Non-Abelian Reduction in Deformation Quantization

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1997

Abstract

We consider a G-invariant star-product algebra $\widehat{A}$ on a symplectic manifold (M, ω) obtained by a canonical construction of deformation quantization. Under assumptions of the classical Marsden-Weinstein theorem we define a reduction of the algebra $\widehat{A}$ with respect to the G-action. The reduced algebra turns out to be isomorphic to a canonical star-product algebra on the reduced phase space B. In other words, we show that the reduction commutes with the canonical G-invariant deformation quantization. A similar statement in the framework of geometric quantization is known as the Guillemin-Sternberg conjecture (by now completely proved).

AMS subject classification: primary 58F05; secondary: 81S10, 70H33.
Key words: deformation quantization, Hamiltonian group action, moment map, classical and quantum reduction.

1 Introduction

We consider a symplectic manifold (M, ω) with a Hamiltonian action of a compact Lie group G. Let $\mathcal{G}$, G^* denote the Lie algebra of G and its dual space. Supposing that the classical moment map $\mu : M \to \mathcal{G}^*$ exists globally on M, we denote by $e_1, e_2, \dots, e_n$ a basis of $\mathcal{G}$, $X_{e_1}, X_{e_2}, \dots, X_{e_n}$ the corresponding vector fields on M and by $\mu_i = \langle \mu, e_i \rangle$ their Hamiltonians. Clearly,

$$\{\mu_i, \mu_j\} = X_{e_i}\mu_j = c_{ij}^k \mu_k \tag{1.1}$$

where c_{ij}^k are structure constants of $\mathcal{G}$ and $\{\ ,\ \}$ the Poisson bracket on M.

Next we suppose that the assumptions of the classical Marsden-Weinstein reduction theorem [1] are fulfilled:

1. $\mu = 0$ is a non-critical value of the moment map,
2. the action of G on the level manifold $M_0 = \{\mu = 0\}$ is free.

Item 1 means that the differentials $d\mu_1, d\mu_2, \dots, d\mu_n$ are lineary independent on M_0, so by implicit function theorem M_0 is a smooth manifold. From (1.1) it follows that M_0 is preserved under G-action. Item 2 implies that M_0 is a principal G-bundle over a base $B = M_0/G$ which is the orbit space of G-action on M_0.

The algebraical version of the classical reduction procedure is as follows. We consider the algebra of classical observables $A = C^\infty(M)$ equipped with a commutative pointwise product of functions and the Poisson bracket defined by the symplectic form ω. Define a subalgebra of G-invariant functions

$$A_0 = \{a \in A : \{\mu_i, a\} = 0\} \subset A \tag{1.2}$$

and an ideal $J \subset A_0$ consisting of functions $a \in A_0$ which may be represented in the form $a = b^i \mu_i$, so

$$J = \{a \in A_0 : \quad a = b^i \mu_i,\ b^i \in A\}. \tag{1.3}$$

The reduced algebra is a quotient

$$R = A_0/J. \tag{1.4}$$

Like the original algebra A, R has a structure of the Lie-Poisson algebra. It means that there is a commutative product on R, as well as the Lie algebra structure $\{\cdot,\cdot\}$ compatible with the product:

$$\{ab, c\} = \{a, c\}b + a\{b, c\}, \tag{1.5}$$

inherited from A_0.

Theorem 1 (Marsden-Weinstein) *There exists a symplectic form ω_B on the reduced phase space B such that its lifting to M_0 coinsides with the restriction of ω to M_0. The reduced algebra R is isomorphic to the Lie-Poisson algebra $C^\infty(B)$ with respect to the form ω_B.*

A typical example of a classical reduction may be constructed starting with any principal G-bundle P over a symplectic manifold (B, ω_B). Let λ be a connection one-form on P. It means that λ is $\mathcal{G}$-valued one-form on P with the properties:

$$i(X_{e_i})\lambda = e_i, \quad g^*\lambda = \mathrm{Ad}_{g^{-1}}\lambda. \tag{1.6}$$

Here X_{e_i} denotes a vector field on P corresponding to $e_i \in \mathcal{G}$.

Let $M = P \times \mathcal{G}^*$ with the action of G

$$g(p, \xi) = (pg, \mathrm{Ad}_g^*\xi)$$

where pg denotes the right action of G on the principal bundle. This action preserves the form

$$\omega = \omega_B - d\langle \xi, \lambda \rangle \tag{1.7}$$

which is non-degenerate for ξ small enough. The moment map is

$$\mu(p, \xi) = \xi.$$

In fact this example describes the general case: in the assumptions of the Marsden-Weinstein theorem there exists an equivariant diffeomorphism f of a G-invariant neighborhood of M_0 in M to a G-invariant neighborhood of $P \times 0$ in $P \times \mathcal{G}$ such that

$$\omega = f^*(\omega_B - d\langle \xi, \lambda \rangle)$$

(the so-called *normal form theorem* [4]).

Passing to a quantum reduction, we first recall briefly a canonical construction of deformation quantization (Section 2). More details may be found in [2] or in the book [3]. The main technical lemmas concerning canonical G-invariant deformation quantization are proved in Section 3. They allow to define a quantum moment map: we show in Section 3 that the following theorem holds.

Theorem 2 *Under assumptions of the classical Marsden-Weinstein theorem the quantum Hamiltonians $\widehat{\mu}_1, \widehat{\mu}_2, \ldots, \widehat{\mu}_n$ obtained by the canonical G-invariant quantization of the classical Hamiltonian functions $\mu_1, \mu_2, \ldots, \mu_n$ satisfy the relations*

$$\frac{i}{h}[\widehat{\mu}_i, \widehat{\mu}_j] = c_{ij}^k \widehat{\mu}_k. \tag{1.8}$$

Note that (1.8) is a quantum version of (1.1). The existence of the quantum moment map was proved in the thesis of A. Astashkevich [5] under somewhat different assumptions.

Now we define a reduction for the algebra $\widehat{A}$ of quantum observables similarly to the classical case. We consider an invariant subalgebra

$$\widehat{A}_0 = \{\widehat{a} \in \widehat{A} : [\widehat{\mu}_i, \widehat{a}] = 0\} \tag{1.9}$$

and a left ideal $\widehat{J} \subset \widehat{A}_0$ generated by $\widehat{\mu}_i$, that is

$$\widehat{J} = \{\widehat{a} \in \widehat{A}_0 : \widehat{a} = \widehat{b}^i \circ \widehat{\mu}_i,\ \widehat{b}^i \in \widehat{A}\}$$

where $\circ$ denotes a multiplication in $\widehat{A}$. The reduced algebra is by definition a quotient

$$\widehat{R} = \widehat{A}_0 / \widehat{J}. \tag{1.10}$$

The aim of this article is to prove the following reduction theorem for canonical deformation quantization.

Theorem 3 *Under assumptions of the classical Marsden-Weinstein theorem the reduced algebra $\widehat{R}$ is isomorphic to the algebra obtained by the canonical deformation quantization of the classical reduced algebra $R = C^\infty(B)$.*

In other words, the following diagram is commutative up to isomorphisms

$$\begin{array}{ccc} A \cong C^\infty(M) & \xrightarrow{Q} & \widehat{A} \\ \downarrow & & \downarrow \\ R \cong C^\infty(B) & \xrightarrow{Q} & \widehat{R}. \end{array}$$

Here horizontal arrows mean canonical deformation quantization (G-invariant for the upper arrow), while vertical arrows mean reduction procedure for the classical (left) and quantum (right) cases. We prove this theorem first for a particular case of the cotangent bundle of a Lie group (Section 4). The general case is considered in Section 5 using a modification of the canonical quantization procedure for non-trivial coefficients bundles.

Note, that the reduction procedure in geometric quantization was intensively studied in recent years. The final proof of the Guillemin-Sternberg conjecture [6] may be found in [7, 8]. In contrast, the literature concerned with the reduction in deformation quantization is very poor. Besides the pioneer work [9] where a number of interesting physical examples was considered using the $*$-product approach we mention the book [3, Chapter 8] where a weaker result was obtained only for the Abelian case.

2 Canonical Deformation Quantization

The canonical construction which we are going to describe briefly deals with sections of the so-called formal *Weyl algebras bundle* W over a symplectic manifold M. Such a section is a "function"

$$a = a(x, y, h) = \sum_{k,|\alpha|=0}^{\infty} h^k a_{k,\alpha}(x) y^\alpha \tag{2.1}$$

where $x \in M,\ y \in T_xM,\quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})$ is a multy-index, $y^\alpha = (y^1)^{\alpha_1}(y^2)^{\alpha_2}\ldots(y^{2n})^{\alpha_{2n}}$, where $y^1, \ldots, y^{2n}$ are coordinates of the vector y in a local frame of the tangent bundle TM. The series in (2.1) is formal, we assign to its terms a total degree $2k + |\alpha|$ and order the terms with respect to this degree. The space of all sections denoted by $C^\infty(M, W)$ form an associative algebra with respect to a fibrewise *Weyl (Moyal) product*

$$a \circ b = \exp\left(-\frac{ih}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) a(x, y, h)\, b(x, z, h)|_{z=y}\,. \tag{2.2}$$

We also consider W-valued differential forms, that is the sections $C^\infty(W \otimes \Lambda)$, where Λ is the exterior algebra of T^*M. Any diffeomorphism f of M acts on sections $a \in C^\infty(M, W \otimes \Lambda)$ by pulling them back:

$$(f^*a)(x, y, dx, h) = a\left(f(x), \frac{\partial f}{\partial x}y, df(x), h\right).$$

If f is a symplectomorphism, f^* preserves the product (1.2), thus f^* is an automorphism of the algebra $C^\infty(M, W \otimes \Lambda)$. Thus, a Hamiltonian vector field X defines a derivation of the algebra of sections

$$\mathcal{L}_X a = \frac{d}{dt} f_t^* a \Big|_{t=0}$$

called the *Lie derivative*. Here f_t is the Hamiltonian flow generated by X.

We define a derivation δ of the algebra $C^\infty(M, W \otimes \Lambda)$ by

$$\delta a = -\frac{i}{h}[\omega_{ij} y^i dx^j, a] = dx^i \wedge \frac{\partial a}{\partial y^i},$$

and the "adjoint" δ^*

$$\delta^* a = y^k i\left(\frac{\partial}{\partial x^k}\right) a.$$

Any symplectic connection on M induces a connection ∂ on the bundle W acting on sections (2.1) by covariant derivation of the coefficients $a_{k,\alpha}(x)$ as tensor fields on M. In local Darboux coordinates ∂ may be written in the form

$$\partial a = d_x a + \frac{i}{2h}[\Gamma_{ijk} y^i y^j dx^k, a]$$

where Γ_{ijk} are connection coefficients of the symplectic connection and d_x means the exterior differential with respect to x.

We will consider more general connections on W :

$$Da = \partial a + \frac{i}{h}[\gamma, a] = \partial a - \delta a + \frac{i}{h}[r, a]$$

where γ and r are global sections of the bundle $W \otimes \Lambda^1$ with $\deg r \geq 3$. These forms are defined up to a central summand. To fix it we impose a normalizing condition

$$\gamma|_{y=0} = r|_{y=0}\,. \tag{2.3}$$

The 2-form

$$\Omega = \frac{1}{2} R_{ij} y^i y^j + \partial\gamma + \frac{i}{h}\gamma \circ \gamma \in C^\infty(M, W \otimes \Lambda^2)$$

is called the curvature of D. Here $R_{ij} = 1/2 R_{ijkl} dx^k \wedge dx^l$ is the curvature of the symplectic connection.

The canonical deformation quantization is based on the following facts [2, 3].

Proposition 4 *There exists a unique form* $r \in C^\infty(M, W \otimes \Lambda^1)$ *with* $\deg r \geq 3$ *satisfying two conditions:*

$$\delta^* r = 0 \tag{2.4}$$

$$\Omega = -\omega. \tag{2.5}$$

The first condition implies (2.3). The second one means that the curvature is a central form, so

$$D^2 a = \frac{i}{h}[\Omega, a] \equiv 0$$

for any section a. The connection D with this property is called *Abelian.* Having a connection D with the properties (2.4), (2.5), we define an algebra

$$W_D = \{a \in C^\infty(M, W) : Da \equiv 0\}$$

of flat sections.

Proposition 5 *There is one-to-one correspondence between flat sections and functions from* $C^\infty(M)[[h]]$ *given by*

$$W_D \in a \to \sigma(a) = a|_{y=0} \in C^\infty(M)[[h]].$$

The inverse map

$$Q : C^\infty(M)[[h]] \to W_D$$

is called *canonical quantization.* It depends on the choice of the symplectic connection ∂, but the corresponding algebras W_D turn out to be isomorphic. Moreover, this isomorphism may be taken in a particular form which we are going to describe. First we introduce *gauge transformations* of the algebra of all sections $C^\infty(M, W)$. To this end consider a "section"

$$U = 1 + \sum_{2k+|\alpha|>0} h^k u_{k,\alpha}(x) y^\alpha. \tag{2.6}$$

In contrast to (2.1) here k may be any integer number, positive or negative. The only restriction is that the total degree $2k + |\alpha|$ should be positive. Such formal series also form an algebra with respect to the fibrewise product

(2.2). So, U is a section of an extended formal Weyl algebras bundle which we denote W^+. Clearly, U is invertible since its leading term is 1.

For a given U of the type (2.6) define an inner isomorphism T of the algebra $C^\infty(M, W^+)$ by

$$Ta = U \circ a \circ U^{-1}. \tag{2.7}$$

We call such an isomorphism a gauge transformation.

Proposition 6 *Let ∂_1, ∂_2 be two symplectic connections and $W_{D_1}, W_{D_2} \subset C^\infty(M, W) \subset C^\infty(M, W^+)$ corresponding algebras of flat sections. Then there exists a section $U \subset C^\infty(M, W^+)$ of the type (2.6) such that the gauge transformation T maps the algebra W_{D_1} onto W_{D_2}.*

Writing

$$D_2 = D_1 + \frac{i}{h}[\Delta, \cdot]$$

with a global section $\Delta\gamma \in C^\infty(M, W \otimes \Lambda^1)$ satisfying

$$\deg\Delta\gamma \geq 3, \quad \Delta\gamma|_{y=0} = 0, \tag{2.8}$$

we may find U as a solution of the equation

$$D_1 U \circ U^{-1} + \frac{i}{h}\Delta\gamma = 0 \tag{2.9}$$

having a unique solution under the normalizing condition

$$U|_{y=0} = 1. \tag{2.10}$$

Note that for two sections

$$\widehat{a}_1 = Q_1(a) \in W_{D_1}, \quad \widehat{a}_2 = Q_2(a) \in W_{D_2}$$

obtained by two different quantizations of the same function $a \in C^\infty(M)[[h]]$, the equality

$$\widehat{a}_2 = U \circ \widehat{a}_1 \circ U^{-1} \tag{2.11}$$

does not hold in general.

3 G-invariant Canonical Quantization

If we have a Hamiltonian action of a Lie group G, we will consider *G-invariant Abelian connections* D, so that

$$D(g^* a) = g^*(Da)$$

for any section $a \in C^\infty(M, W \otimes \Lambda)$ or

$$D\mathcal{L}_{X_e} a = \mathcal{L}_{X_e}(Da)$$

for any vector field X_e defined by an element e of the Lie algebra.

The following facts are easy consequences of the propositions of the previous section.

1. If ∂ is a G-invariant symplectic connection, then the corresponding Abelian connection D is also G-invariant.
2. For two G-invariant Abelian connections D_1, D_2 the form $\Delta\gamma$ is G-invariant, that is
$$g^*\Delta\gamma = \Delta\gamma, \quad \mathcal{L}_X \Delta\gamma = 0. \tag{3.1}$$
3. The solution U of (2.9), (2.10) defining an isomorphism T by (2.7) between W_{D_1} and W_{D_2} is G-invariant.

Lemma 7 *Let H_e be a Hamiltonian function of the vector field $X_e, e \in \mathcal{G}$. If $dH_e \neq 0$, then for a G-invariant Abelian connection D and for any section $a \in C^\infty(M, W \otimes \Lambda)$*

$$\mathcal{L}_{X_e} a = (i(X_e)D + Di(X_e))a + \frac{i}{h}[Q(H_e), a]. \tag{3.2}$$

Proof. The statement is local, so we may choose the Darboux local coordinates such that

$$X_e = \frac{\partial}{\partial x^1}, \quad H_e = \omega_{1i} x^i$$

(see, e.g. [3, Theorem 2.3.4]). For a special choice

$$D = d_x + \frac{i}{h}[\omega_{ij} y^i dx^j, \cdot\,]$$

we have

$$Q(H_e) = \omega_{1i}(x^i + y^i).$$

The corresponding one-parameter group g_t acts on sections as

$$g_t^* a = a(x^1 + t, x^2, \ldots, x^n, y, dx, h),$$

so that Equality (3.2) becomes evident.

For another choice of the Abelian connection $D_1 = D + \frac{i}{h}[\Delta\gamma, \cdot]$ with g_t-invariant $\Delta\gamma$ we would have

$$\mathcal{L}_{X_e} a = (i(X_e)D_1 + D_1 i(X_e))a + \frac{i}{h}[Q(H) - i(X_e)\Delta\gamma]$$

because the addition of $[\Delta\gamma, \,\cdot\,]/h$ to D and $-i(X_e)\Delta\gamma$ to $Q(H)$ simultaneously doesn't change the right-hand side. It remains to show that the section $Q(H) - i(X)\Delta\gamma$ is equal to $Q_1(H)$. Because of (2.8)

$$(Q(H) - i(X_e)\Delta\gamma)\big|_{y=0} = Q(H)\big|_{y=0} = \omega_{1i}x^i,$$

so it is sufficient to show that $Q(H) - i(X_e)\Delta\gamma$ is flat with respect to D_1, that is

$$-Di(X_e)\Delta\gamma + \frac{i}{h}[\Delta\gamma, Q(H) - i(X_e)\Delta\gamma] = 0 \qquad (3.3)$$

since $DQ(H) = 0$.

Applying (3.2) which is proved for D to $\Delta\gamma$ which is G-invariant, we obtain

$$Di(X_e)\Delta\gamma + \frac{i}{h}[Q(H), \Delta\gamma] = -i(X_e)D\Delta\gamma,$$

so the left-hand side of (3.3) becomes

$$i(X_e)D\Delta\gamma + \frac{i}{h}[i(X_e)\Delta\gamma, \Delta\gamma] = i(X_e)(D\Delta\gamma + \frac{i}{h}\Delta\gamma^2).$$

But

$$D\Delta\gamma + \frac{i}{h}\Delta\gamma^2 \equiv 0$$

since both D and D_1 have the same curvature $-\omega$, whence (3.3) follows. $\square$

Lemma 8 *Let X_e, H be the same as in the previous lemma. Let D_1, D_2 be two G-invariant Abelian connections, U the solution of* (2.9), (2.10). *Then*

$$Q_2(H) = U \circ Q_1(H) \circ U^{-1}. \qquad (3.4)$$

Proof. Since U is a G-invariant section, we have by the previous lemma using (2.9)

$$\begin{aligned} \mathcal{L}_{X_e} U &= i(X_e) D_1 U + \frac{i}{h}[Q_1(H), U] \\ &= -\frac{i}{h} i(X_e)\Delta\gamma \circ U + \frac{i}{h} Q_1(H) \circ U - \frac{i}{h} U \circ Q_1(H) = 0, \end{aligned}$$

yielding (3.4). □

Lemma 9 *G-invariant canonical quantization commutes with the group action, that is*

$$Q(g^* a) = g^* Q(a) \qquad (3.5)$$

for any $\in C^\infty(M)[[h]]$.

Proof. Clearly,

$$(g^* Q(a))|_{y=0} = (g^* Q(a)|_{y=0}) = g^* a.$$

Next,

$$D(g^* Q(a)) = g^* D(Q(a)) \equiv 0$$

since D is G-invariant and $Q(a)$ is flat. So, (3.5) follows from the uniqueness of the quantization procedure (Proposition 5). □

The infinitesimal version of (3.5) is

$$Q(\mathcal{L}_X a) = \mathcal{L}_X Q(a). \qquad (3.6)$$

We apply the properties of the G-invariant canonical quantization to obtain the quantum moment map. So far the group G may by any finite-dimensional Lie group (not necessarily compact or semisimple). All we need is the existence of the classical moment map μ which has no critical values in some neighborhood $V \subset \mathcal{G}^*$ of 0. Restricting μ to $\mu^{-1}(V)$, we may assume that μ has no critical points at all.

Proof of Theorem 2. Applying Q to (1.1), we obtain

$$Q(X_i \mu_j) = c_{ij}^k Q(\mu_k).$$

By Lemma 9

$$Q(X_i \mu_j) = Q(\mathcal{L}_{X_i} \mu_j) = \mathcal{L}_{X_i} Q(\mu_j),$$

yielding

$$\mathcal{L}_{X_i} \widehat{\mu}_j = c_{ij}^k \widehat{\mu}_k.$$

Now, by Lemma 7

$$\mathcal{L}_{X_i} \widehat{\mu}_j = i(X_i) D\widehat{\mu}_j + \frac{i}{h}[Q(\mu_i), \widehat{\mu}_j] = \frac{i}{h}[\widehat{\mu}_i, \widehat{\mu}_j]$$

since $D\widehat{\mu}_j \equiv 0$ implying (1.8). □

Having the quantum Hamiltonians, we define the quantum reduction of the algebra $\widehat{A} = W_D(M)$ as was described in the introduction.

4 The Case of the Lie Group

From now on the group G is supposed to be compact. We consider its right and left action on $F = T^*G \cong G \times \mathcal{G}^*$ and the canonical deformation quantization on F wich is invariant with respect to this $G \times G$ action. An invariant star-product on T^*G was first constructed in [10]. Here we describe this construction in our terms and then apply it to prove the reduction theorem for this particular case.

The group G may be considered as a principal G-bundle (whose base is a point), so the construction of the Hamiltonian action of G on $G \times \mathcal{G}^*$ described in the introduction may be applied. The connection form λ in this case is the Maurer-Cartan left-invariant form λ_L. We also need the Maurer-Cartan right-invariant form $\lambda_R = \mathrm{Ad}_g \lambda_L$ (for matrix Lie groups $\lambda_L = g^{-1}dg, \ \lambda_R = dg\, g^{-1}$). Introduce a symplectic form

$$\omega_F = -d\langle \xi, \lambda_L \rangle = -d\langle \eta, \lambda_R \rangle \tag{4.1}$$

where $\xi \in \mathcal{G}^*, \ \eta = \mathrm{Ad}^*_{g^{-1}} \xi$. This form is preserved by left and right actions defined by

$$L_u(g, \xi) = (ug, \xi), \quad R_u(g, \xi) = (gu, \mathrm{Ad}^*_u \xi) \tag{4.2}$$

with moment maps

$$\mu_L(g,\xi) = \eta, \quad \mu_R(g,\xi) = \xi.$$

The functions ξ and η satisfy the following relations

$$\{\xi_i, \xi_j\} = c_{ij}^k \xi_k, \quad \{\eta_i, \eta_j\} = -c_{ij}^k \eta_k, \quad \{\xi_i, \eta_j\} = 0. \tag{4.3}$$

For deformation quantization we consider the tangent bundle

$$T(G \times \mathcal{G}^*) \cong G \times \mathcal{G}^* \times \mathcal{G} \times \mathcal{G}^*$$

with the left and right actions of G defined by (4.2)

$$L_u(g,,\xi,t,\tau) = (ug,\xi,t,\tau), \;\; R_u(g,\xi,t,\tau) = (gu, \mathrm{Ad}_u^*\xi, \mathrm{Ad}_{u^{-1}}t, \mathrm{Ad}_u^*\tau), \tag{4.4}$$

The sections of the Weyl algebras bundle are functions $a = a(g,\xi,t,\tau,h)$ on $G \times \mathcal{G}^* \times \mathcal{G} \times \mathcal{G}^*$ considered as formal series with respect to t, τ, h. The fibrewise product $\circ$ is the Moyal product (2.2) corresponding to the standard Poisson bracket on $\mathcal{G} \times \mathcal{G}^*$

$$\{a,b\} = \frac{\partial a}{\partial \tau_i}\frac{\partial b}{\partial t^i} - \frac{\partial a}{\partial t^i}\frac{\partial b}{\partial \tau_i}.$$

By Theorem 2 any $G \times G$-invariant quantization gives flat sections

$$\widehat{\xi}_i = Q(\xi_i); \quad \widehat{\eta}_i = Q(\eta_i)$$

defining quantum Hamiltonians for $G \times G$ action. Thus, they should satisfy the following commutation relations similar to (4.3):

$$[\widehat{\xi}_i, \widehat{\xi}_j] = -ihc_{ij}^k \widehat{\xi}_k, \quad [\widehat{\eta}_i, \widehat{\eta}_j] = ihc_{ij}^k \widehat{\eta}_k, \quad [\widehat{\xi}_i, \widehat{\eta}_j] = 0. \tag{4.5}$$

Moreover,

$$\frac{i}{h}[\widehat{\xi}_i, \widehat{a}] = \mathcal{L}_{R_i}\widehat{a}, \quad \frac{i}{h}[\widehat{\eta}_i, \widehat{a}] = \mathcal{L}_{L_i}\widehat{a} \tag{4.6}$$

where $\mathcal{L}_{L_i}$ and $\mathcal{L}_{R_i}$ are Lie derivatives corresponging to $e_i \in \mathcal{G}$ via left and right actions on $G \times \mathcal{G}^* \times \mathcal{G} \times \mathcal{G}^*$.

Theorem 10 *There exists a unique $G \times G$-invariant canonical deformation quantization Q satisfying the following two conditions:*

1. for any $a(g) \in C^\infty(G)$

$$\widehat{a} = Q(a) = a(ge^t), \qquad (4.7)$$

2. $\widehat{\xi}_i$ *and* $\widehat{\eta}_i$ *are linear forms in* $\xi + \tau$.

Proof. First of all observe that the section $\widehat{\xi}_i$ of the form $\langle \xi + \tau, f_i(g,t)\rangle$ where $f_i(g,t)$ is a function with values in $\mathcal{G}$ is uniquely defined by its adjoint action on $a(ge^t) = Q(a(g))$

$$\frac{i}{h}[\widehat{\xi}_i, a(ge^t)] = \langle \frac{\partial a}{\partial t}, f_i\rangle.$$

Because of (4.6) the latter expression should be equal to

$$\mathcal{L}_{R_i} a(ge^t) = \frac{d}{dz} a(ge^t e^{ze_i})|_{z=0}.$$

The Campbell-Hausdorff formula for $e^t e^{ze_i}$ implies immediately

$$f_i(g,t) = \frac{\mathrm{ad}_t}{1 - \exp(-\mathrm{ad}_t)} e_i,$$

so we have the only possibility for $\widehat{\xi}_i$ compatible with the assumptions of the theorem. Similarly,

$$\widehat{\eta}_i = \langle \xi + \tau, \frac{\mathrm{ad}_t}{\exp(\mathrm{ad}_t) - 1} \mathrm{Ad}_{g^{-1}} e_i\rangle.$$

It is easy to check that these $\widehat{\xi}_i$ and $\widehat{\eta}_i$ satisfy commutation relations (4.5). Indeed, the commutator of two linear forms in $\xi + \tau$ is again a linear form in $\xi + \tau$. Thus, to prove (4.5), we need to check that commutators of both sides of (4.5) with any section of the form $a(ge^t)$ coincide. But this is the case because of (4.6).

Now we present an Abelian connection D_F on the Weyl algebras bundle so that $a(ge^t)$ and $\widehat{\xi}_i$, $\widehat{\eta}_i$ constructed above are flat with respect to D_F. To this end consider the form

$$\gamma = \langle d\xi, t\rangle - \langle \widehat{\eta}, \lambda_R\rangle = d\xi_i t^i - \widehat{\eta}_i \lambda_R^i$$

on $G \times \mathcal{G}^*$ with values in W. Denoting by

$$da = d\xi \frac{\partial a}{\partial \xi_i} + \lambda_R^i \mathcal{L}_{R_i} a$$

the differential of the section $a(g, \xi, t, \tau)$ with respect to g, ξ and using (4.5), (4.6) along with the relation

$$d\lambda_R = \frac{1}{2}[\lambda_R, \lambda_R]$$

for the Maurer-Cartan form, one immediately checks that

$$d\gamma + \frac{i}{h}\gamma \circ \gamma \equiv 0,$$

so that

$$D_F a = da + \frac{i}{h}[\gamma, a]$$

is an Abelian connection. To compute its curvature, we have to normalize γ replacing it by

$$\gamma_n = \gamma - \gamma|_{t=0,\tau=0} = \gamma + \langle \eta, \lambda_R \rangle,$$

resulting in

$$\Omega = d\gamma_n + \frac{i}{h}\gamma_n \circ \gamma_n = d\langle \eta, \lambda_R \rangle = d\langle \xi, \lambda_L \rangle.$$

This completes the proof. □

We will consider the reduction procedure of the algebra $W_{D_F}(F)$ with respect to the *right* action of G. The following proposition is crucial for the quantum reduction.

Proposition 11 *For the constructed quantization $\widehat{\eta}_i$ belong to the ideal generated by $\widehat{\xi}_i$, more precisely,*

$$\widehat{\eta}_i = (\widehat{\mathrm{Ad}_{g^{-1}}})_i^j \circ \widehat{\xi}_j. \tag{4.8}$$

Proof. Denoting by $a_i^j(g)$ the entries of the matrix $\mathrm{Ad}_{g^{-1}}$, we have by Theorem 10

$$(\widehat{\mathrm{Ad}_{g^{-1}}})_i^j = \widehat{a}_i^j = a_i^j(ge^t).$$

Thus,

$$\widehat{a}_i^j \circ \widehat{\xi}_j = \widehat{a}_i^j \widehat{\xi}_j + \frac{ih}{2} \frac{\partial \widehat{a}_i^j}{\partial t^k} \frac{\partial \widehat{\xi}_j}{\partial \tau_k} = \widehat{a}_i^j \widehat{\xi}_j + \frac{1}{2} [\widehat{a}_i^j, \widehat{\xi}_j]$$

since $\widehat{\xi}$ is linear in τ. Show, that the commutator in the last expression vanishes. Indeed,

$$[\widehat{\xi}_j, \widehat{a}_i^j] = \mathcal{L}_{R_j} \widehat{a}_i^j = (\mathrm{ad}_{e_j})_k^j \widehat{a}_i^k$$

and

$$(\mathrm{ad}_{e_j})_k^j = c_{jk}^j = -\mathrm{tr}\, \mathrm{ad}_{e_k} = 0$$

because of the unimodularity of a compact Lie group.

Now,

$$\widehat{a}_i^j \circ \widehat{\xi}_j|_{t=0,\tau=0} = \widehat{a}_i^j \widehat{\xi}_j|_{t=0,\tau=0} = a_i^j \xi_j = \eta_i,$$

proving that $\widehat{a}_i^j \circ \widehat{\xi}_j = Q(\eta_i)$. □

Finally, we prove the reduction theorem for the right action.

Theorem 12 *For any right-invariant canonical deformation quantization on $F = G \times \mathcal{G}^*$ the reduced algebra consists of constants only.*

Proof. Since any two canonical right-invariant quantizations are isomorphic and this isomorphism preserves the Hamiltonians $\widehat{\xi}_i$ of the infinitesimal right action, we may assume that we are dealing with the special quantization constructed in Theorem 10. For this quantization the right-invariant subalgebra $\widehat{A}_0 \subset \widehat{A} = W_{D_F}(F)$ is $\widehat{A}_0 = Q(a(\eta, h))$ and the ideal $\widehat{J} \subset \widehat{A}_0$ is generated by $\widehat{\eta}_i$.

The algebra $\widehat{A}_0$ may be described in a more convenient way using the so-called *Weyl correspondence.* For a polynomial $a(\eta, h) \in A_0$ define the corresponding polynomial $a^W(\widehat{\eta}, h) \in \widehat{A}_0$ replacing in $a(\eta, h)$ each monomial by the symmetric $\circ$-product of $\widehat{\eta}_i$. There is a standard way to extend this correspondence to more general functions. For a function $a(\eta, h) \in C_0^\infty(\mathcal{G}^*)[[h]]$ let

$$\widetilde{a}(x) = \int e^{-i\langle \eta, x\rangle} a(\eta, h) d\eta$$

be its Fourier transform (here $x \in \mathbb{R}^n$ is interpreted as an element of $\mathcal{G}$). Then we put

$$a^W(\widehat{\eta}, h) = (2\pi)^{-n} \int e^{i\langle \widehat{\eta}, x\rangle} \widetilde{a}(x, h) dx \tag{4.9}$$

where the exponential function $\exp(i\langle\widehat{\eta}, x\rangle) \in \widehat{A}_0$ is defined via differential equation

$$\frac{dU(s)}{ds} = i\langle\widehat{\eta}, x\rangle \circ U(s); \quad U(0) = 1,$$

so that $\exp(i\langle\widehat{\eta}, x\rangle)$ is defined as $U(1)$. The function $a(\eta, h)$ is called the Weyl symbol of the flat section $a^W(\widehat{\eta}, h) \in W_{D_F}$. The product $\circ$ in W_{D_F} induces the composition rule for the Weyl symbols as follows. The differential equation implies the usual Campbell-Hausdorff formula

$$e^a \circ e^b = e^{a+b+CH(a,b)},$$

$$CH(a, b) = \frac{1}{2}[a, b] + \frac{1}{12}[[a, b], b] + \frac{1}{12}[a, [a, b]] + \dots$$

Taking $a = i\langle\widehat{\eta}, x\rangle,\ b = i\langle\widehat{\eta}, y\rangle$, we have by (4.5)

$$[\langle\widehat{\eta}, x\rangle, \langle\widehat{\eta}, y\rangle] = -ih\langle\widehat{\eta}, [x, y]\rangle,$$

implying

$$\exp(i\langle\widehat{\eta}, x\rangle) \circ \exp(i\langle\widehat{\eta}, y\rangle) = \exp(i\langle\eta, x + y + \frac{1}{h}CH(hx, hy)\rangle).$$

This leads to the following $*$-product on symbols

$$a(\eta, h) * b(\eta, h) = \exp\langle\eta, \frac{i}{h}CH(-ih\frac{\partial}{\partial\xi}, -ih\frac{\partial}{\partial\zeta})\rangle\ a(\xi, h)b(\zeta, h)|_{\xi=\zeta=\eta}. \quad (4.10)$$

From this formula it is evident that the higher-order terms vanish at $\eta = 0$. Thus, the left ideal (as well as the right one) generated by $\eta_1, \eta_2, \dots, \eta_n$ via this $*$-product consists of functions vanishing at $\eta = 0$ and vice versa. By the Weyl rule, this ideal corresponds to the left ideal in $\widehat{A}_0$, generated by $\widehat{\eta}_1, \widehat{\eta}_2, \dots, \widehat{\eta}_n$. Moreover, it is easy to describe the projection

$$\pi : \widehat{A}_0 \to \widehat{R} = \widehat{A}_0/\widehat{J}. \quad (4.11)$$

For a flat section $\widehat{a} \in \widehat{A}_0$ we take its Weyl symbol $a(\eta, h)$, that is we represent $\widehat{a}$ as $a^W(\widehat{\eta}, h)$ (note, that $a(\eta, h) \neq \widehat{a}|_{t=0,\tau=0}$ in general) and then set

$$\pi\widehat{a} = a(0, h) \in \mathbb{C}[[h]] \cong \widehat{R},$$

completing the proof. □

5 General Case

Let M be a symplectic manifold with a Hamiltonian (right) action of a compact Lie group G satisfying the assumptions of the classical Marsden-Weinstein theorem. We use a typical model described in the introduction. So, M is considered as a fibration over a symplectic base (B, ω_B) with a typical fibre $F = G \times \mathcal{G}^*$ and the group G acts on fibres by right translations (4.2). More precisely, we have to restrict ourselves to a tubular G-invariant neighborhood of $M_0 = \mu^{-1}(0)$ and to the corresponding neighborhood $V \in \mathcal{G}^*$. Without loss of generality we consider a special G-invariant deformartion quantization adopted to this fibering structure. To this end introduce a bundle $\mathcal{K}$ over B associated to a principal G-bundle M_0 taking the algebra $W_{D_F}(F)$ constructed in the previous section as a fibre of $\mathcal{K}$ with a left action T_u on $a = a(g, \xi, t, \tau) \in W_{D_F}(F)$ defined by

$$T_u a = L^*_{u^{-1}} a = a(u^{-1} g, \xi, t, \tau).$$

The Lie algebra $\mathcal{G}$ of G acts then as

$$t(e_i)a = \frac{d}{dt} a(e^{-e_i t} g, \xi, t, \eta)\bigg|_{t=0} = -\mathcal{L}_{L_i} a = -\frac{i}{h}\left[\widehat{\eta}_i, a\right], \tag{5.1}$$

where (4.6) was used. According to a general construction a section of the associated bundle is defined by local sections $p : U \to M_0$ of the principal bundle M_0 and corresponding local functions

$$a(x) = a(x, g, \xi, t, \tau, h), \ \ x \in U$$

satisfying the following transition rule: for another local section $R_f p = pf, \quad f = f(x) \in G$ the corresponding function is

$$T_f a = a(x, f^{-1}(x) g, \xi, t, \tau, h).$$

The connection λ on the principal bundle M_0 defines an associated connection according to the rule

$$\partial_{\mathcal{K}} a = d_x a(x) + t(p^* \lambda) a(x) = d_x a - \frac{i}{h}[\langle \widehat{\eta}, p^* \lambda \rangle, a]. \tag{5.2}$$

This definition does not depend on the choice of the local section p defining correctly a derivation of the algebra of sections $C^\infty(\mathcal{K})$. The curvature of this connection is

$$-\,d\langle \widehat{\eta}, p^*\lambda\rangle + \frac{i}{h}\langle \widehat{\eta}, p^*\lambda\rangle \circ \langle \widehat{\eta}, p^*\lambda\rangle = -\langle \widehat{\eta}, \kappa\rangle \tag{5.3}$$

where

$$\kappa = d(p^*\lambda) + \frac{1}{2}[p^*\lambda, p^*\lambda]$$

is a local expression for the curvature form of λ. The curvature is a global section of $\mathcal{K} \otimes \Lambda^2$.

Note, that the sections $\widehat{\eta}_i$ are local (that is they depend on the choice of the section p of the principal bundle). On the contrary, the sections $\widehat{\xi}_i$ are global, and moreover, they are flat with respect to the connection (5.2). Clearly, $\widehat{\xi}_i$ are generators of the right action of G on fibres of $\mathcal{K}$. There are two subbundles

$$\mathcal{K}_J \subset \mathcal{K}_0 \subset \mathcal{K}$$

associated to M_0 with fibres

$$\widehat{A}_J \subset \widehat{A}_0 \subset \widehat{A} = W_{D_F}(F).$$

Here $\widehat{A}_0$ is the subalgebra of right-invariant elements of $\widehat{A}$, thus $a \in \widehat{A}_0$ depends only on $\widehat{\eta}_i$ (in any local representation) while $\widehat{A}_J$ is an ideal of $\widehat{A}_0$ generated by $\widehat{\eta}_i$. The quotient $\mathcal{K}/\mathcal{K}_J = \mathcal{K}_R$ is a trivial bundle $\mathbb{C}[[h]]$ by Theorem 10. We will use the projection $\pi : \mathcal{K} \to \mathcal{K}_R$ (see Section 4). It may be thought of as a substitution $\widehat{\eta}_i = 0$ in the sections $a^W(x, \widehat{\eta}_i, h) \in C^\infty(\mathcal{K}_0)$. Clearly, the connection $\partial_{\mathcal{K}}$ may be restricted to subbundles $\mathcal{K}_0, \mathcal{K}_J$ because the connection form in (5.2) takes values in $\mathcal{K}_J$.

Consider now the Weyl algebras bundle on B with coefficients in $\mathcal{K}$, that is

$$W(B, \mathcal{K}) = W(B) \otimes \mathcal{K}$$

where the tensor product is taken with respect to $\mathbb{C}[[h]]$. The sections of $W(B, \mathcal{K})$ locally have the form

$$a(x, y) = a(x, y, g, \xi, t, \tau, h) \tag{5.4}$$

where $x \in B$, $y \in T_x B$. For fixed x, y this function is a flat section of $W_{D_F}(F)$. A symplectic connection on B defines a connection ∂_B on the bundle $W(B)$ (see Section 2) and futher, a connection ∂ on the bundle $W(B, \mathcal{K})$

$$\partial = \partial_B \otimes 1 + 1 \otimes \partial_{\mathcal{K}}.$$

Our goal is to construct an Abelian connection $D_{B,\mathcal{K}}$ on $W(B, \mathcal{K})$ whose curvature is $\Omega_{B,\mathcal{K}} = -\omega_B$. We look for $D_{B,\mathcal{K}}$ in the form

$$D_{B,\mathcal{K}} = \partial + \frac{i}{h}[\gamma, \cdot\,]$$

where $\gamma \in C^\infty(W(B, \mathcal{K}_0) \otimes \Lambda^1)$ is a global section which should satisfy the following equation

$$\partial\gamma + \frac{i}{h}\gamma^2 = \omega_B + \langle \widehat{\eta}, \kappa \rangle - \frac{1}{2} R_{ij} y^i y^j \tag{5.5}$$

where R_{ij} is the curvature of ∂_B. This equation has a unique solution under the normalization

$$\delta^* \gamma := y^k \left(\frac{\partial}{\partial x^k} \right) \gamma = 0 \tag{5.6}$$

for $|\xi|$ small enough [3, theorem 8.2.1, theorem 6.5.1]. As usual, we consider a subalgebra $W_{D_{B,\mathcal{K}}}(B, \mathcal{K})$ of flat sections.

Theorem 13 *The algebra $W_{D_{B,\mathcal{K}}}(B, \mathcal{K})$ coincides with the algebra $W_{D_M}(M)$ of flat sections on M with respect to an Abelian connection D_M whose curvature is*

$$\Omega_M = -\omega_B + d\langle \xi, \lambda \rangle = -\omega_M.$$

Proof. The section (5.4) may be considered as a section of $W(M)$. If it belongs to $W_{D_{B,\mathcal{K}}}(B, \mathcal{K})$ the following two conditions should be satisfied: 1) $D_F a = 0$ which means it is flat along the fibres, 2) $D_{B,\mathcal{K}} a = 0$. They both are equivalent to the only condition

$$D_M a := D_F a + D_{B,\mathcal{K}} a = 0.$$

Clearly, $D_M^2 \equiv 0$ since D_F and $D_{B,\mathcal{K}}$ are Abelian connections. To compute the curvature of D_M, we have to normalize the connection form $\langle \widehat{\eta}, p^* \lambda \rangle$ in (5.2) substructing its constant term

$$\langle \widehat{\eta}, p^* \lambda \rangle|_{t=0,\tau=0} = \langle \eta, p^* \lambda \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} p^* \lambda \rangle$$

(the form γ is normalized because of (5.6)). This leads to

$$\Omega_M = \Omega_F + \Omega_{B,\mathcal{K}} + d\langle \xi, \mathrm{Ad}_{g^{-1}} p^*\lambda\rangle = -\omega_B + d\langle \xi, \lambda_L + \mathrm{Ad}_{g^{-1}} p^*\lambda\rangle,$$

proving the theorem since

$$\lambda_L + \mathrm{Ad}_{g^{-1}} p^*\lambda = \lambda$$

because of (1.6). □

Proof of Theorem 3. Without loss of generality we may consider the reduction of the algebra $\widehat{A} = W_{D_{B,\mathcal{K}}}(B,\mathcal{K})$. Clearly, the invariant subalgebra $\widehat{A}_0$ is $W_{D_{B,\mathcal{K}}}(B,\mathcal{K}_0)$, while the ideal $\widehat{J}$ is $W_{D_{B,\mathcal{K}}}(B,\mathcal{K}_J)$ (note that the connections ∂, $D_{B,\mathcal{K}}$ preserve the subbundles $W(B,\mathcal{K}_0)$ and $W(B,\mathcal{K}_J)$). We need to show that the projection π maps $W_{D_{B,\mathcal{K}}}(B,\mathcal{K}_0)$ to $W_{D_B}(B)$ (recall that π acts by substitution $\widehat{\eta}_i = 0$). First apply π to (5.5). Since

$$\partial\gamma = \partial_B\gamma + \frac{i}{h}[\langle\widehat{\eta}, p^*\lambda\rangle, \gamma], \tag{5.7}$$

we have

$$\pi(\partial\gamma) = \partial_B(\pi\gamma)$$

because the second term in (5.7) vanishes under π. The term $\langle\widehat{\eta}, \kappa\rangle$ in (5.5) also vanishes and we obtain

$$\partial_B(\pi\gamma) + \frac{i}{h}(\pi\gamma)^2 = -\omega_B - \frac{1}{2}R_{ij}y^iy^j.$$

This equation along with $\delta^*(\pi\gamma) = 0$ (the consequence of (5.6)) means that $\pi\gamma$ defines the Abelian connection on $W(B)$:

$$D_B = \partial_B + \frac{i}{h}[\pi\gamma, \cdot\,].$$

Now, let $a \in W_{D_{B,\mathcal{K}}}(B,\mathcal{K}_0)$. Thus, it satisfies the equation

$$D_{B,\mathcal{K}}a = \partial a + \frac{i}{h}[\gamma, a] = 0.$$

Applying π to both sides, we obtain

$$\partial_B(\pi a) + \frac{i}{h}[\pi\gamma, \pi a] = D_B(\pi a) = 0,$$

which means that $\pi a \in W_{D_B}(B)$ proving the theorem. □

Acknowlegements

This work was done during the authors visits to three different places. I would like to thank Professor M. Gromov for the invitation to the Courant Institute of Mathematical Sciences of the New York University where the work was begun, Professor M. Flato for the invitation to the Laboratoir de Physique Mathématique of the Université de Bourgogne where the work was continued and Professor B.-W. Schulze for the invitation to the Institut für Mathematik of the Universität Potsdam where it was finished. I am deeply indebted to Professor B. Tsygan who has pointed out that the first version of the proof was incomplete.

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