\partial x} = U_+^{-1} \, f \, U_-$$

and, taking into account (4.8),

$$V(x) = -\int_x^{2\pi} U_+^{-1}(y)\, f(y)\, U_-(y)\, dy.$$

Now, returning to (4.11) and replacing $f(y)$, we come to the integral equation

$$w_{12}(x) = -\int_x^{2\pi} U_+(x)U_+^{-1}(y)\,(b_{12}(y) - w_{12}(y)b_{21}(y)w_{12}(y))\,U_-(y)U_-^{-1}(x)dy.$$

This equation may be solved by iterations. From Theorem 3.1 and what has been said at the end of Section 3, we deduce that

$$\begin{aligned} \|U_+(x)U_+^{-1}(y)\| &\leq C \ \exp(d\,(x-y)\,\tau), & (4.12)\\ \|U_-(y)U_-^{-1}(x)\| &\leq C \ \exp(d\,(x-y)\,\tau) & (4.13)\end{aligned}$$

for $\tau \gg 1$ and $x \leq y$. In particular, these expressions are uniformly bounded for $\tau \gg 1$ and $0 \leq x \leq y \leq 2\pi$. The initial iteration

$$-\int_x^{2\pi} U_+(x)U_+^{-1}(y)\, b_{12}(y)\, U_-(y)U_-^{-1}(x)\, dy$$

may be estimated by means of (4.12), (4.13) as

$$\begin{aligned} C\int_x^{2\pi} \exp(2\,d\,(x-y)\,\tau)\ dy &\leq \frac{C}{2d\tau} \\ &= O\left(\frac{1}{\tau}\right). \end{aligned}$$

When combined with the boundedness of (4.12) and (4.13), this estimate implies the convergence of the iterations and the desired estimate (4.10).

Similarly, for w_{21} we obtain an integral equation

$$w_{21}(x) = \int_0^x U_-(x)U_-^{-1}(y)\,(b_{21}(y) - w_{21}(y)b_{12}(y)w_{21}(y))\,U_+(y)U_+^{-1}(x)dy$$

and then repeat the previous arguments.

□

Turning to the block-diagonal part, we denote by $Z_\pm(x,\tau)$ the entries of Z. More precisely, we take them as solutions of the Cauchy problems

$$\left.\begin{aligned} \frac{\partial Z_+}{\partial x} &= (\lambda_+ + b_{12}w_{21})\,Z_+, \\ Z_+|_{x=0} &= 1 \end{aligned}\right. \qquad (4.14)$$

and

$$\begin{aligned} \frac{\partial Z_-}{\partial x} &= \left(\lambda_- + b_{21} w_{12}\right) Z_-, \\ Z_-|_{x=0} &= 1. \end{aligned} \tag{4.15}$$

The crucial property of the coefficients in (4.14) and (4.15) is that for $\tau \gg 1$ the matrix

$$\lambda_- + b_{21} w_{12} = \lambda_- + O\left(\frac{1}{\tau}\right)$$

is stable and so is

$$-\left(\lambda_+ + b_{12} w_{21}\right).$$

In particular, this implies estimates (3.3), (3.8) for Z_- and (3.9) for Z_+.

We have thus constructed a solution of the form (4.3), with

$$W(x,\tau) = O\left(\frac{1}{\tau}\right)$$

uniformly in x. It does not satisfy the initial condition $U(0,\tau) = 1$, but this drawback can be easily corrected. Indeed,

$$\begin{aligned} V(x,\tau) &= U(x,\tau) U^{-1}(0,\tau) \\ &= (1 + W(x,\tau)) Z(x,\tau) (1 + W(0,\tau))^{-1} \end{aligned}$$

is the desired solution. For the monodromy matrix, we obtain

$$\begin{aligned} M(\tau) &= V(2\pi,\tau) \\ &= (1 + W(2\pi,\tau)) Z(2\pi,\tau)(1 + W(0,\tau))^{-1} \\ &= \left(1 + O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} Z_+(2\pi,\tau) & 0 \\ 0 & Z_-(2\pi,\tau) \end{pmatrix} \left(1 + O\left(\frac{1}{\tau}\right)\right). \end{aligned} \tag{4.16}$$

Finally, we apply (4.16) to compute the asymptotic expansion of the phase function

$$\varphi(\tau) = \frac{1}{2} \log\det\left(M(\tau) + M^{-1}(\tau) - 2\right)$$

for $\Re\tau \to \pm\infty$ and $|\Im\tau| \le C$. All the calculations will be performed modulo πi. From (4.16) it follows that

$$\begin{aligned} M(\tau) + M^{-1}(\tau) - 2 = \left(1 + O\left(\frac{1}{\tau}\right)\right) \Big\{ Z(2\pi,\tau) \left(1 + O\left(\frac{1}{\tau}\right)\right) \\ + \left(1 + O\left(\frac{1}{\tau}\right)\right) Z^{-1}(2\pi,\tau) - 2\left(1 + O\left(\frac{1}{\tau}\right)\right) \Big\} \left(1 + O\left(\frac{1}{\tau}\right)\right) \end{aligned}$$

implying

$$\varphi(\tau) = \frac{1}{2}\ \log\det\Big\{ Z(2\pi,\tau)\left(1+O\left(\frac{1}{\tau}\right)\right) + \left(1+O\left(\frac{1}{\tau}\right)\right) Z^{-1}(2\pi,\tau) - 2\left(1+O\left(\frac{1}{\tau}\right)\right)\Big\} + O\left(\frac{1}{\tau}\right).$$

A straightforward computation shows that the expression in curly brackets transforms further to

$$\begin{aligned}
&\begin{pmatrix} Z_+(2\pi,\tau) & 0 \\ 0 & 1 \end{pmatrix} \Big\{ \begin{pmatrix} 1 & 0 \\ 0 & Z_-(2\pi,\tau) \end{pmatrix} \left(1+O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & Z_-(2\pi,\tau) \end{pmatrix} \\
&\quad + \begin{pmatrix} Z_+^{-1}(2\pi,\tau) & 0 \\ 0 & 1 \end{pmatrix} \left(1+O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} Z_+^{-1}(2\pi,\tau) & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad - 2 \begin{pmatrix} Z_+^{-1}(2\pi,\tau) & 0 \\ 0 & 1 \end{pmatrix} \left(1+O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & Z_-(2\pi,\tau) \end{pmatrix} \Big\} \begin{pmatrix} 1 & 0 \\ 0 & Z_-^{-1}(2\pi,\tau) \end{pmatrix}.
\end{aligned}$$

Now, $Z_+^{-1}(2\pi,\tau)$ and $Z_-(2\pi,\tau)$ decay exponentially for $\Re\tau \to +\infty$. Indeed, applying (3.3) for $x = 2\pi$ and $y = 0$, we get

$$\begin{aligned}
\|Z_-(2\pi,\tau)\| &\le C\ \exp(-2\pi d\tau) \\
&= O\left(\frac{1}{\tau}\right);
\end{aligned}$$

the same is true for $Z_+^{-1}(2\pi,\tau)$, as may be seen from (3.9) for $x = 0$ and $y = 2\pi$. Hence, the previous expression can be rewritten as

$$\begin{pmatrix} Z_+(2\pi,\tau) & 0 \\ 0 & 1 \end{pmatrix} \left(1+O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} 1 & 0 \\ 0 & Z_-^{-1}(2\pi,\tau) \end{pmatrix}$$

so that

$$\varphi(\tau) = \frac{1}{2}\ \log\det Z_+(2\pi,\tau) - \frac{1}{2}\ \log\det Z_-(2\pi,\tau) + O\left(\frac{1}{\tau}\right).$$

Finally, using the Liouville formula (2.2) for Z_+ and Z_-, we arrive at

$$\begin{aligned}
\varphi(\tau) &= \frac{1}{2}\int_0^{2\pi} \mathrm{tr}\,(\lambda_+ + b_{12}w_{21})dx - \frac{1}{2}\int_0^{2\pi} \mathrm{tr}\,(\lambda_- + b_{21}w_{12})dx + O\left(\frac{1}{\tau}\right) \\
&= \frac{1}{2}\int_0^{2\pi} \left(\mathrm{tr}\,(a_+(x)\tau + b_{11}(x)) - \mathrm{tr}\,(a_-(x)\tau + b_{22}(x))\right) dx + O\left(\frac{1}{\tau}\right).
\end{aligned} \tag{4.17}$$

Similarly an asymptotic formula for $\varphi(\tau)$ may be obtained as $\Re\tau \to -\infty$ and $|\Im\tau| \le C$. The result will be given by (4.17) with the opposite sign. We summarise these results as follows.

Theorem 4.2 *Let $\Re\tau \to \pm\infty$ and $|\Im\tau| \le C$. Then the following asymptotic formulas hold:*

$$\begin{aligned}\varphi(\tau) &= \pm\frac{1}{2}\int_0^{2\pi}\left(\operatorname{tr}\left(a_+(x)\tau + b_{11}(x)\right) - \operatorname{tr}\left(a_-(x)\tau + b_{22}(x)\right)\right)dx + \pi i N_\pm \\ &\quad + O\left(\frac{1}{\tau}\right). \end{aligned} \tag{4.18}$$

The integers $N_\pm$ remain undetermined. We may fix one of them, then the other will depend on the path to be used for analytic extension.

5 The index formula

Combining Theorem 4.2 with the calculations of Section 1, we give a simple interpretation of the boundary terms in the index formula.

First we introduce a special value $\tau_0 \in \mathbb{C}$ to be referred to as the *centre*. This is a solution of the equation

$$\int_0^{2\pi}\left(\operatorname{tr}\left(a_+(x)\tau_0 + b_{11}(x)\right) - \operatorname{tr}\left(a_-(x)\tau_0 + b_{22}(x)\right)\right)dx = 0. \tag{5.1}$$

Lemma 5.1 *Equation* (5.1) *has a unique solution.*

Proof. The equation is linear, so we only need to show that the coefficient

$$a = \int_0^{2\pi}\left(\operatorname{tr} a_+(x) - \operatorname{tr} a_-(x)\right)dx$$

does not vanish. Indeed, $\Re\ \operatorname{tr} a_+(x) > 0$ since all the eigenvalues of $a_+(x)$ have positive real parts. The same is true for $-\Re\ \operatorname{tr} a_-(x) > 0$, whence $\Re a > 0$ proving the lemma.

□

The following consequence of Theorem 4.2 justifies the designation 'centre'.

Corollary 5.2 *Let $\tau \mapsto 2\tau_0 - \tau$ be a symmetry transformation with respect to the centre τ_0. Then the function*

$$\begin{aligned} f(\tau) &= \det\left(M(\tau) + M^{-1}(\tau) - 2\right) \\ &= e^{2\varphi(\tau)} \end{aligned}$$

is asymptotically even, that is, for $|\Re\tau|$ large enough and $|\Im\tau| \le C$, we have

$$f(\tau) = f(2\tau_0 - \tau)\left(1 + O\left(\frac{1}{\tau}\right)\right). \tag{5.2}$$

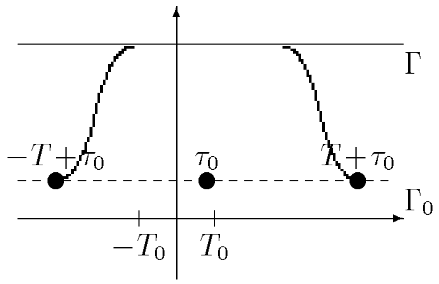

Fig. 1: Variation of $\varphi(t)$ along Γ.

Let now Γ be a horizontal line (weight line) not containing ramification points of $\varphi(\tau)$, and $\tau_0 \in \mathbb{C}$ be a point (centre). Our next objective is to introduce a number $\Delta_{\Gamma,\tau_0}\varphi(\tau)$ which will be called the *variation* of $\varphi(\tau)$ along Γ with respect to the centre τ_0. To this end, consider another horizontal line Γ_0 passing through τ_0. In the closed strip between Γ and Γ_0 there are a finite number of ramification points of $\varphi(\tau)$. In particular, for $|\Re\tau| > T_0$ with T_0 large enough, there are no ramification points in this strip. We consider a contour starting at the point $-T+\tau_0 \in \Gamma_0$ with T positive and large enough, such that $|\Re(-T+\tau_0)| > T_0$, then going along Γ in the region where $|\Re\tau| < T_0$, and finishing at the point $T+\tau_0$ (see Fig. 1).

By $\varphi(T+\tau_0) - \varphi(-T+\tau_0)$ we denote the variation of $\varphi(\tau)$ along this contour. Set

$$\Delta_{\Gamma,\tau_0}\varphi(\tau) = \lim_{T\to+\infty} \left(\varphi(T+\tau_0) - \varphi(-T+\tau_0)\right).$$

Because of (5.2) this number is an integer multiple of πi.

Theorem 5.3 *The boundary contribution in the index formula* (0.1) *is equal to*

$$\frac{1}{2\pi i}\ \Delta_{\Gamma,\tau_0}\varphi(\tau).$$

For the proof, we first need the following lemma.

Lemma 5.4 *The term containing the 'eta' invariant is equal to*

$$-\frac{1}{2}\ \eta(A_c) = \frac{1}{2\pi i}\ \Delta_{\Gamma,0}\varphi(\tau).$$

Proof. Consider

$$Q(\tau) = \overline{\mathrm{Tr}}\ \frac{\partial}{\partial\tau}\left(A_c^{-1}(\tau)A_c'(\tau) - i\gamma\frac{\partial}{\partial\tau}A_c^{-1}(\tau)A_c'(\tau)\right).$$

By Theorem 2.2, this quantity is equal to

$$\frac{\partial^2}{\partial\tau^2}\left(\varphi(\tau) - i\gamma\frac{\partial}{\partial\tau}\varphi(\tau)\right).$$

According to the definition of $\overline{\mathrm{Tr}}$ (see Melrose [Mel95]) and $\eta(A_c)$, we obtain

$$-\frac{1}{2}\,\eta(A_c) = \lim_{T\to\infty}\int_{-T}^{T} d\tau_1 \int_0^{\tau_1} Q(\tau+i\gamma)d\tau,$$

the right-hand side being understood as a constant term in the asymptotic expansion when $T \to \infty$. Thus,

$$-\frac{1}{2}\,\eta(A_c) = \frac{1}{2\pi i}\lim_{T\to\infty}\left(\varphi(\tau) - i\gamma\frac{\partial}{\partial\tau}\varphi(\tau)\right)\Bigg|_{\tau=-T+i\gamma}^{\tau=T+i\gamma}$$

and the variation of $\varphi(\tau)$ is taken along the weight line Γ (for $(\partial/\partial\tau)\varphi(\tau)$, the variation does not depend on the path). In the region $|\Re\tau| > T_0$ where $\varphi(\tau)$ is holomorphic in the strip between Γ and the real axis, we may use the Taylor formula, thus obtaining

$$\varphi(\tau) - i\gamma\frac{\partial}{\partial\tau}\varphi(\tau) = \varphi(\tau - i\gamma) + R_2(\tau,\gamma)$$

where $R_2(\tau,\gamma)$ is a remainder term which tends to 0 for $\Re\tau \to \pm\infty$ and $|\Im\tau| \le C$. Hence it follows that

$$-\frac{1}{2}\,\eta(A_c) = \frac{1}{2\pi i}\lim_{T\to\infty}\left(\varphi(T) - \varphi(-T)\right),$$

however, the variation is now taken along the contour in Fig. 1 with $\tau_0 = 0$. This completes the proof.

□

Proof of Theorem 5.3. Consider the difference

$$\begin{aligned}&\Delta_{\Gamma,\tau_0}\varphi(\tau) - \Delta_{\Gamma,0}\varphi(\tau)\\ &\quad = \lim_{T\to\infty}\left((\varphi(T+\tau_0) - \varphi(T)) - (\varphi(-T+\tau_0) - \varphi(-T))\right)\end{aligned}$$

where the variations in parentheses are taken along the segments $[T, T+\tau_0]$ and $[-T, -T+\tau_0]$ (see Fig. 2).

To calculate the limit, we invoke the asymptotic formula (4.18). This yields the value

$$\tau_0\int_0^{2\pi}\left(\mathrm{tr}\,a_+(x) - \mathrm{tr}\,a_-(x)\right)dx$$

or

$$-\int_0^{2\pi}\left(\mathrm{tr}\,b_{11}(x) - \mathrm{tr}\,b_{22}(x)\right)dx,$$

the last equality being due to the definition of τ_0. Comparing this expression with (1.7) (mind the change of notation), we see that they cancel.

□

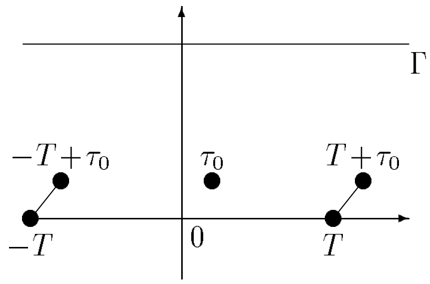

Fig. 2: Two auxiliary segments.

6 Higher-order operators

Let

$$A_c(\tau)u = \sum_{j+k\le m} A_{j,k}(x)\tau^j \left(-i\frac{\partial}{\partial x}\right)^k u$$

be the conormal symbol of an mth order elliptic operator. Without loss of generality we assume that $A_{m,0}(x) \equiv 1$.

When dealing with higher-order ordinary differential operators, one uses a standard trick known as the *reduction* to a system of first-order operators. Assuming $\tau \ne 0$, we introduce new unknown functions

$$\left(-i\frac{\partial}{\partial x}\right)^k u := \tau^k u_k, \quad k = 0, 1, \dots, m-1, \tag{6.1}$$

so that $u_0 \equiv u$. Then the operator $u \mapsto \tau^{-(m-1)} A_c(\tau)u$ may be rewritten in the form

$$\tau u_0 + \sum_{k=1}^{m} A_{m-k,k}(x)\left(-i\frac{\partial}{\partial x}\right) u_{k-1} + \sum_{j+k\le m-1} A_{j,k}(x)\tau^{j+k-(m-1)} u_k.$$

Together with the relations

$$i\,\frac{\partial u_{k-1}}{\partial x} + \tau u_k = 0$$

which are due to (6.1), the above expression defines an operator $\mathcal{A}(\tau)$ acting in the space of vector-valued functions

$$\vec{u} = \begin{pmatrix} u_0 \\ u_1 \\ \dots \\ u_{m-1} \end{pmatrix}$$

as a matrix

$$\begin{pmatrix} \tau + A_1\,\partial/\partial x + B_1 & A_2\,\partial/\partial x + B_2 & \dots & A_m\,\partial/\partial x + B_m \\ i\,\partial/\partial x & \tau & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tau \end{pmatrix}, \tag{6.2}$$

where

$$A_k(x) = -iA_{m-k,k}(x)$$

and

$$B_k(x,\tau) = \sum_{j=0}^{m-k} A_{j,k-1}(x)\tau^{j+k-m}. \tag{6.3}$$

Thus, we come to a first-order differential operator

$$\mathcal{A}(\tau) = \tau + A(x)\frac{\partial}{\partial x} + B(x,\tau) \tag{6.4}$$

where $B(x,\tau)$ is a polynomial in inverse powers of τ. The matrix $A(x)$ is completely determined by the principal symbol a_0 of the operator $A_c(\tau)$ and the constant term of $B(x,\tau)$ is determined by the next term a_1 in the complete symbol. The asymptotical analysis of the monodromy matrix of operator (6.4) remains valid, modulo $O(1/\tau)$, for $B(x,\tau)$ of the form (6.3). We may simply omit all the terms in (6.3) but the constants in τ.

Lemma 6.1 *Let the weight line Γ not contain the origin. Then the 'eta' invariants for $A_c(\tau)$ and for $\mathcal{A}(\tau)$ coincide.*

Remark 6.2 The assumption $0 \notin \Gamma$ involves no loss of generality because we may change the origin.

Proof. To avoid cumbersome matrix formulas, let us restrict our attention to the case $m = 2$. In this case matrix (6.2) has the form

$$\mathcal{A}(\tau) = \begin{pmatrix} \tau + A_1\,\partial/\partial x + B_1 & A_2\,\partial/\partial x + B_2 \\ i\,\partial/\partial x & \tau \end{pmatrix},$$

with

$$\begin{aligned} A_1(x) &= -iA_{1,1}(x); \\ A_2(x) &= -iA_{0,2}(x) \end{aligned}$$

and

$$\begin{aligned} B_1(x,\tau) &= A_{1,0}(x) + A_{0,0}(x)\tau^{-1}; \\ B_2(x,\tau) &= A_{0,1}(x). \end{aligned}$$

One immediately checks that

$$\tau\left(\tau + A_1\frac{\partial}{\partial x} + B_1\right) - \left(A_2\frac{\partial}{\partial x} + B_2\right)i\frac{\partial}{\partial x} = A_c(\tau),$$

and consequently

$$\mathcal{A}(\tau)\begin{pmatrix} \tau & 0 \\ -i\,\partial/\partial x & 1/\tau \end{pmatrix} = \begin{pmatrix} A_c(\tau) & * \\ 0 & 1 \end{pmatrix} \tag{6.5}$$

where the 'asterisk' means any expression whose explicit form is not essential. We rewrite this matrix identity in an abbreviated form

$$\mathcal{A}(\tau)C(\tau) = B(\tau)$$

where $B(\tau)$ means the right-hand side of (6.5) and $C(\tau)$ stands for the second factor on the left-hand side.

Clearly,

$$\overline{\mathrm{Tr}}\, B'(\tau)B^{-1}(\tau) = \overline{\mathrm{Tr}}\, A_c'(\tau)A_c^{-1}(\tau).$$

On the other hand,

$$\begin{aligned}\overline{\mathrm{Tr}}\, B'(\tau)B^{-1}(\tau) &= \overline{\mathrm{Tr}}\, \left(\mathcal{A}'(\tau)C(\tau) + \mathcal{A}(\tau)C'(\tau)\right) C^{-1}(\tau)\mathcal{A}^{-1}(\tau)\\ &= \overline{\mathrm{Tr}}\, \mathcal{A}'(\tau)\mathcal{A}^{-1}(\tau) + \overline{\mathrm{Tr}}\, \mathcal{A}(\tau)C'(\tau)C^{-1}(\tau)\mathcal{A}^{-1}(\tau).\end{aligned}$$

Since $\overline{\mathrm{Tr}}$ is a trace functional, we conclude that

$$\begin{aligned}\overline{\mathrm{Tr}}\, \mathcal{A}(\tau)C'(\tau)C^{-1}(\tau)\mathcal{A}^{-1}(\tau) &= \overline{\mathrm{Tr}}\, C'(\tau)C^{-1}(\tau)\\ &= 0\end{aligned}$$

which implies

$$\overline{\mathrm{Tr}}\, \mathcal{A}'(\tau)\mathcal{A}^{-1}(\tau) = \overline{\mathrm{Tr}}\, A_c'(\tau)A_c^{-1}(\tau).$$

The same is true for $\widetilde{\mathrm{Tr}} = \overline{\mathrm{Tr}}\, \partial/\partial\tau$, for $\widetilde{\mathrm{Tr}}$ is also a trace functional. Thus,

$$\begin{aligned}&\overline{\mathrm{Tr}}\, \left(\mathcal{A}'(\tau)\mathcal{A}^{-1}(\tau) - i\gamma\frac{\partial}{\partial\tau}\mathcal{A}'(\tau)\mathcal{A}^{-1}(\tau)\right)\\ &\quad = \overline{\mathrm{Tr}}\, \left(A_c'(\tau)A_c^{-1}(\tau) - i\gamma\frac{\partial}{\partial\tau}A_c'(\tau)A_c^{-1}(\tau)\right)\end{aligned}$$

which means the coincidence of the 'eta' invariants.

□

It follows that, for a higher-order operator, the 'eta' invariant term is $\Delta_{\Gamma,0}\varphi(\tau)$, where $\varphi(\tau)$ is a phase function for the operator (6.2).

Unfortunately, we do not know any interpretation of additional integral terms in (0.1) for higher-order operators in terms of the monodromy matrix.

7 Particular cases

We have seen that, for the function $f(\tau) = \det\left(M(\tau) + M^{-1}(\tau) - 2\right)$, the asymptotic equality holds

$$f(T+\tau_0) = f(-T+\tau_0)\left(1 + O\left(\frac{1}{T}\right)\right) \tag{7.1}$$

as $T \to +\infty$. Here $\tau_0 \in \mathbb{C}$ is a point which we call the 'centre'. An interesting particular case is when equality (7.1) is precise, i.e., $f(\tau)$ is an even function on the line $\Gamma_0 = \{\Im\tau = \Im\tau_0\}$ with respect to the centre τ_0. If such is the case, Theorem 5.3 admits further simplification.

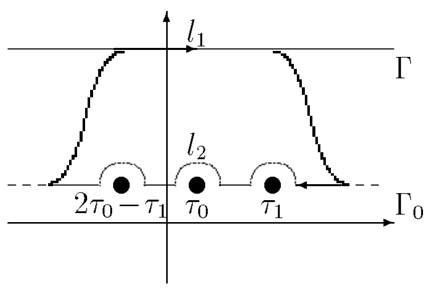

Fig. 3: The contour $l = l_1 \cup l_2$.

Theorem 7.1 *Suppose $f(\tau)$ is an even function with respect to the centre τ_0, that is*

$$f(T + \tau_0) = f(-T + \tau_0) \tag{7.2}$$

for each real T. Then,

$$\frac{1}{2\pi i}\,\Delta_{\Gamma,\tau_0}\varphi(\tau) = \left(p + \frac{1}{2}q\right)\operatorname{sgn}\left(\Im\tau_0 - \gamma\right)$$

where p is the number of ramification points of $\varphi(\tau)$ (counted along with their multiplicities) in the strip between the lines Γ and Γ_0 and q is the number of ramification points on the line Γ_0.

Proof. To be specific, let $\Im\tau_0 < \gamma$. Consider a closed contour $l = l_1 \cup l_2$ where l_1 is the contour described in Section 5 to define $\Delta_{\Gamma,\tau_0}\varphi(\tau)$, and l_2 goes along the line Γ_0 by passing the ramification points lying on Γ_0 along small semicircles (see Fig. 3). Clearly,

$$\begin{aligned}\frac{1}{2\pi i}\,\Delta_{\Gamma,\tau_0}\varphi(\tau) &= \frac{1}{2\pi i}\,\Delta_{l_1}\varphi(\tau)\\ &= -p - \frac{1}{2\pi i}\,\Delta_{l_2}\varphi(\tau).\end{aligned}$$

We next observe that the variation of $\varphi(\tau)$ along l_2 is equal to the sum of variations along all the semicircles. Indeed, the variations along the segments of Γ_0 cancel because of (7.2). When the radii of the semicircles tend to 0, the variations along them tend to πi times the number q of ramification points on Γ_0 counted together with their multiplicities. This is the desired conclusion.

□

Since the result is very simple, it is desirable to have simple sufficient conditions for (7.2) to be fulfilled. One of these is the symmetry condition of [SSS97] for the conormal symbol: there exist isomorphisms $\sigma_0(x)$ and $\sigma_1(x)$ of the bundles E^0 and E^1, such that

$$A_c(-T + \tau_0) = \sigma_1(x)\,A_c(T + \tau_0)\,\sigma_0(x)$$

for each real T. Indeed, in this case we have

$$M(-T+\tau_0) = \sigma_0^{-1}(0)\, M(T+\tau_0)\, \sigma_0(0) \tag{7.3}$$

which means that the monodromy matrix is an even function up to conjugation. It follows that (7.2) is fulfilled.

There are some other cases which cannot be reduced to (7.3). For example, if

$$M(-T+\tau_0) = \sigma_0^{-1}(0)\, M^{-1}(T+\tau_0)\, \sigma_0(0) \tag{7.4}$$

instead of (7.3), then (7.2) also holds.

We finish the paper with a couple of examples where condition (7.4) is satisfied.

Example 7.2 (scalar case) In this case $M(\tau)$ is a scalar function of the form $\exp(a\tau + b)$ and τ_0 is the root of the equation $a\tau + b = 0$. Thus,

$$\begin{aligned} M(-T+\tau_0) &= \exp(-aT) \\ &= (\exp(aT))^{-1} \\ &= (\exp(a(T+\tau_0)+b))^{-1} \\ &= M^{-1}(T+\tau_0), \end{aligned}$$

as desired.

□

Example 7.3 (constant coefficients) Let

$$A_c(\tau) = \tau - iA\frac{\partial}{\partial x}$$

with a constant matrix A. Then

$$M(\tau) = \exp\left(-2\pi i\, A^{-1}\tau\right)$$

and we deduce that (7.4) holds with $\tau_0 = 0$ and $\sigma_0 = 1$.

□

In particular, geometric operators of the Dirac type have constant coefficients under appropriate coordinates and bundle trivialisations (cf. [APS75, Mel93]).

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