A Lefschetz Fixed Point Theorem for Manifolds with Conical Singularities

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Abstract

We establish an Atiyah–Bott–Lefschetz formula for elliptic operators on manifolds with conical singular points.

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1 Introduction

In the present paper, we suggest a Lefschetz formula for endomorphisms of elliptic complexes defined on manifolds with conical singularities. The corresponding formula for smooth manifolds was for the first time obtained in the famous paper by Atiyah and Bott [2]. This paper occured to be of a great interest and the number of works devoted to Atiyah-Bott-Lefschetz theory at present hardly can be evaluated. A new methods for proof of Atiyah-Bott-Lefschetz formula were presented, this paper was generalized in different directions, etc. We list here only some papers of the above mentioned type. This is a paper by Gilkey [9], where the heat equation approach to the Lefschetz theorem was developed, the papers by Bismut [3], [4] (probability approach), the papers by Fedosov [7], [8], as well as by Sternin-Shatalov [21], [22] (semiclassical approach). Note that in the latter of the listed papers one can find an elementary proof of the semiclassical Lefschetz formula for endomorphisms of elliptic complexes obtained by quantization of arbitrary canonical transformations[1].

[1] In contrast to "classical" Atiyah-Bott-Lefschetz formula which was proved for the so-called geometrical endomorphisms of complexes (see [2]).

Needless to say, the classical Atiyah–Bott–Lefschetz formula is a special case of this general theorem. We also want to indicate the paper by Brenner-Shubin [5], where the Lefschetz formula for manifolds with boundary is established and which, by the way, contains a good review on this topic and rather full a bibliography.

As was indicated in the preceding, in the present paper we obtain the Lefschetz formula for manifolds with conical singularities. Elliptic theory for such manifolds is presently well developed (e.g., see Schulze [16] and the bibliography therein). Hence, the problem of evaluating the Lefschetz number in this situation is quite natural.

Let us indicate the main feature of our theory. The Lefschetz formula in this situation is given by a sum of contributions of the interior part of the manifold (which has the standard Atiyah–Bott form) and the contributions of the singular points. The latter are expressed in terms of analytic families, namely, the conormal symbol of the elliptic operator and the conormal symbols of the operators constituting the endomorphism of the complex.

A preliminary version of this paper was published as a preprint [13].

2 Preliminaries

For the reader's convenience, here we briefly recall some well-known definitions and statements concerning manifolds with conical singularities and (pseudo)differential operators on such manifolds. A detailed exposition of these and related topics can be found elsewhere (e.g., see [16, 17] and the literature cited therein).

1. We start with the geometrical definition of a manifold with conical singularities.

let Ω be a compact closed C^∞ manifold of dimension $\dim\Omega = n-1$. A *model cone with base* Ω is, by definition, the set

$$K \equiv K_\Omega = ([0,1)\times\Omega)/(\{0\}\times\Omega).$$

The points of K will be denoted by (r,ω), $r\in[0,1)$, $\omega\in\Omega$; note that for any $\omega\in\Omega$, the pair $(0,\omega)$ represents the same point of K, namely, the *vertex.* We also consider the sets

$$\overset{\circ}{K} = (0,1)\times\Omega$$

(the cone with vertex deleted) and the *blow-up*

$$K^\wedge = [0,1)\times\Omega$$

of K, where the vertex is blown up to produce the boundary $\partial K \simeq \Omega$.

Now we stand on the viewpooint of ringed spaces.

On K, we consider two structure rings, namely, the ring

$$C^\infty(K) \stackrel{\text{def}}{=} C^\infty(K^\wedge)$$

of smooth functions of $r \in [0,1)$ and $\omega \in \Omega$ and the ring $\operatorname{Diff}(K)$ of differential operators of the form[2]

$$\widehat{D} = D\left(\overset{2}{r}, \overset{2}{\omega}, ir\overset{1}{\frac{\partial}{\partial r}}, -i\overset{1}{\frac{\partial}{\partial \omega}}\right)$$

with coefficients in $C^\infty(K)$[3]. The essential point here is the fact that the elements of $\operatorname{Diff}(K)$ are *Mellin (or cone degenerated) differential operators* with respect to the variable r; in other words, the differentiation $\partial/\partial r$ occurs in these operators only in the combination $r\partial/\partial r$.

Note that

$$C^\infty(K) \subset C^\infty(\overset{\circ}{K}), \qquad \operatorname{Diff}(K) \subset \operatorname{Diff}(\overset{\circ}{K}),$$

where $\operatorname{Diff}(\overset{\circ}{K})$ is the ring of all differential operators with smooth coefficients on K.

Definition 1 Let M be a Hausdorff space, and let $\{\alpha_1, \dots, \alpha_N\} \subset M$ be a finite subset. One says that M is a *manifold with conical singularities* $\{\alpha_1, \dots, \alpha_N\}$ if the following conditions are satisfied.

(i) The set $\overset{\circ}{M} = M \setminus \{\alpha_1, \dots, \alpha_N\}$ is a C^∞ manifold.

(ii) Two subrings, $\mathcal{A} \subset C^\infty(\overset{\circ}{M})$ and $\mathcal{B} \subset \operatorname{Diff}(\overset{\circ}{M})$, are given (in the following we use the notation $\mathcal{A} = C^\infty(M)$ and $\mathcal{B} = \operatorname{Diff}(M)$). These rings admit partitions of unity subordinate to any locally finite cover of M such that each point α_j, $j = 1, \dots, N$, does not belong to the boundary of any element of the cover. Moreover, for any open set $U \subset M$ such that

$$\overline{U} \cap \{\alpha_1, \dots, \alpha_N\} = \emptyset,$$

one has

$$C^\infty(M)\Big|_U = C^\infty(U), \qquad \operatorname{Diff}(M)\Big|_U = \operatorname{Diff}(U).$$

(iii) For each $j = 1, \dots, N$, there exists a neighborhood $U_j \subset M$ of α_j, a closed compact C^∞ manifold Ω_j, and a homeomorphism

$$\psi_j \colon U_j \to K_j \tag{1}$$

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[2]To simplify the exposition, in the following we sometimes use the same notation for points $\omega \in \Omega_j$ and their coordinate representations $\omega = (\omega_1, \dots, \omega_{n-1})$ in some local chart on Ω_j. This will not lead to a misunderstanding.

[3]The numbers (Feynman indices) over operators indicate the order of their action; see [11, 14]

of U_j onto the model cone K_j with base Ω_j such that the restriction

$$\psi_j\Big|_{U_j\setminus\{\alpha_j\}} : U_j \setminus \{\alpha_j\} \to \mathring{K}_j$$

is a diffeomorphism and

$$\psi_j\left(C^\infty(M)\right) = C^\infty(K_j), \quad \psi_j\left(\mathrm{Diff}(M)\right) = \mathrm{Diff}(K_j)$$

(the left-hand sides of these equations are well defined since $\psi_j|_{U_j\setminus\{\alpha_j\}}$ is a diffeomorphism).

In the neighborhood U_j of the conical point α_j, we shall use either the *conical coordinates* (r,ω), $r \in [0,1)$, $\omega \in \Omega_j$, or the *cylindrical coordinates* (t,ω), where

$$t = \ln\frac{1}{r} \in (0,+\infty].$$

Let M be a manifold with conical singularities $\{\alpha_1,\ldots,\alpha_N\}$. We can apply the blow-up procedure to each of the points α_j, $j = 1,\ldots,N$, using the homeomorphism (1) and the blow-up defined for the model cones. Thus, we obtain the blow-up $M^\wedge$ of the manifold M. Note that

$$\partial M^\wedge = \bigcup_{j=1}^{N} \Omega_j$$

(the disjoint union) and that $C^\infty(M) = C^\infty(M^\wedge)$.

We refer to the ring $\mathrm{Diff}(M)$ as the *ring of Mellin differential operators* on M. The principal symbols of Mellin differential operators on M are well defined on the *compressed cotangent bundle* $\tilde{T}^*M^\wedge$ (see [12]). We write

$$T^*M \stackrel{\mathrm{def}}{=} \tilde{T}^*M^\wedge.$$

The compressed cotangent bundle is a special case of a vector bundle over a manifold with conical singularities. Specifically, we adopt the following definition.

Definition 2 Let M be a manifold with conical singularities $\{\alpha_1,\ldots,\alpha_N\}$. A *vector bundle* over M is a vector bundle over the blow-up $M^\wedge$.

2. Let M be a compact manifold with conical singularities $\{\alpha_1,\ldots,\alpha_N\}$. We intend to define Mellin pseudodifferential operators on M. These operators, as well as Mellin differential operators, naturally act in weighted Sobolev spaces. To define

these spaces, let us first consider the case of a model cone, K, with base Ω. Let $s \in \mathbf{R}$ and $\gamma \in \mathbf{R}$. For any $u \in C_0^\infty(K)$, we define a norm $\|u\|_{s,\gamma}$ by setting

$$\|u\|_{s,\gamma}^2 = \int r^{-2\gamma} \left\| \left(1 + \left(ir\frac{\partial}{\partial r} \right)^2 + \Delta_\Omega \right)^{s/2} u \right\|_{L^2(\Omega)}^2 \frac{dr}{r},$$

where Δ_Ω is the positive Beltrami–Laplace operator on Ω (with respect to some Riemannian metric).

To define weighted Sobolev spaces on M, we assign some weight γ_j to each conical point α_j. Thus we obtain the *weight exponent vector*

$$\gamma = (\gamma_1, \ldots, \gamma_N).$$

We write $\gamma_i = \gamma(\alpha_i)$, $i = 1, \ldots, N$.

Now the norm $\|\cdot\|_{s,\gamma}$ of the weighted Sobolev space $H^{s,\gamma}(M)$ is defined as follows with the help of a partition of unity. In a neighborhood of each conical point α_i, we use the norm $\|\cdot\|_{s,\gamma(\alpha_i)}$ on the corresponding model cone, while outside a neighborhood of the set of conical points, any of the (equivalent) usual Sobolev norms $\|\cdot\|_s$ defined on a smooth manifold is used. The space $H^{s,\gamma}(M,E)$, where E is a vector bundle over M, can now be defined in an obvious way.

3. Now we are in a position to define pseudodifferential operators. The conventional approach, which goes back to [10], is to define them modulo infinitely smoothing operators. However, this is not appropriate for the aims of the present paper, since we intend to compute traces, which do depend on the smoothing part. Therefore, we give a definition which is not explicitly invariant in that it involves a partition of unity subordinate to a cover of M by coordinate neighborhoods but has the advantage of defining a unique operator rather than an equivalence class modulo smoothing operators.

Let M be a manifold with conical singularities $\{\alpha_1, \ldots, \alpha_N\}$. Consider a finite cover of M by coordinate neighborhoods,

$$M = \bigcup_{j=1}^{p} U_j, \tag{2}$$

where U_j is a conical coordinate neighborhood around α_j, $j = 1, \ldots, N$, and the U_j, $j > N$, are some coordinate neighborhoods on $\overset{\circ}{M}$. Thus, for $1 \le j \le N$, the coordinates on U_j are (r, ω), $r \in [0,1)$, $\omega \in \Omega_j$, and for $j > N$, the coordinates on U_j are just local coordinates $(x_1, \ldots, x_n)$ on $\overset{\circ}{M}$.

We suppose that $\alpha_j \notin U_k$ for $j \neq k$. Let $\{e_j\}_{j=1}^p$ be a C^∞ partition of unity subordinate to the cover (2), and let $\{f_j\}_{j=1}^p$ be a collection of C^∞ functions such that

$$f_j e_j = e_j, \quad \operatorname{supp} f_j \subset U_j, \quad j = 1, \dots, p.$$

We consider operators of the form

$$\widehat{H} = \sum_{j=1}^{p} f_j \widehat{H}_j e_j, \tag{3}$$

where the $\widehat{H}_j$ are mth-order pseudodifferential operators defined in local coordinates. More precisely, for $j > N$, the $\widehat{H}_j$ are usual pseudodifferential operators,

$$\widehat{H}_j = H_j\left(\overset{2}{x}, -i\,\overset{1}{\frac{\partial}{\partial x}}\right), \tag{4}$$

with symbols $H_j(x, q)$ belonging to the classical symbol space $S^m(\mathbf{R}^n_x \times \mathbf{R}^n_q)$ [10]; for $1 \leq j \leq N$, the $\widehat{H}_j$ are Mellin pseudodifferential operators,

$$\widehat{H}_j = \widehat{H}_j\left(\overset{2}{r}, ir\overset{1}{\frac{d}{dr}}\right), \tag{5}$$

where $\widehat{H}_j(r, p)$ is an operator-valued symbol with values in the algebra of pseudodifferential operators on Ω_j and satisfying the estimates

$$\left\| (1 + p^2 + \Delta_{\Omega_j})^{-\frac{m-|\alpha|}{2}} \frac{\partial^{\alpha+p} \widehat{H}_j}{\partial p^\alpha \partial r^\beta}(r, p) \right\|_{H^s(\Omega_j) \to H^s(\Omega_j)} \leq C_{\alpha\beta s}, \tag{6}$$

$\alpha, \beta = 0, 1, 2, \dots$, $s \in \mathbf{R}$, for $r \in [0, 1]$ and $p \in \mathcal{L}_{\gamma(\alpha_j)} = \{\operatorname{Im} p = \gamma(\alpha_j)\}$. (Needless to say, the operator $\widehat{H}_j$ can be further specialized as

$$\widehat{H}_j = H_j\left(\overset{2}{r}, \overset{2}{\omega}, ir\overset{1}{\frac{\partial}{\partial r}}, -i\,\overset{1}{\frac{\partial}{\partial \omega}}\right)$$

if we choose some system of local coordinates $(\omega_1, \dots, \omega_{n-1})$ on Ω_j; here the symbol H_j belongs to the space $S^m(T^* K_j)$.)

Proposition 1 (see [16]) *The operator* (3) *is bounded in the spaces*

$$\widehat{H} : H^{s,\gamma}(M) \to H^{s-m,\gamma}(M)$$

for any $s \in \mathbf{R}$.

Note that the consideration of Mellin pseudodifferential operators in the spaces $H^{s,\gamma}(M)$ with given γ can always be reduced to the case in which

$$\gamma(\alpha_i) = 0, \qquad i = 1, \dots, N.$$

First, consider the local situation. Let $s \in \mathbf{R}$ and $\gamma \in \mathbf{R}$. The multiplication by r^γ is obviously an isomorphism between $H^{s,0}(K)$ and $H^{s,\gamma}(K)$, where K is the model cone. Next,

$$r^{-\gamma}\widehat{H}\left(\overset{2}{r}, \overset{1}{ir\frac{d}{dr}}\right) r^\gamma = \widehat{H}\left(\overset{2}{r}, \overset{1}{ir\frac{d}{dr}} + i\gamma\right),$$

so that after applying this isomorphism, we must require the estimates (6) to be satisfied for the newly obtained symbol on the real axis. The globalization of this trick is perfectly easy: one takes an arbitrary nonvanishing function ψ on M such that $\psi = r^{\gamma(\alpha_j)}$ near α_j, $j = 1, \dots, N$. Then the multiplication by ψ is an isomorphism between $H^{s,0}(M)$ and $H^{s,\gamma}(M)$.

Now let us recall the notion of the *conormal symbol* [16] of a Mellin pseudodifferential operator on a manifold with conical singularities. Let M be a manifold with conical singularities $\{\alpha_1, \dots, \alpha_N\}$, and let $\widehat{D}$ be a Mellin pseudodifferential operator on M. Then the conormal symbol of $\widehat{D}$ at a conical point α_j is the family of pseudodifferential operators on Ω_j defined as follows. Near α_j, we can represent $\widehat{D}$ in the form

$$\widehat{D} = \widehat{D}\left(\overset{2}{r}, \overset{1}{ir\frac{d}{dr}}\right), \tag{7}$$

where $\widehat{D}(r,p)$ is the operator-valued symbol of $\widehat{D}$ (with values in pseudodifferential operators on Ω). The conormal symbol is defined by

$$\widehat{D}_0(p) = \widehat{D}(0,p), \quad p \in \mathbf{R}^1. \tag{8}$$

This definition is, in fact, coordinate-independent.

4. In conclusion, let us recall the definition of an elliptic differential operator on a manifold M with conical singularities.

Definition 3 ([16]) Let $\widehat{D}$ be an mth-order differential operator on a manifold M with conical singularities $\{\alpha_1, \dots, \alpha_N\}$. One says that $\widehat{D}$ is *elliptic* if the following conditions are satisfied:

(i) $\widehat{D}$ is elliptic in the usual sense on $\overset{\circ}{M} = M \setminus \{\alpha_1, \dots, \alpha_N\}$;

(ii) the conormal symbol $\hat{D}_{j0}(p)$ of the operator $\hat{D}$ at each conical point α_j is an elliptic family with parameter p in the sense of Agranovich–Vishik [1] in some sector containing the real axis.

It is well known that then $\widehat{D}_{j0}(p)$ is finite-meromorphically invertible, that is, $\widehat{D}_{j0}(p)^{-1}$ is a meromorphic operator family with finite-dimensional principal parts of the Laurent series at the poles, and moreover, there are finitely many poles of $\widehat{D}_{j0}(p)^{-1}$ in each strip $|\operatorname{Im} p| \leq c$.

Furthermore, if M is compact, then for any weight exponent vector γ such that the weight line $\mathcal{L}_{\gamma(\alpha_j)}$ does not contain poles of $\widehat{D}_{j0}(p)^{-1}$, the operator

$$\widehat{D} : H^{s,\gamma}(M) \to H^{s-m,\gamma}(M)$$

is a *Fredholm operator for* any $s \in \mathbf{R}$. This can be proved by conctructing a regularizer; we shall use a regularizer of a special form in Section 4.

3 Statement of the Results

1. Let M be a compact manifold with conical singularities $\{\alpha_1, \dots, \alpha_N\}$, and let E_1, E_2, ..., E_k be finite-dimensional vector bundles over M. Suppose that we are given an endomorphism of an elliptic complex on M (see [15]), that is, a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{s,\gamma}\left(E^1\right) & \xrightarrow{\hat{D}^1} & \cdots & \xrightarrow{\hat{D}^{k-1}} & H^{s-m_1-\ldots-m_{k-1},\gamma}\left(E^k\right) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{T}_1} & & & & \downarrow{\scriptstyle \hat{T}_k} & & \\
0 & \longrightarrow & H^{s,\gamma}\left(E^1\right) & \xrightarrow{\hat{D}^1} & \cdots & \xrightarrow{\hat{D}^{k-1}} & H^{s-m_1-\ldots-m_{k-1},\gamma}\left(E^k\right) & \longrightarrow & 0
\end{array} \tag{9}$$

whose rows are (one and the same) elliptic complex on M. Then, as is usual in Lefschetz theory, the operators $\hat{T}_j$ act on the cohomology $H^j\left(\hat{D}^{\cdot}\right)$, and one can define the Lefschetz number $L\left(\hat{D}^{\cdot},\hat{T}^{\cdot}\right)$ by

$$L\left(\hat{D}^{\cdot},\hat{T}^{\cdot}\right)=\sum_{j=1}^{k}(-1)^{j-1}\mathrm{Trace}\ \hat{T}_j\Big|_{H^j(D^{\cdot})}.$$

Our goal is to compute the Lefschetz number for the case in which all the $\hat{T}_j$ are geometrical endomorphisms induced by some diffeomorphism of M.

We recall that for the case in which the manifold in question has no singular points, this problem was already solved long ago [2]. Since the relationship between the fixed points of the diffeomorphism in question and the Lefschetz number is of local nature, we shall be interested mainly in the contribution of the *conical fixed points*.

2. For simplicity, we consider only short complexes of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{s,\gamma}(E) & \xrightarrow{\hat{D}} & H^{s-m,\gamma}(F) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \hat{T}_1} & & \downarrow{\scriptstyle \hat{T}_2} & & \\
0 & \longrightarrow & H^{s,\gamma}(E) & \xrightarrow{\hat{D}} & H^{s-m,\gamma}(F) & \longrightarrow & 0
\end{array} \tag{10}$$

where $\hat{D}$ is an elliptic operator on M.

The definition of the Lefschetz number for the diagram (10) becomes

$$L\left(\hat{D},\hat{T}^{\cdot}\right)=\mathrm{Trace}\ \hat{T}_1\Big|_{\mathrm{Ker}\,D}-\mathrm{Trace}\ \hat{T}_2\Big|_{\mathrm{Coker}\,D}.$$

In the diagram (10), the $\hat{T}_j$ are geometrical morphisms, that is,

$$\left(\hat{T}_j u\right)(x)=A_j(x)\,u(g(x)), \tag{11}$$

where

$$g\ :\ M\rightarrow M$$

is a mapping of the underlying manifold and the $A_j(x)$ are bundle homomorphisms, and x is a point of the manifold M. Since M has a special structure near the singular points, the mapping g must preserve this structure, and we shall describe the structure of this mapping in detail.

Since the structure ring of the conical manifold near a conical point is the ring of functions $F(r, \omega)$ smooth in both variables up to the point $r = 0$, it follows that each conical point is necessarily taken by g to a conical point, and we see that g must be given near any conical point α by the expressions

$$g \,:\, \begin{cases} r' = rA(r, \omega), \\ \omega' = B(r, \omega), \end{cases} \tag{12}$$

where the function $A(r, \omega) \neq 0$ and the mapping $B(r, \omega)$ are smooth up to $r = 0$, or, in the cylindrical representation,

$$g \,:\, \begin{cases} t' = t + a(e^{-t}, \omega), \\ \omega' = b(e^{-t}, \omega) \end{cases} \tag{13}$$

with the same smoothness requirement on the functions $a(r, \omega)$ and $b(r, \omega)$. Let us introduce the following definition.

Definition 4 The transformation g given by (13) is said to be *nondegenerate* if

(i) for any interior fixed point $x = g(x) \in \overset{\circ}{M}$, one has

$$\det (1 - g_*(x)) \neq 0;$$

(ii) for any fixed conical point $\alpha = g(\alpha)$, one has either

$$a(0, \omega) > 0 \text{ for all } \omega \in \Omega \tag{14}$$

(then the fixed point is said to be *attractive*), or

$$a(0, \omega) < 0 \text{ for all } \omega \in \Omega \tag{15}$$

(then the fixed point is said to be *repulsive*).

Clearly, a nondegenerate transformation has at most finitely many fixed points.

Note that the mappings (11) can be rewritten as the Fourier–Maslov integral operators [11, 14] with the local expression

$$\hat{T}_j = A_j\left(\overset{2}{x}\right) \exp \left[\left(g\left(\overset{2}{x}\right) - \overset{2}{x} \right) \overset{1}{\frac{\partial}{\partial x}} \right], \; j = 1, 2,$$

where $x=(r,\omega)$, or, in a more detailed form,

$$\hat{T}_j = A_j\left(\overset{2}{e^{-t}},\overset{2}{\omega}\right)\exp\left[a\left(\overset{2}{e^{-t}},\overset{2}{\omega}\right)\overset{1}{\frac{\partial}{\partial t}}\right]\exp\left\{\tilde{b}\left(\overset{2}{e^{-t}},\overset{2}{\omega}\right)\overset{1}{\frac{\partial}{\partial\omega}}\right\},\ j=1,2, \tag{16}$$

where $\tilde{b}\left(e^{-t},\omega\right)=b\left(e^{-t},\omega\right)-\omega$.

We define the *conormal symbol* of the operator (11) at the point α by

$$\hat{T}_{j0}(p)=A_j\left(0,\overset{2}{\omega}\right)\exp\left[ia\left(0,\overset{2}{\omega}\right)p\right]\exp\left\{\tilde{b}\left(0,\overset{2}{\omega}\right)\overset{1}{\frac{\partial}{\partial\omega}}\right\},\ j=1,2.$$

3. Now we are in a position to state the main result of the present paper.

Theorem 1 *Let the mapping g be nondegenerate. Then the Lefschetz number of the diagram* (10) *has the form*

$$L\left(\hat{D},\hat{T}^{\cdot}\right)=\mathcal{L}_{\text{cone}}+\mathcal{L}_{\text{int}},$$

where

$$\mathcal{L}_{\text{int}}=\sum_{x=g(x)\in\overset{\circ}{M}}\frac{\operatorname{Trace}A_1(x)-\operatorname{Trace}A_2(x)}{\left|\det\left(1-g_*(x)\right)\right|} \tag{17}$$

is the usual Lefschetz contribution of interior fixed points, and

$$\mathcal{L}_{\text{cone}}=\sum_{\alpha=g(\alpha)\in M\setminus\overset{\circ}{M}}\mathcal{L}_\alpha$$

is the sum of contributions of conical fixed points. Here the contribution $\mathcal{L}_\alpha$ of a conical fixed point α is given by

$$\begin{aligned}\mathcal{L}_\alpha &= \lim_{h\to 0}\ Trace\frac{1}{2\pi i}\int_{\mathcal{L}_{\gamma(\alpha)}}\widehat{T}_{10}(p)\widehat{D}_0^{-1}(p)(1+(h^2\Delta_\Omega)^N)^{-1}\\ &\times\ (1+h^{2N}(p-i\gamma(\alpha))^{2N})^{-1}\frac{\partial\widehat{D}_0}{\partial p}(p)\,dp,\end{aligned}$$

where $\widehat{T}_{10}(p)$ and $\widehat{D}_0(p)$ are the conormal symbols of $\widehat{T}_1$ and $\widehat{D}$ at the point α, respectively, and N is a sufficiently large positive integer.

4. Under additional conditions, the expression for the contribution of the conical fixed points can be further simplified.

Let $\hat{B}(p)$ be an analytic Fredholm family of operators depending on the parameter $p \in \mathbf{C}$ such that $\hat{B}(p)$ is finite-meromorphically invertible with a discrete set of poles $p_j \in \mathbf{C}$, $j = 1, 2, 3, \ldots$, of the inverse family.

Definition 5 We say that $\hat{B}(p)$ is a *family of power type* if the following conditions are satisfied.

(i) There exist constants C_1 and N_1 such that

$$\sum_{|\operatorname{Im} p_j| \le \gamma} 1 \le C_1 \gamma^{N_1}, \quad \gamma \ge 0.$$

(ii) The orders of the poles are uniformly bounded. This means that there exists a number N_2 such that for each j the operator function

$$(p - p_j)^{N_2} \hat{B}^{-1}(p)$$

is holomorphic near the singular point p_j of the family $\hat{B}^{-1}(p)$.

(iii) Let $B_j(p)$ be the singular part of the Laurent series of $\hat{B}^{-1}(p)$ around p_j (it is necessarily a family of finite-dimensional operators). Then there exist constants C_3 and N_3 such that

$$\dim B_j(p) \le C_3 |\operatorname{Im} p_j|^{N_3}$$

and constants C_4 and N_4 such that

$$\left\| \frac{\partial^k}{\partial p^k} \left[(p - p_j)^{N_2} \hat{B}^{-1}(p) \right] \Big|_{p = p_j} \right\| \le C_4 |\operatorname{Im} p_j|^{N_4}$$

for $k = 0, 1, \ldots, N_2$.

(iv) There exists a sequence of positive numbers $R_j \to +\infty$ such that $\hat{B}^{-1}(p)$ has no poles on the circle of radius R_j centered at zero and

$$\left\| \hat{B}^{-1}(p) \right\| \le C_5 R_j^{N_5}, \qquad |p| = R_j,$$

with some constants C_5 and N_5.

The conormal symbol of the Beltrami-Laplace operator

$$r^{-2}\left[\left(r\frac{\partial}{\partial r}\right)^2+\frac{\partial^2}{\partial\varphi^2}\right]$$

corresponding to the metric $ds^2 = dr^2 + r^2 dbvarphi^2$ on the two-dimensional cone is an example of a family of power type (see also Examples below). Actually, this symbol is

$$\hat{B}\,(p) = -p^2 + \frac{\partial^2}{\partial\varphi^2},$$

and its inverse has poles at $p = ik$, $k \in \mathbf{Z}$. So, the requrement (i) is fulfilled. Later on, all the multiplicities of these poles do not exceed 2, so that the requirement (ii) is fulfilled as well. The requirements (iii) and (iv) are also valid; the easy check of this fact is left to the reader.

Theorem 2 *Under the conditions of Theorem* 1*, suppose that the conormal symbols of the operator* $\widehat{D}$ *at the singular fixed points of* g *are families of power type. Then the contribution of the conical fixed points can be represented in the form*

$$\mathcal{L}_{\text{cone}} = \mathcal{L}_{\text{cone},+} + \mathcal{L}_{\text{cone},-},$$

where

$$\mathcal{L}_{\text{cone},+} = \sum_{\alpha\in \operatorname{Sing}_a M}\left\{\sum_{\operatorname{Im} p_j>\gamma(\alpha)} \operatorname{Res}_{p_j} \operatorname{Trace} \hat{T}_{1,0}(\alpha,p)\hat{D}_0^{-1}(\alpha,p)\frac{\partial \hat{D}_0}{\partial p}(\alpha,p)\right\}, \qquad (18)$$

$$\mathcal{L}_{\text{cone},-} = -\sum_{\alpha\in \operatorname{Sing}_r M}\left\{\sum_{\operatorname{Im} p_j<\gamma(\alpha)} \operatorname{Res}_{p_j} \operatorname{Trace} \hat{T}_{1,0}(\alpha,p)\hat{D}_0^{-1}(\alpha,p)\frac{\partial \hat{D}_0}{\partial p}(\alpha,p)\right\} \qquad (19)$$

are the contributions of attractive and repulsive fixed singular points, respectively (here $\operatorname{Sing}_a M$ *and* $\operatorname{Sing}_r M$ *are the sets of fixed singular points of these two types). In* (18) *and* (19)*,* $\hat{T}_{1,0}(\alpha,p)$ *and* $\hat{D}_0(\alpha,p)$ *are the conormal symbols of the operators* $\hat{T}_1$ *and* $\hat{D}$*, respectively, at the conical point* α*.*

The requirement that the conormal symbols of $\hat{D}$ be families of power type is not too restrictive; in fact, the authors do not know any examples for which this condition is violated.

4 Proof of Theorems 1 and 2

We conduct the proof for the case in which M has only one conical point α, which is then necessarily a fixed point of g. The generalization to the case of several conical points is trivial; one must only have in mind that in the latter case some of the points (or even all of them) may not be fixed. As it will be shown below, the contributions of such points are zero.

1. Without loss of generality, in the following we assume that $\gamma(\alpha) \equiv \gamma = 0$; otherwise, we perform the similarity transformation

$$\hat{D} \to r^{-\gamma} \hat{D} r^{\gamma}, \qquad \hat{T}_j \to r^{-\gamma} \hat{T}_j r^{\gamma},$$

which does not affect the Lefschetz number and shifts the argument p of all symbols by $i\gamma$, thus reducing the situation to this special case.

We shall compute the Lefschetz number $L\left(\hat{D}, \hat{T}^{\cdot}\right)$ of the diagram (10) by the well-known formula (e.g., see [6])

$$L\left(\hat{D}, \hat{T}^{\cdot}\right) = \operatorname{Trace}\left(\hat{T}_1\left(1 - \hat{R}_{\text{gl}} \hat{D}\right)\right) - \operatorname{Trace}\left(\hat{T}_2\left(1 - \hat{D} \hat{R}_{\text{gl}}\right)\right), \qquad (20)$$

where $\hat{R}_{\text{gl}}$ is some global regularizer of the operator $\hat{D}$ on the manifold M. More precisely, we shall use a family of regularizers depending on two small parameters $\lambda > 0$ and $h > 0$ and take advantage of the fact that $L\left(\hat{D}, \hat{T}^{\cdot}\right)$ is independent of λ and h to obtain explicit formulas by passing to the limit as $\lambda \to 0$ and $h \to 0$. From (20) it follows, in particular, that there is no contributions to the Lefschetz number from non-fixed conical points of the manifold M since such points produce zero terms in the integrals expressing the traces on the right in this formula.

2. The regularizer $\hat{R}_{\text{gl}}$ will be assembled from local regularizers with the help of a special partition of unity. First, let us consider the construction of the local regularizer near the conical point. Here we use the special conical coordinate $r \in \mathbf{R}_+$ and treat M as the direct product $K = \mathbf{R}_+ \times \Omega$, with the neighborhood of $r = 0$ being essential. The operator $\hat{D}$ here has the form

$$\hat{D} = \hat{D}\left(\overset{2}{r}, \overset{1}{ir\frac{d}{dr}}\right),$$

where $\hat{D}(r, p)$ is the operator-valued symbol of $\hat{D}$ in this decomposition, defined for sufficiently small r. By assumption, the real line $\{\operatorname{Im} p = 0\}$ is free of the poles of the operator family $\hat{D}_0^{-1}(p)$, so that the operator

$$\hat{D}_0 = \hat{D}_0\left(ir\frac{d}{dr}\right)$$

is boundedly invertible as an operator from $H^{s,0}(K)$ to $H^{s-m,0}(K)$ for any $s \in \mathbf{R}$.

Lemma 1 *There exists an $\varepsilon > 0$ and an elliptic mth-order operator*

$$\hat{D}_\varepsilon = \hat{D}_\varepsilon \left(\overset{2}{r}, i \overset{1}{r \frac{d}{dr}} \right)$$

defined on the entire K such that

(i) $\hat{D}_\varepsilon (r, p) = \hat{D} (r, p)$ *for* $r \leq \varepsilon$*;*

(ii) the inverse

$$\hat{D}_\varepsilon^{-1} \colon H^{s,0} (K) \to H^{s+m,0} (K)$$

exists and is bounded for any $s \in \mathbf{R}$*.*

Proof. Let $\rho \in C_0^\infty \left(\overline{\mathbf{R}}_+ \right)$ be a function such that $\rho (r) = 1$ for $r \leq 1$ and $\rho (r) = 0$ for $r \geq 2$. Then the operator

$$\hat{D}_\varepsilon = \hat{D}_0 + \rho (r/\varepsilon) \left[\hat{D} - \hat{D}_0 \right]$$

is well defined on the entire K for sufficiently small ε (when the support of $\rho (r/\varepsilon)$ is contained in the domain where $\hat{D} (r, p)$ is defined). Moreover, $\hat{D} - \hat{D}_0 = r\hat{F}$, where $\hat{F}$ is an mth-order operator with smooth coefficients, and we have

$$\left\| \hat{D}_\varepsilon - \hat{D}_0 \right\|_{H^{m,0}(K) \to L^{2,0}(K)} \leq C\varepsilon, \tag{21}$$

(one need not differentiate the coefficients of $\hat{D}_\varepsilon - \hat{D}_0$ to estimate the L_2 norm). Since

$$\hat{D}_0 \ : \ H^{m,0} (K) \to L^{2,0} (K)$$

is invertible, it follows from the estimate (21) that so is $\hat{D}_\varepsilon$ for sufficiently small ε. Let us choose and fix such an ε. We claim that assertion *(ii)* of Lemma 1 is valid. Indeed, $\hat{D}_\varepsilon$ is elliptic, and moreover, it is a Fredholm operator of index zero because it is homotopic to $\hat{D}_0$ by the homotopy

$$\hat{D}_{\varepsilon,\tau} = \hat{D}_0 + \tau \rho (r/\varepsilon) \left[\hat{D} - \hat{D}_0 \right].$$

Now, for $s > m$ we readily see that the operator $\hat{D}_\varepsilon$ is invertible: its kernel remains trivial in the narrower space, and then the cokernel is trivial since the index is zero. For $m > s$, the space in which $\hat{D}_\varepsilon$ acts is broader, but we can use ellipticity: all solutions to $\hat{D}_\varepsilon u = 0$ must belong to $\bigcap_s H^{s,0} (K)$, and so again the kernel is trivial. The proof of Lemma 1 is complete.

We shall use the regularizer $\hat{R}_1 = \hat{D}_\varepsilon^{-1}$ in a small neighborhood of the conical point. Clearly, we have

$$\hat{R}_1 = \hat{R}_1\left(\overset{2}{r}, ir\overset{1}{\frac{d}{dr}}\right) = R_1\left(\overset{2}{x}, -i\overset{1}{\frac{\partial}{\partial x}}\right),$$

where $\hat{R}_1(r,p)$ is an operator-valued symbol of order $-m$ and $R_1(x,\xi)$ is a symbol of order $-m$ (we write $x=(t,\omega)$ and denote by $\xi=(p,q)$ the dual variables).[4]

3. Away from some neighborhood of the conical point, we will use a pseudodifferential regularizer

$$\hat{R}_3 = R_3(x, -i\partial/\partial x)$$

constructed in the standard manner (here x are arbitrary (local) coordinates on the manifold M). More precisely, we choose $\hat{R}_3 = \hat{R}_3(h)$ to depend on the parameter h in a special way (see below). In the "intermediate zone," we use some operator $\hat{R}_2(h)$ (which is a modification of $\hat{R}_1$).

Now let us precisely describe the zones and the construction of the regularizers.

We cover M by three zones as shown in Figure 1. We see that in zones I and II, the operator $\hat{R}_1$ gives an exact local inverse of $\hat{D}$; note also that the "boundary layer" between zones II and III is immovable, whereas that between zones I and II moves to the singular point as $\lambda \to 0$.

Let $\chi = \chi(\xi)$ be a function sufficiently rapidly tending to 1 as $|\xi| \to \infty$ (the precise form of the function $\chi(\xi)$ will be indicated later on in the proof). We set $\hat{\chi} = \chi(-ih\partial/\partial x)$ and $\hat{R}_2(h) = \hat{R}_1 \circ \hat{\chi}$. Next, we define the global regularizer by the formula

$$\begin{aligned}\hat{R}_{\mathrm{gl}} &= \psi_1\left(\frac{r}{\lambda}\right) \circ \hat{R}_1 \circ f_1\left(\frac{r}{\lambda}\right) + \psi_2(r,\lambda) \circ \hat{R}_2(h) \circ f_2(r,\lambda) \\ &\quad + \psi_3(r) \circ \hat{R}_3(h) \circ f_3(r), \end{aligned} \tag{22}$$

where

$$1 = f_1\left(\frac{r}{\lambda}\right) + f_2(r,\lambda) + f_3(r) \tag{23}$$

is a partition of unity subordinate to the cover by zones I–III and each ψ_j is a cutoff function equal to 1 on the support of the corresponding f_j and intended to ensure that the operator (22) is well defined (recall that the $\hat{R}_j$, $j=1,2,3$, are defined only locally on M).

[4]To simplify the subsequent, rather cumbersome, calculations with the regularizers, we proceed as if the entire Ω were covered by a single coordinate system, ω. More detailed computations, which can be done easily, include a partition of unity on Ω.

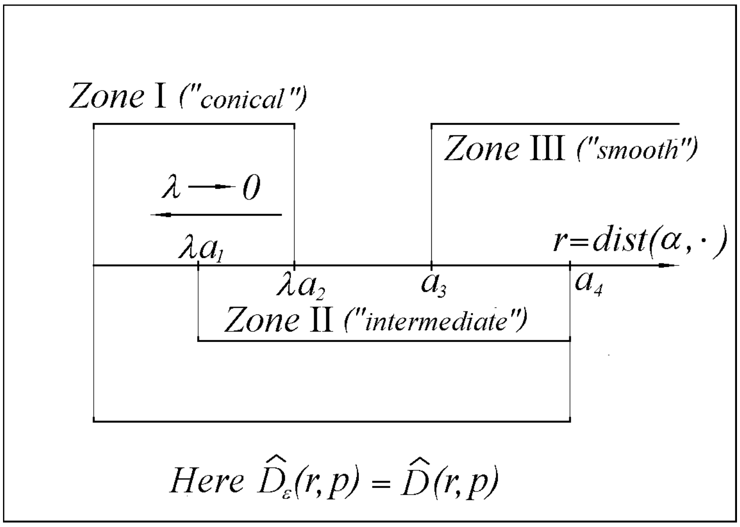

Figure 1. Cover of M by zones.

Standard computations show that then (22) is indeed a regularizer, but we need to impose some additional conditions on the partition of unity (23) and on the associated functions ψ_j so as to evaluate the Lefschetz number (20) effectively.

Condition 1 For any $\lambda \in (0,1]$, *we require that*

$$[\operatorname{supp} f_j \cup g\,(\operatorname{supp} f_j)] \cap \operatorname{supp}\,(1-\psi_j) = \emptyset,\ j=1,2,3.$$

Lemma 2 *The functions f_j and ψ_j can be chosen so that* (23) *holds, the regularizer $\hat{R}_{\rm gl}$ is well defined, and Condition* 1 *is satisfied.*

Proof. Since g is a conical diffeomorphism, we have

$$cr \le r'(r,\omega) \le Cr,\ \omega\in\Omega,$$

with some positive constants c and C for sufficiently small r. Consider the arrangements of supports shown in Figure 2.

It is important to have $b_4 \le \varepsilon$, so that $\hat{R}_1$ be a local inverse of $\hat{D}$ where it is used. Note that with this arrangement, the condition $\operatorname{supp} f_j \cap \operatorname{supp}\,(1-\psi_j)$ is satisfied automatically. It remains to ensure the implications

$$\begin{aligned} r &\le \lambda a_2 \Rightarrow r'(r,\omega) < b_2;\\ \lambda a_1 &\le r \le a_4 \Rightarrow \lambda b_1 < r'(r,\omega) < b_4;\\ r &\ge a_3 \Rightarrow r'(r,\omega) < b_3. \end{aligned} \qquad (24)$$

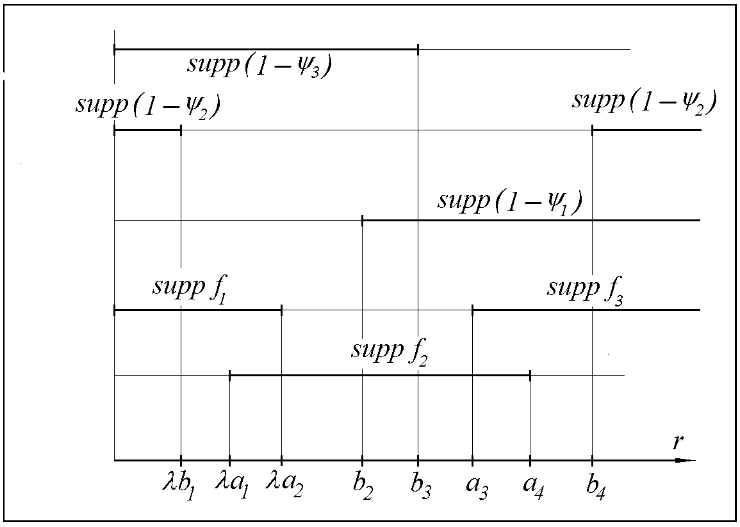

Figure 2. Supports of auxiliary functions.

To this end, we require that

$$\begin{aligned} b_2 &< Ca_2, \\ b_1 &< ca_1,\ b_4 < Ca_4, \\ b_3 &< ca_3. \end{aligned} \tag{25}$$

Now we satisfy (25) by successively choosing

$$\begin{aligned} b_4 &< \varepsilon,\ a_4 < \min\left(b_4, b_4/C\right),\ a_3 < a_4,\ b_3 < \min\left(a_3, ca_3\right), \\ b_2 &< b_3,\ a_2 < \min\left(b_2, b_2/C\right),\ a_1 < a_2,\ b_1 < \min\left(a_1, ca_1\right). \end{aligned}$$

The proof of Lemma 2 is complete.

Let

$$\hat{D} \circ \hat{R}_j = 1 - \hat{Q}_j,\ \ \hat{R}_j \circ \hat{D} = 1 - \hat{Q}'_j,\ j = 1, 2, 3. \tag{26}$$

Here the $\hat{Q}_j$ and $\hat{Q}'_j$ are smoothing operators (for $\hat{R}_2 = \hat{R}_2(h) = \hat{R}_1 \circ \hat{\chi}$ it will be ensured by an appropriate choice of the function χ). Note that[5] $\hat{Q}_1 = 0$ and $\hat{Q}'_1 = 0$ for $r \leq \varepsilon$. Then routine computations show that

$$\hat{D} \circ \hat{R}_{\rm gl} = 1 - \hat{Q},\ \ \hat{R}_{\rm gl} \circ \hat{D} = 1 - \hat{Q}', \tag{27}$$

[5]The precise meaning of these words is that $\hat{Q}_1 f = 0$ for $r \leq \varepsilon$ if $\operatorname{supp} f \subset [0, \varepsilon]$, and the same for $\hat{Q}'_1$.

where

$$\hat{Q} = \sum_{j=2}^{3} \psi_j \circ \hat{Q}_j \circ f_j - \sum_{j=1}^{3} \left[\hat{D}, \psi_j\right] \circ \hat{R}_j \circ f_j, \tag{28}$$

$$\hat{Q}' = \sum_{j=2}^{3} \psi_j \circ \hat{Q}'_j \circ f_j + \sum_{j=1}^{3} \psi_j \circ \hat{R}_j \circ \left[\hat{D}, f_j\right], \tag{29}$$

and accordingly,

$$\begin{aligned} L\left(\hat{D}, \hat{T}\right) = {} & \mathrm{Trace}\left(\hat{T}_1 \sum_{j=2}^{3} \psi_j \circ \hat{Q}'_j \circ f_j\right) + \mathrm{Trace}\left(\hat{T}_1 \sum_{j=1}^{3} \psi_j \circ \hat{R}_j \circ \left[\hat{D}, f_j\right]\right) \\ & - \mathrm{Trace}\left(\hat{T}_2 \sum_{j=2}^{3} \psi_j \circ \hat{Q}_j \circ f_j\right) + \mathrm{Trace}\left(\hat{T}_2 \sum_{j=1}^{3} \left[\hat{D}, \psi_j\right] \circ \hat{R}_j \circ f_j\right) \end{aligned} \tag{30}$$

Note that supp $\left[\hat{D}, \psi_j\right] \subset$ supp $(1 - \psi_j)$ and the last term in (30) vanishes by virtue of Condition 1. By the same condition, it follows that when evaluating the traces, the functions ψ_j may be omitted in the first three terms, so that

$$\begin{aligned} L\left(\hat{D}, \hat{T}\right) = {} & \text{“Trace”}\left(\hat{T}_1 \sum_{j=2}^{3} \hat{Q}'_j \circ f_j\right) - \text{“Trace”}\left(\hat{T}_2 \sum_{j=2}^{3} \hat{Q}_j \circ f_j\right) \\ & + \text{“Trace”}\left(\hat{T}_1 \sum_{j=1}^{3} \hat{R}_j \circ \left[\hat{D}, f_j\right]\right), \end{aligned} \tag{31}$$

where the quotes on the Trace indicate the fact that, even though the operator in question may not be in the trace class, we evaluate the trace integral of its kernel (of course, this integral is necessarily convergent). Note that

$$\left[\hat{D}, f_2(r, \lambda)\right] = -\left[\hat{D}, f_1(r/\lambda)\right] - \left[\hat{D}, f_3(r)\right],$$

where the supports of the terms on the right-hand side have an empty intersection, and so it is reasonable to rewrite (31) in the form

$$L\left(\hat{D}, \hat{T}\right) = \mathcal{L}_{\mathrm{smooth}}(\lambda, h) + \mathcal{L}_{\mathrm{cone}}(\lambda, h), \tag{32}$$

where

$$\begin{aligned} \mathcal{L}_{\mathrm{smooth}}(\lambda, h) = {} & \text{“Trace”}\left(\hat{T}_1 \sum_{j=2}^{3} \hat{Q}'_j(h) \circ f_j\right) - \text{“Trace”}\left(\hat{T}_2 \sum_{j=2}^{3} \hat{Q}_j(h) \circ f_j\right) \\ & + \text{“Trace”}\left(\hat{T}_1 \left(\hat{R}_3(h) - \hat{R}_2(h)\right) \left[\hat{D}, f_3\right]\right) \end{aligned} \tag{33}$$

and

$$\mathcal{L}_{\text{cone}}(\lambda, h) = \text{"Trace"} \left(\hat{T}_1 \left(\hat{R}_1 - \hat{R}_2(h)\right) \left[\hat{D}, f_1\left(\frac{r}{\lambda}\right)\right]\right). \tag{34}$$

From now on, we omit the quotes on the word "Trace".

To begin with, we shall evaluate $\mathcal{L}_{\text{smooth}}$. Since $a_0(0,\omega) \neq 0$, we can assume that $a_0(r,\omega) \neq 0$ for $r \leq \varepsilon$, $\omega \in \Omega$, and so g has no fixed points on the support of $f_1 + f_2$. Thus, all the contributions to the Lefschetz number from the smooth part of the manifold come from the terms containing $\hat{Q}_3$ and $\hat{Q}'_3$. The computation of these terms can be done by using in the definition of $\hat{R}_3$ a cutoff function depending on h similar to that introduced for $\hat{R}_2$ and then evaluating the integral by means of the stationary phase method. The answer is the standard contribution of the interior fixed points plus $O(h)$. We omit the computations and refer the reader to [21]. Let us consider those terms in $\mathcal{L}_{\text{smooth}}$ which involve $\hat{R}_2$. These are

$$\text{Trace}\left(\hat{T}_1 \hat{Q}'_2(h) f_2(r,\lambda)\right) - \text{Trace}\left(\hat{T}_2 \hat{Q}_2(h) f_2(r,\lambda)\right) \equiv I(h,\lambda) \tag{35}$$

and

$$\text{Trace}\left(\hat{T}_1 \left(\hat{R}_3(h) - \hat{R}_2(h)\right) \left[\hat{D}, f_3\right]\right) \equiv I_1(h).$$

Consider $I(h,\lambda)$. Recall that

$$\hat{Q}_2(h) = 1 - \hat{D}\hat{R}_2(h), \quad \hat{Q}'_2(h) = 1 - \hat{R}_2(h)\hat{D}. \tag{36}$$

Let

$$\hat{R}_2(h) = \hat{R}_1 \hat{\chi},$$

where $\hat{\chi} = \chi\left(h\hat{\xi}\right) = \chi(h\hat{p}, h\hat{q})$. Then

$$\begin{aligned} \hat{Q}_2 &= 1 - \hat{D}\hat{R}_1\hat{\chi} = 1 - \hat{\chi}, \\ \hat{Q}'_2 &= 1 - \hat{R}_1\hat{\chi}\hat{D} = 1 - \hat{R}_1\hat{D}\hat{\chi} + \hat{R}_1\left[\hat{D}, \hat{\chi}\right] \\ &= 1 - \hat{\chi} + \hat{R}_1\left[\hat{D}, \hat{\chi}\right]. \end{aligned} \tag{37}$$

The symbol $\chi(h\xi)$ will be chosen so that $1 - \hat{\chi}$, as well as $\hat{R}_1\left[\hat{D}, \hat{\chi}\right]$, will be smoothing operators for each fixed $h \in (0,1]$; then $\hat{R}_2(h)$ is indeed a regularizer on $\operatorname{supp} f_2$. However, since $1 - \hat{\chi}$ does not contain r as a factor, this property is lost as $\lambda \to 0$, and $I(h,0)$ is formally a difference of two divergent integrals. Therefore, we have to regularize the expression (35). We use the subscript "0" to indicate the operator with coefficients frozen at $r = 0$; for example,

$$\hat{D}_0 = \hat{D}\left(0, ir\frac{\partial}{\partial r}\right) = \hat{D}_0\left(ir\frac{\partial}{\partial r}\right), \quad \hat{T}_{i0} = \hat{T}_i\left(0, ir\frac{\partial}{\partial r}\right) = \hat{T}_{i0}\left(ir\frac{\partial}{\partial r}\right),$$

where $\hat{D}_0(p)$ and $\hat{T}_{i0}(p)$ are the conormal symbols of $\hat{D}$ and $\hat{T}_i$, respectively, etc. In particular, $\hat{R}_{10} = \hat{R}_0 = \hat{D}_0^{-1}$.

Lemma 3 *The following equation takes place:*

$$\text{Trace}\left(\hat{T}_{10}\hat{Q}'_{20}f_2\right) = \text{Trace}\left(\hat{T}_{20}\hat{Q}_{20}f_2\right). \tag{38}$$

Proof. We have

$$\begin{aligned}
\text{Trace}\left(\hat{T}_{10}\hat{Q}'_{20}f_2\right) &= \text{Trace}\left(\hat{T}_{10}\left(1-\hat{R}_0\hat{\chi}\hat{D}_0\right)f_2\right)\\
&= \text{Trace}\left(\hat{T}_{10}\left(\hat{R}_0\hat{D}_0-\hat{R}_0\hat{\chi}\hat{D}_0\right)f_2\right)\\
&= \text{Trace}\left(\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)\hat{D}_0f_2\right)\\
&= \text{Trace}\left(\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)\left[\hat{D}_0,f_2\right]\right)\\
&\quad+\text{Trace}\left(\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)f_2\hat{D}_0\right).
\end{aligned} \tag{39}$$

Now

$$\text{Trace}\left(\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)\left[\hat{D}_0,f_2\right]\right) = 0,$$

since all elements in this expression except for f_2 do not depend on r and f_2 is a compactly supported function, so that the integral of each of its derivatives vanishes (cf. [18]). We cyclically permute the factors in the second term:

$$\begin{aligned}
\text{Trace}\left(\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)f_2\hat{D}_0\right) &= \text{Trace}\left(\hat{D}_0\hat{T}_{10}\hat{R}_0\left(1-\hat{\chi}\right)f_2\right)\\
&= \text{Trace}\left(\hat{T}_{20}\hat{D}_0\hat{R}_0\left(1-\hat{\chi}\right)f_2\right)\\
&= \text{Trace}\left(\hat{T}_{20}\left(1-\hat{\chi}\right)f_2\right),
\end{aligned}$$

which proves the lemma, since $1-\hat{\chi} = \hat{Q}_{20}$.

Now we regularize the expression (35) as follows:

$$\begin{aligned}
I(h,\lambda) &= \text{Trace}\left(\left[\hat{T}_1\hat{Q}'_2(h)-\hat{T}_{10}\hat{Q}'_{20}(h)\right]f_2\right)\\
&\quad-\text{Trace}\left(\left[\hat{T}_1\hat{Q}_2(h)-\hat{T}_{10}\hat{Q}_{20}(h)\right]f_2\right).
\end{aligned} \tag{40}$$

From now on we work in the coordinates (t,ω). Let $Q\left(e^{-t},\omega,p,q,h\right)$ be the symbol of $\hat{Q}_2(h)$. Let also $Q'\left(e^{-t},\omega,p,q,h\right)$ be the symbol of $\hat{Q}'_2(h)$. Then

$$\begin{aligned} I(h,\lambda) &= \left(\frac{1}{2\pi}\right)^n \int \left\{ e^{i\left[a\left(e^{-t},\omega\right)p+\tilde{b}\left(e^{-t},\omega\right)q\right]} F\left(e^{-t},\omega,p,q,h\right) \right. \\ &- \left. e^{i\left[a(0,\omega)p+\tilde{b}(0,\omega)q\right]} F\left(0,\omega,p,q,h\right) \right\} f_2(t,\lambda)\, dt d\omega dp dq, \end{aligned} \tag{41}$$

where

$$\tilde{b}\left(e^{-t},\omega\right) = b\left(e^{-t},\omega\right) - \omega,$$

$$\begin{aligned} F\left(e^{-t},\omega,p,q,h\right) &= \operatorname{trace}\left[A_1\left(e^{-t},\omega\right) Q'_2\left(e^{-t+a\left(e^{-t},\omega\right)}, b\left(e^{-t},\omega\right), p, q, h\right)\right] \\ &- \operatorname{trace}\left[A_2\left(e^{-t},\omega\right) Q_2\left(e^{-t+a\left(e^{-t},\omega\right)}, b\left(e^{-t},\omega\right), p, q, h\right)\right] \end{aligned} \tag{42}$$

and trace stands for the trace of a matrix.

We can rewrite $I(h,\lambda)$ in the form

$$\begin{aligned} I(h,\lambda) &= \left(\frac{1}{2\pi}\right)^n \int_0^1 d\tau \int \exp\left\{ i\left[a\left(\tau e^{-t},\omega\right)p + \tilde{b}\left(\tau e^{-t},\omega\right)q\right]\right\} e^{-t} \\ &\times \left\{ \left(p\frac{\partial a}{\partial r} + q\frac{\partial \tilde{b}}{\partial r}\right) F\left(\tau e^{-t},\omega,p,q,h\right) + \frac{\partial F}{\partial r}\left(\tau e^{-t},\omega,p,q,h\right)\right\} \\ &\times f_2(t,\lambda)\, dt d\omega dp dq. \end{aligned} \tag{43}$$

Due to the presence of the factor e^{-t}, the integral $I(h,\lambda)$ converges as $\lambda \to 0$ to the integral of the same form with $f_2(t,\lambda)$ replaced by $1 - f_3(t)$. We denote this integral by $I(h)$.

Next, it is an easy exercise (cf. [18]) to show that

$$\mathcal{L}_{\text{cone}}(\lambda,h) \to \mathcal{L}_{\text{cone}}(h) = \frac{1}{2\pi i}\operatorname{Trace}\int \hat{T}_{10}(p)\,\hat{R}_0(p)\left(1-\hat{\chi}(p)\right)\frac{\partial \hat{D}_0}{\partial p}(p)\,dp \tag{44}$$

as $\lambda \to 0$. Summarizing, we have the following expression for the Lefschetz number:

$$\begin{aligned} L\left(\hat{D},\hat{T}\right) &= \frac{1}{2\pi i}\operatorname{Trace}\int \hat{T}_{10}(p)\,\hat{R}_0(p)\left(1-\hat{\chi}(p)\right)\frac{\partial \hat{D}_0}{\partial p}(p)\,dp \\ &+ I(h) + I_1(h) + \{\text{C.I.P}\} + O(h), \end{aligned}$$

where C.I.P. stands for the contribution of the interior fixed points written in the standard form. Now we shall choose $\hat{\chi}(h\xi)$ in a special way so as to

a) ensure that $I(h) = O(h)$ and $I_1(h) = O(h)$;

b) evaluate the first term, which gives the contribution of the conical point.

Namely, we set

$$\psi(\xi) = 1 - \chi(\xi) = \frac{1}{(1+p^N)(1+q^N)}, \tag{45}$$

where $N = 2N_1$ is a sufficiently large even number to be chosen later. Here, of course, for brevity we write

$$q^N = \left(q_1^2 + \ldots + q_{n-1}^2\right)^{N/2}. \tag{46}$$

The function

$$\chi(\xi) = \frac{p^N + q^N + p^N q^N}{(1+p^N)(1+q^N)} \tag{47}$$

possesses the following properties:

a) $\chi(\xi)$ has an Nth-order zero at the origin;

b) $\chi(\xi) = 1 + O\left(\left(1+p^N\right)^{-1}\left(1+q^N\right)^{-1}\right)$ as $|\xi| \to \infty$;

c) $\chi(\xi)$ is bounded with all its derivatives;

d) moreover, for $\alpha + |\beta| > 0$ the derivative $\chi^{(\alpha,\beta)}(\xi)$ has a zero of order $N - \alpha - |\beta|$ at the origin and is at least $O\left(\left(1+p^N\right)^{-1}\left(1+q^N\right)^{-1}\right)$ at infinity.

We rewrite

$$\begin{aligned} I(h) &= \left(\frac{1}{2\pi h}\right)^n \int_0^1 d\tau \int \exp\left\{\frac{i}{h}\left[a\left(\tau e^{-t}, \omega\right) p + \tilde{b}\left(\tau e^{-t}, \omega\right) q\right]\right\} e^{-t} \\ &\qquad \times U\left(e^{-t}, \omega, \frac{p}{h}, \frac{q}{h}, h\right)(1 - f_3(t))\, dt\, d\omega\, dp\, dq, \end{aligned}$$

where U is the expression in the braces in (43). Since $a\left(\tau e^{-t}, \omega\right) \neq 0$ on the support of the integrand, we can integrate by parts with respect to p. We then obtain $I(h) = O(h)$ provided that

$$\frac{\partial^{n+1}}{\partial p^{n+1}}\left(U\left(e^{-t}, \omega, \frac{p}{h}, \frac{q}{h}, h\right) e^{-t}(1 - f_3(t))\right)$$

is bounded uniformly in h by an integrable function of (p,q,t). It suffices to prove the same estimate for Q_2' and Q_2. Since

$$Q_2\left(e^{-t},\omega,\frac{p}{h},\frac{q}{h}\right)=\frac{1}{(1+p^N)(1+q^N)},$$

the desired estimate for Q_2 is guaranteed. Next, we have

$$Q_2'\left(e^{-t},\omega,p,q\right)=1-\chi(hp,hq)+\operatorname{smbl}\hat{R}_1\left[\hat{D},\hat{\chi}\right].$$

Only the last term needs special consideration. We have

$$\hat{R}_1\left[\hat{D},\hat{\chi}\right]=-\hat{R}_1\left[\hat{D},\hat{\psi}\right].$$

Let

$$\hat{A}=\left(1+(hp)^N\right)\left(1+(hq)^N\right)=\left[\psi\left(h\hat{\xi}\right)\right]^{-1}.$$

We have

$$\left[\hat{D},\psi\left(h\hat{\xi}\right)\right]=\left[\hat{D},\hat{A}^{-1}\right]=-\hat{A}^{-1}\left[\hat{D},\hat{A}\right]\hat{A}^{-1}=-\psi\left(h\hat{\xi}\right)\left[\hat{D},\hat{A}\right]\psi\left(h\hat{\xi}\right).$$

Next,

$$\begin{aligned}\left[\hat{D},\hat{A}\right] &= h^N\left[\hat{D},\hat{p}^N\right]+h^N\left[\hat{D},\hat{q}^N\right]+h^{2N}\left[\hat{D},\hat{q}^N\hat{p}^N\right]\\ &= h^N\left[\hat{D},\hat{p}^N\right]+h^N\left[\hat{D},\hat{q}^N\right]+h^{2N}\hat{q}^N\left[\hat{D},\hat{p}^N\right]+h^{2N}\left[\hat{D},\hat{q}^N\right]\hat{p}^N.\end{aligned}$$

To commute $\hat{p}^N$ and $\hat{q}^N$ with $\hat{D}$, we have in fact to commute them with the coefficients of $\hat{D}$. The commutators with the coefficients are differential operators, which we represent in such a form that multiplication by x acts *before* differentiation. Thus, we represent

$$\left[\hat{D},\hat{A}\right]=\Phi\left(\overset{3}{\overline{h\hat{\xi}}},\overset{2}{x},\overset{1}{\hat{\xi}}\right),$$

where $\hat{\xi}$ comprises the differentiations that originally were present in $\hat{D}$ and $h\hat{\xi}$ involves those arising in the commutators of $\hat{A}$ with the coefficients of $\hat{D}$. More precisely, $\left[\hat{D},\hat{A}\right]$ is a sum of terms of the form

$$\begin{aligned}h^j\left(\overset{3}{\overline{h\hat{p}}}\right)^{N-j}E\left(\overset{2}{x},\overset{1}{\hat{\xi}}\right),\ h^j\left(\overset{3}{\overline{h\hat{q}}}\right)^{N-j}E\left(\overset{2}{x},\overset{1}{\hat{\xi}}\right),\ j &= 1,\dots,N;\\ h^{j+k}\left(\overset{3}{\overline{h\hat{p}}}\right)^{N-j}\left(\overset{3}{\overline{h\hat{q}}}\right)^{N-k}E\left(\overset{2}{x},\overset{1}{\hat{\xi}}\right),\ j+k &= 1,\dots,N,\end{aligned}$$

where the $E\left(x, \hat{\xi}\right)$ are mth-order differential operators; these terms can be rewritten in the form

$$h^{-m} h^{j} \left(h\hat{p}\right)^{N-j} F\left(x, h\hat{\xi}, h\right), \text{ etc.,}$$

where the $F(x, \eta, h)$ are mth-order polynomials with coefficients regularly depending on h (in fact, they are homogeneous mth-order polynomials in (η, h)). Finally, we see that $\hat{R}_1 \left[\hat{D}, \psi\left(h\hat{\xi}\right)\right]$ is a sum of terms of the form[6]

$$\begin{aligned} S\left(\overset{2}{x}, \overset{1}{\hat{\xi}}, h\right) &= \left[\!\left[h^{-m} \hat{R}_1 \left(\overset{2}{x}, \overset{1}{\hat{\xi}}\right) \Xi\left(h, h\,\overset{1}{\hat{\xi}}\right) \right]\!\right] \\ &\quad \times \left[\!\left[F_1\left(\overset{2}{x}, \overset{1}{\overline{h\hat{\xi}}}, h\right) \psi\left(h\hat{\xi}\right) \right]\!\right] \\ &\equiv \left[\!\left[S_1\left(\overset{2}{x}, \overset{1}{\hat{\xi}}, h\right) \right]\!\right] \circ \left[\!\left[S_2\left(\overset{2}{x}, \overset{1}{\hat{\xi}}, h\right) \right]\!\right], \end{aligned}$$

where $\Xi(h, \xi)$ is one of the expressions

$$h^{j} p^{N-j} \psi(\xi), \; h^{j} q^{N-j} \psi(\xi), \text{ or } h^{j+k} p^{N-j} q^{N-k} \psi(\xi).$$

To estimate the p-derivatives of

$$S(x, \xi/h, h) = S(x, p/h, q/h, h),$$

we use the ψDO composition formula

$$S(x, \xi/h, h) = S_1\left(\overset{2}{x}, \frac{\xi - ih\,\overset{1}{\partial/\partial x}}{h}, h\right) S_2(x, \xi/h, h). \tag{48}$$

Let us make use of the following lemma.

Lemma 4 *Let $H_1(x,\xi)$ and $H_2(x,\xi)$ be arbitrary smooth functions of x and ξ. Then for any $M > 0$, there exists an $M_1 > 0$ such that the estimates*

$$\left| \frac{\partial^{\alpha+\beta} H_j(x,\xi)}{\partial x^{\alpha} \partial \xi^{\beta}} \right| \le C_{\alpha\beta} \left(1 + |\xi|\right)^{\nu_j - \beta}$$

[6]Here $[\![\cdot]\!]$ denote the so-called *autonomous* brackets. This mean that the order of action of the operators determined by the Feynman indices inside these brackets is valid only inside them; the bracketed expressions are then used as undividable entities. More detailed information about functions of noncommuting operators can be found, for example, in [11, 14].

for $|\alpha| = 0, 1, 2, \ldots,\ |\beta| = 1, 2, \ldots M_1,\ j = 1, 2,$ *imply the estimates*

$$\left|\frac{\partial^{\alpha+\beta} H(x,\xi)}{\partial x^\alpha \partial \xi^\beta}\right| \leq \tilde{C}_{\alpha\beta} (1 + |\xi|)^{\nu_1+\nu_2-\beta}$$

for $|\alpha| = 0, 1, 2, \ldots,\ |\beta| = 1, 2, \ldots M,$ *where*

$$H(x,\xi) = H_1\left(\overset{2}{x}, \overline{\xi - ih\frac{\partial}{\partial x}}^{\,1}\right) H_2(x,\xi)$$

(note that M_1 *is independent of* ν_1 *and* ν_2*).*

The proof is a standard exercise in the spirit of the well-known Kohn and Nirenberg paper [10], and we omit it altogether.

Let us apply this lemma to the estimate (48). We have, say,

$$S_1\left(x, \frac{\xi}{h}, h\right) = h^{-m} R_1\left(x, \frac{\xi}{h}\right) \frac{h^j p^{N-j} h^k q^{N-k}}{(1+p^N)(1+q^N)}. \tag{49}$$

Then

$$\begin{aligned} \left|S_1\left(x, \frac{\xi}{h}, h\right)\right| &\leq Ch^{-m}\left(1 + \left|\frac{p}{h}\right| + \left|\frac{q}{h}\right|\right)^{-m} \frac{h^j p^{N-j} h^k q^{N-k}}{(1+|p|)^N (1+|q|)^N} \\ &= C\frac{h^j p^{N-j} h^k q^{N-k}}{(1+|p|+|q|)^m (1+|p|)^N (1+|q|)^N} \\ &\leq C\frac{1}{(1+|p|+|q|)^m} \end{aligned} \tag{50}$$

if $2N \geq m$. To obtain this, rather coarse estimate, one uses the inequalities

$$\frac{|p|}{h+|p|+|q|} \leq 1, \quad \frac{|q|}{h+|p|+|q|} \leq 1, \quad \frac{h}{h+|p|+|q|} \leq 1,$$

of which the first two inequalities are applied alternately as many times as possible and the third is applied when no factors p and q remain in the numerator; then we use the inequality

$$(1+|p|)^{-1}(1+|q|)^{-1} \leq (1+|p|+|q|)^{-1}.$$

For the other two variants of S_1, the estimate (50) is valid provided that $N \geq m$. By differentiating S_1 and using a similar technique, we obtain

$$\left|\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial\xi^{\beta}}\left[S_1\left(x,\frac{\xi}{h},h\right)\right]\right|\le C_{\alpha\beta}\left(1+|p|+|q|\right)^{-m} \tag{51}$$

for any α and for $|\beta|\le N$. Next,

$$S_2\left(x,\xi/h,h\right)=F_1\left(x,\xi,h\right)\left(1+p^N\right)^{-1}\left(1+q^N\right)^{-1},$$

and so

$$\left|\frac{\partial^{\alpha+\beta}}{\partial x^{\alpha}\partial\xi^{\beta}}\left[S_2\left(x,\frac{\xi}{h},h\right)\right]\right|\le \tilde{C}_{\alpha\beta}\left(1+|p|+|q|\right)^{-N+m} \tag{52}$$

for any α and β. Passing to even coarser estimates, we can write, instead of (51) and (52),

$$\left|\partial_x^{\alpha}\partial_p^{\beta}S_1\right|\le C_{\alpha\beta}\left(1+|p|+|q|\right)^{N/3-m-|\beta|},\ 0\le|\beta|\le\frac{N}{3}, \tag{53}$$

and

$$\left|\partial_x^{\alpha}\partial_p^{\beta}S_2\right|\le C_{\alpha\beta}\left(1+|p|+|q|\right)^{-2N/3+m-|\beta|},\ 0\le|\beta|\le\frac{N}{3}. \tag{54}$$

By choosing N sufficiently large and applying Lemma 4, we ensure uniform integrable bounds for as many derivatives of $S\left(x,\xi/h,h\right)$ as desired.

Now let us show that the integral

$$I(h)=\operatorname{Trace}\left\{\hat{T}_1\left(\hat{R}_3-\hat{R}_2\right)\left[\hat{D},f_3\right]\right\}$$

can also be evaluated by the stationary phase method (the same argument then applies to any similar operators arising in the smooth part of the manifold). We have $\hat{R}_2=\hat{R}_1\chi\left(h\hat{\xi}\right)$, where

$$\chi\left(\xi\right)=1-\psi\left(\xi\right)=1-\frac{1}{\left(1+p^N\right)\left(1+q^N\right)},$$

N is even, and $\hat{R}_1$ is a regularizer independent of h. Likewise, assume that $\hat{R}_3=\hat{R}\chi_1\left(h\xi\right)$, where $\hat{R}$ is a regularizer independent of h and $\chi_1\left(h\xi\right)$ has the following two properties[7] (as well as $\chi\left(h\xi\right)$):

[7]For brevity, we in fact discuss a model situation: since local regularizers are usually constructed in different coordinate systems, at least one of the functions χ and χ_1 will depend on coordinates after the change of variables. This makes the derivation of the estimates even more awkward. Moreover, to make the estimates global on Ω, by q^N one must understand the symbol of the $(N/2)$th power of the positive Laplace–Beltrami operator on Ω.

(i) $\chi_1(\xi)$ has a zero of high order at the origin;

(ii) $\psi_1(\xi) = 1 - \chi_1(\xi)$ decays at infinity at least as fast as $(1+|\xi|)^{-N}$ together with all derivatives, where N is sufficiently large.

Since $\hat{R}_1$ and $\hat{R}$ are both regularizers of the same operator $\hat{D}$, it follows that the difference $\hat{R}_1 - \hat{R}$ is a pseudodifferential operator of high negative order. By the cyclic invariance of the trace, we have

$$I = \operatorname{Trace}\left\{\hat{T}_1\left(\hat{R}_3 - \hat{R}_2\right)\left[\hat{D}, f_3\right]\right\} = \operatorname{Trace}\left\{\left[\hat{D}, f_3\right]\hat{T}_1\left(\hat{R}_3 - \hat{R}_2\right)\right\}$$

and, after some computations,

$$I = \left(\frac{1}{2\pi h}\right)^n \int\int e^{\frac{i}{h}\xi(g(x)-x)}\left[F\left(x, \frac{\xi}{h}\right)\chi(\xi) - F_1\left(x, \frac{\xi}{h}\right)\chi_1(\xi)\right] dx d\xi,$$

where F and F_1 are classical symbols of order -1 whose difference is a classical symbol of high negative order $-M$. We have

$$\begin{aligned} F\left(x, \frac{\xi}{h}\right)\chi(\xi) - F_1\left(x, \frac{\xi}{h}\right)\chi_1(\xi) &= \left\{F\left(x, \frac{\xi}{h}\right) - F_1\left(x, \frac{\xi}{h}\right)\right\}\chi(\xi) \\ &\quad + F_1\left(x, \frac{\xi}{h}\right)\left(\chi(\xi) - \chi_1(\xi)\right). \end{aligned} \tag{55}$$

Let us consider the derivatives of the first term. We have (the terms in which χ is differentiated can be estimated in a similar way)

$$\left|\left[F^{(\alpha,\beta)}\left(x, \frac{\xi}{h}\right) - F_1^{(\alpha,\beta)}\left(x, \frac{\xi}{h}\right)\right]\chi(\xi)\right| \le Ch^{-|\beta|}\left(1 + \frac{\xi}{h}\right)^{-M-|\beta|}|\chi(\xi)|$$

$$= \frac{Ch^M}{(h+|\xi|)^{M+|\beta|}}|\chi(\xi)| \le \begin{cases} C_1, \ |\xi| \le 1, \\ C_2(1+|\xi|)^{-M-|\beta|}, \ |\xi| > 1, \end{cases}$$

provided that the order of zero of $\chi(\xi)$ at the origin is at least $|\beta|$. The second term can be estimated separately for $|\xi| \le 1$ and $|\xi| > 1$:

$$\begin{aligned} \left|F_1\left(x, \frac{\xi}{h}\right)\left(\chi(\xi) - \chi_1(\xi)\right)\right| &\le \left|F_1\left(x, \frac{\xi}{h}\right)\chi(\xi)\right| + \left|F_1\left(x, \frac{\xi}{h}\right)\chi_1(\xi)\right| \\ &\le C\left(1 + \left|\frac{\xi}{h}\right|\right)^{-1}\left(|\chi(\xi)| + |\chi_1(\xi)|\right) \le \text{const} \end{aligned}$$

for $|\xi| < 1$ (similar estimates for the derivatives use the fact that χ and χ_1 vanish to a high order at zero). Moreover,

$$\left| F_1^{(\alpha,\beta)}\left(x, \frac{\xi}{h}\right)\left(\chi\left(\xi\right) - \chi_1\left(\xi\right)\right)\right| = \left| F_1^{(\alpha,\beta)}\left(x, \frac{\xi}{h}\right)\left(\psi\left(\xi\right) - \psi_1\left(\xi\right)\right)\right|$$
$$\leq \frac{Ch^{-|\beta|}}{\left(1 + \frac{|\xi|}{h}\right)^{1+|\beta|}}\left(1 + |\xi|\right)^{-N-|\beta|} \leq \tilde{C}h\left(1 + |\xi|\right)^{-N-|\beta|}$$

for $|\xi| \geq 1$.

The proof of Theorem 1 is complete.

It remains to prove Theorem 2. The contribution of the conical point is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{con}} &= \lim_{h\to 0} \operatorname{Trace} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \hat{T}_{10}\left(p\right) \hat{R}_0\left(p\right)\left(1 - \hat{\chi}\left(p\right)\right) \frac{\partial \hat{D}_0}{\partial p}\left(p\right)\, dp \\ &= \lim_{h\to 0} \operatorname{Trace} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \hat{T}_{10}\left(p\right) \hat{R}_0\left(p\right) \frac{1}{1 + \left(h\hat{q}\right)^N} \frac{1}{1 + h^N p^N} \\ &\qquad \times \frac{\partial \hat{D}_0}{\partial p}\left(p\right)\, dp = \lim_{h\to 0} \mathcal{L}_{\mathrm{con}}\left(h\right). \qquad (56)\end{aligned}$$

To be definite, assume that $a\left(0, \omega\right) > 0$. Let $a = \min_{\omega} a\left(0, \omega\right)$. Since

$$\hat{T}_{10}\left(p\right) = A_{10}\left(\overset{2}{\omega}\right) e^{ipa\left(0,\overset{2}{\omega}\right)} e^{i\tilde{b}\left(0,\overset{2}{\omega}\right)\left(-i\overset{1}{\partial/\partial\omega}\right)},$$

it follows that

$$\left\|\frac{\partial^\alpha \hat{T}_{10}}{\partial p^\alpha}\left(p\right)\right\|_{H^s(\Omega)\to H^s(\Omega)} \leq C_{\alpha,s} e^{-a\operatorname{Im} p},\ \operatorname{Im} p > 0,\ \alpha = 1, 2, \ldots$$

for any s. By virtue of the growth conditions imposed on the conormal symbol, we can calculate the integral (56) by the residue theorem as the sum of residues of the integrand in the upper half-plane. We obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{con}}\left(h\right) &= \sum_{\operatorname{Im} p_j > 0} \operatorname{Trace} \operatorname{Res}_{p_j} \hat{T}_{10}\left(p\right) \hat{R}_0\left(p\right) \frac{1}{1 + \left(h\hat{q}\right)^N} \frac{1}{1 + h^N p^N} \\ &\qquad \times \frac{\partial \hat{D}_0}{\partial p}\left(p\right) + \operatorname{Trace} \sum_{l=0}^{N/2-1} \hat{T}_{10}\left(\frac{\zeta_l}{h}\right) \hat{R}_0\left(\frac{\zeta_l}{h}\right) \frac{1}{1 + \left(h\hat{q}\right)^N}\end{aligned}$$

$$\times \frac{1}{h\prod_{s\neq l}(\zeta_s-\zeta_l)}\frac{\partial \hat{D}_0}{\partial p}\left(\frac{\zeta_l}{h}\right), \tag{57}$$

where the ζ_l, $l=0,\dots,N/2-1$, are the $N/2$th-order roots of -1 in the upper half-plane.

The sum (57) converges by virtue of the condition imposed on the conormal symbol. Moreover, the second sum, which contains $N/2$ terms, tends to zero as $h\to 0$ along some subsequence since

$$\left\|\hat{T}_{10}\left(\frac{\zeta_{hl}}{h}\right)\right\| \le C_\alpha \exp\left(-\frac{a}{h}\operatorname{Im}\zeta_l\right) \le C\exp\left(-\frac{a\sin\frac{\pi}{N}}{h}\right)$$

and by virtue of the polynomial growth condition imposed on $\hat{R}_0(p)=\hat{D}_0^{-1}(p)$. Furthermore, the first sum tends to the contribution indicated in the statement of the theorem. The proof is complete.

5 Examples

Now let us consider some examples. In all our examples, $\hat{D}$ will be the Laplace operator

$$\hat{D}=\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial\varphi^2} \;:\; H^{s,\gamma_1,\gamma_2}(S)\to H^{s-2,\gamma_1,\gamma_2}(S) \tag{58}$$

acting in the one-dimensional trivial bundle on the surface of the circular spindle

$$S=\left\{\left\{[-\infty,+\infty]\times S^1\right\}/\left[\{-\infty\}\times S^1\right]\right\}/\left[\{+\infty\}\times S^1\right].$$

Here the function spaces $H^{s,\gamma_1,\gamma_2}(S)$ are defined by the norm

$$\|u\|^2_{s,\gamma_1,\gamma_2}=\int_{-\infty}^{+\infty}\int_0^{2\pi}\psi_{\gamma_1,\gamma_2}(t)\left|\left(1+\left(-i\frac{\partial}{\partial\varphi}\right)^2+\left(-i\frac{\partial}{\partial t}\right)^2\right)^{s/2}u\right|^2 d\varphi dt,$$

where $\psi_{\gamma_1,\gamma_2}(t)$ is a smooth function such that

$$\psi_{\gamma_1,\gamma_2}(t)=\begin{cases} e^{-2\gamma_1 t}, & t\ll -1,\\ e^{-2\gamma_2 t}, & t\gg 1,\end{cases}$$

and (t,φ) are the natural coordinates on S.

Note that in view of the definition of Sobolev spaces $H^{s,\gamma}(M)$ for a manifold M with conical singularities, we have

$$H^{s,\gamma_1,\gamma_2}(S) = H^{s,\gamma}(S),$$

where

$$\gamma(-\infty) = \gamma_1 \text{ and } \gamma(+\infty) = -\gamma_2.$$

We suppose that the *conormal symbol* of the operator (58), that is, the analytic family

$$\hat{D}_0\,(p) = -p^2 + \frac{d^2}{d\varphi^2} \tag{59}$$

of operators on the unit circle, is invertible on the "weight lines" $\operatorname{Im} p = \gamma_j$, $j = 1, 2$.

As was shown in [19] [20] (see also [16]), under these conditions the operator (58) is an epimorphism with nontrivial kernel for $\gamma_1 < \gamma_2$, an isomorphism for $\gamma_1 = \gamma_2$, and a monomorphism with nontrivial cokernel for $\gamma_1 > \gamma_2$. For brevity, we consider only the first case.

Let us consider the kernel of the operator (58) in the case $\gamma_1 < \gamma_2$ in more detail. First, note that the poles of the inverse of (59) are

$$p_k = k \in \mathbf{Z},$$

and the kernels of the operators $D_0\,(p_k)$ are

$$\begin{aligned} L_k &= \left\{u = a_k e^{ik\varphi} + b_k e^{-ik\varphi}\right\},\ k \neq 0, \\ L_0 &= \{u = c\}\,. \end{aligned}$$

The kernel of the operator (58) is

$$\operatorname{Ker} \hat{D} \equiv \operatorname{Ker} \hat{D}_{\gamma_1,\gamma_2} = \bigcup_{\gamma_1 < k < \gamma_2} e^{kt} L_k.$$

To simplify the notation, we suppose that $0 \notin (\gamma_1, \gamma_2)$.

Let us now consider three different mappings of S.

5.1 All points of the conormal mapping are fixed

This situation is realized for the mapping

$$g = \begin{cases} t' = t + c, \\ \varphi' = \varphi, \end{cases}$$

$c > 0$. It is easy to see that for such a mapping the singular point $t = -\infty$ is repulsive and the singular point $t = +\infty$ is attractive. Moreover, the mapping $g^\wedge$ is the identity mapping for each of the two singular points of the manifold, and so *all* the points of the circle S^1 are fixed for these mappings.

Straightforward computation of the Lefschetz number shows that the action of $\hat{T}_1$ on the components $e^{kt}L_k$ of the kernel of the operator (58) is given by the matrix

$$\begin{pmatrix} e^{kc} & 0 \\ 0 & e^{kc} \end{pmatrix},$$

and hence, the Lefschetz number of the considered diagram is

$$L\left(\hat{D}, \hat{T}^{\cdot}\right) = 2 \sum_{\gamma_1 < k < \gamma_2} e^{kc}. \tag{60}$$

Let us evaluate the Lefschetz number using our theorems.

To compute the contribution of fixed points, we use the standard cylindrical coordinates in a neighborhood of each point. For the point $+\infty$, these coordinates are (t, φ). Accordingly, the conormal symbol of $\hat{T}$ is

$$\hat{T}_0(p) = e^{ipc},$$

the conormal symbol of $\hat{D}$ is

$$\hat{D}_0(p) = -p^2 + \frac{d^2}{d\varphi^2}$$

and we obtain

$$\mathcal{L}_{\mathrm{cone},+} = \sum_{\operatorname{Im} p_j > -\gamma_2} \operatorname{Trace} \operatorname{Res}_{p_j} \frac{2e^{ipc}p}{p^2 - \frac{d^2}{d\varphi^2}}. \tag{61}$$

(Recall that the weight exponent of the point $+\infty$ is $-\gamma_2$.) We can rewrite (61) in the form

$$\mathcal{L}_{\mathrm{cone},+} = \sum_{k > -\gamma_2} \operatorname{Res}_{p=ik} \frac{2e^{ipc}p}{p^2 + k^2} \operatorname{Trace} P_{k^2},$$

where P_{k^2} is the projection on the subspace of eigenfunctions of $-\frac{d^2}{d\varphi^2}$ with eigenvalue k^2. Thus,

$$\mathcal{L}_{\mathrm{cone},+} = \sum_{k > -\gamma_2} e^{-kc} \cdot \operatorname{Trace} P_{k^2} = \sum_{k < \gamma_2} e^{kc} \operatorname{Trace} P_{k^2}$$

For the point $-\infty$ the weight exponent is γ_1 and the standard cylindrical coordinates are $(\tilde{t} = -t, \varphi)$. Accordingly,

$$\hat{T}_0(p) = e^{-ipc}, \quad \hat{D}_0(p) = -p^2 + \frac{d^2}{d\varphi^2},$$

and

$$\begin{aligned}
\mathcal{L}_{\text{cone},-} &= -\sum_{\operatorname{Im} p_j < \gamma_1} \operatorname{Trace} \operatorname{Res}_{p_j} \frac{2e^{-ipc}p}{p^2 - \frac{d^2}{d\varphi^2}} \\
&= -\sum_{k<\gamma_1} \operatorname{Res}_{p=ik} \frac{2e^{-ipc}p}{p^2+k^2} \operatorname{Trace} P_{k^2} \\
&= -\sum_{k<\gamma_1} e^{kc} \operatorname{Trace} P_{k^2}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathcal{L}_{\text{cone},-} + \mathcal{L}_{\text{cone},+} &= \left(\sum_{k<\gamma_2} - \sum_{k<\gamma_1} \right) e^{kc} \operatorname{Trace} P_{k^2} \\
&= \sum_{\gamma_1 < k < \gamma_2} e^{kc} \operatorname{Trace} P_{k^2} = 2 \sum_{\gamma_1 < k < \gamma_2} e^{kc},
\end{aligned}$$

since, by our assumptions, $0 \notin (\gamma_1, \gamma_2)$. This is just the answer (60).

5.2 The conormal mapping has no fixed points

It is even more remarkable that we can deform the mapping considered in the previous example so that the conormal mapping has no fixed points and the Lefschetz number remains unvanishing and, moreover, tends to that computed in the previous example. To show this, we consider the mapping

$$g = \begin{cases} t' = t + c, \\ \varphi' = \varphi + \varphi_0 \end{cases}$$

for some value of φ_0. In this case, the corresponding conormal mapping is simply the rotation by φ_0 on the unit circle and hence has no fixed points except for the cases $\varphi_0 = 2\pi k$, for which it coincides with the mapping considered in the previous example. Nevertheless, the action of the operator $\hat{T}_1$ on the components $e^{kt}L_k$ of the kernel of the operator (58) is given by the matrix

$$\begin{pmatrix} e^{k(c+i\varphi_0)} & 0 \\ 0 & e^{k(c-i\varphi_0)} \end{pmatrix},$$

and the Lefschetz number is given by

$$L\left(\hat{D},\hat{T}^{\cdot}\right)=2\sum_{\gamma_1<k<\gamma_2}e^{kc}\cos\left(k\varphi_0\right).$$

This number does not vanish except for special values of φ_0 and tends to the Lefschetz number of Example 1 as $\varphi_0 \to 0$. The same answer is given by Theorem 2; we leave the computations to the reader, as well as in the following example.

5.3 The conormal mapping has discrete fixed points

To illustrate this situation, we consider the mapping

$$g=\begin{cases} t'=t+c, \\ \varphi'=-\varphi. \end{cases}$$

Then the conormal mapping for each of the two singular points of the considered manifold has exactly two fixed points, $\varphi=0$ and $\varphi=\pi$. Computations similar to those in the previous subsections lead us to the following matrix representation of the action of $\hat{T}_1$ on the components $e^{kt}L_k$ of the kernel of the operator (58):

$$\hat{T}_1\Big|_{e^{kt}L_k}=\begin{pmatrix} 0 & e^{kc} \\ e^{kc} & 0 \end{pmatrix};$$

hence,

$$L\left(\hat{D},\hat{T}^{\cdot}\right)=0.$$

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