Operator Algebras on Singular Manifolds. IV, V

B.-W. Schulze, B. Sternin, and V. Shatalov

Preprint 97/??
June 1997

Universität Potsdam
Institut für Mathematik
PF 60 15 53
14415 Potsdam

Bert-Wolfgang Schulze,
Boris Sternin, and Victor Shatalov

OPERATOR ALGEBRAS
on
SINGULAR MANIFOLDS

Potsdam 1997

Professor Bert-Wolfgang Schulze
Potsdam University
E-mail:schulze@mpg-ana.uni-potsdam.de

Professors Boris Sternin and Victor Shatalov
Moscow State University
E-mail: sternin@mpg-ana.uni-potsdam.de

This text is a working version Chapters I, II, Part 2 of the book "Operator Algebras on Singular Manifolds" by B.-W. Schulze, B. Sternin, and V. Shatalov to be published in 1998 in Akademie Verlag.

Contents

Chapter I

Structure rings on manifolds with singularities

The material of this chapter is an extended version of the author's works [9].

1.1 Singularities of a point type

In this section, we shall describe the types of singular points on a manifold with singularities with the help of *structure rings* of these singularities. As it was explained in the Introduction, such a description is important since it expresses the type of the considered singular point in the purely interior terms, and since it defines natural classes of operators to be considered with each type of singularity. Here we consider singularities of the point type only (that is, we suppose that the set of singular points is a finite subset of the manifold M in question). As it was mentioned above, all other types of singularity can be obtained by operations of multiplying by smooth compact manifold and of taking a cone over some existing manifold with singularities. These operations will be considered in the second section of this chapter.

1.1.1 General considerations

Local rings

Let M be a compact topological space with a finite number of marked points $\{\alpha_1, \dots, \alpha_N\}$ (these points will be referred below as the *singular points* of the manifold M).

Definition 1 The space M is called a *smooth manifold with singularities at points* $\{\alpha_1, \dots, \alpha_N\}$ if:

- The space $M \setminus \{\alpha_1, \dots, \alpha_N\}$ possesses the structure of C^∞-manifold.
- For each singular point α_j, $j = 1, \dots N$ there exists a neighborhood U_j homeomorphic to a compactified half-cylinder

$$\overline{C}\,(\Omega) = \{\Omega \times [0, +\infty]\} \,/\, \{\Omega \times \{+\infty\}\} \tag{1.1}$$

  with the base Ω being a compact smooth manifold. It is supposed that smooth structures on manifolds $U_j \setminus \alpha_j$ and

$$C\,(\Omega) = \{\Omega \times [0, +\infty)\} \tag{1.2}$$

  coincide with each other.

- This manifold is supplied with a structure ring $\mathcal{R}_j$ of a singularity for each singular point α_j of the manifold (the notion of a structure ring of singularity will be explained below).

Let us consider in detail the notion of a (local) structure ring of a singularity near a singular point of the manifold. Similar to the fact that differential structure of a C^∞-manifold is uniquely determined by the structure sheaf of this manifold (that is, a local ring of C^∞-functions given at each point of the underlying manifold), the type of singularity at each point α_j of the manifold M is determined by a ring of "smooth functions" in a neighborhood of this point. Since the differential structure on the whole half-cylinder (1.1) except for its infinite point is a standard one, this ring must contain functions smooth on the whole half-cylinder $C\,(\Omega)$, and the only thing determining the type of the considered singular point is the behavior of these functions at infinity. To describe this behavior, we introduce special coordinates (t, ω) on $C\,(\Omega)$ defined by representation (1.2). Namely, ω are local coordinates on Ω, and $t \in [0, 1)$. More precisely, we shall consider local rings of functions stabilizing with some speed at infinity. To describe the speed of stabilization, we introduce the following definition:

Definition 2 The strictly positive function $\varphi\,(t) \in C^\infty\,(\mathbf{R}_+)$ such that

$$\lim_{t \to +\infty} \varphi\,(t) = 0$$

will be called a *weight function.*

Now, functions which stabilize with the speed determined by a weight function $\varphi(t)$ can be described as

$$f(t,\omega) = F(\varphi(t),\omega), \tag{1.3}$$

with $F(\tau,\omega)$ smooth up to $\tau = 0$. Clearly, the set of functions of the form (1.3) forms a ring with respect to the usual multiplication. However, this ring is not closed under the differentiation operation (this requirement is quite essential for the structure ring of a singular point) and, hence, the above definitions have to be modified.

So, we can describe the notion of a structure ring of the singularity in the following way:

Definition 3 The subring $\mathcal{R}^{\varphi} \subset C^{\infty}(C(\Omega))$ is called a *structure ring of a singularity of weight* $\varphi(t)$ if:

- this ring is closed with respect to differentiation;
- for any element $f(t,\omega) \in \mathcal{R}^{\varphi}$ there exists a limit
$$f_{\infty}(\omega) = \lim_{t\to+\infty} f(t,\omega);$$
- for every positive ε there exists a $C_{\varepsilon} > 0$ such that the estimate
$$|f(t,\omega) - f_{\infty}(\omega)| \le C_{\varepsilon}[\varphi(t)]^{1-\varepsilon}$$
is valid.

In the case when the weight function $\varphi(t)$ satisfies some additional requirements, one can describe the ring $\mathcal{R}^{\varphi}$ in terms of this function. Namely, suppose that successive logarithmic derivatives

$$\varphi_0(t) = \varphi(t), \ \varphi_j(t) = \varphi_{j-1}^{-1}(t)\,\varphi_{j-1}'(t), \ j = 1, 2, \ldots$$

satisfy the following conditions:

- there exist a number $j_0 \ge 0$ such that $\varphi_{j_0+1}(t) = F(t^{-1})$ with function $F(\tau)$ regular up to the origin;
- the estimates
$$|\varphi(t)\,\varphi_j(t)| \le C_{\varepsilon}[\varphi(t)]^{1-\varepsilon}, \ j = 1, \ldots, j_0$$
are valid for any positive ε with some constant C_{ε}.

In the case when the two above requirements are fulfilled we refer the function $\varphi(t)$ as a *weight function* with a finite depth j_0.

The following affirmation takes place:

Proposition 1 *Suppose that the weight function $\varphi(t)$ satisfies the above requirements. Then the set of functions of the form*

$$f(t,\omega) = F(\varphi(t),\omega) + \varphi(t) P_N[\varphi_1(t),\ldots,\varphi_{j_0}(t)] G\left(t^{-1},\varphi(t),\omega\right),$$

where P_N is a polynomial of its arguments with smooth in ω coefficients and the functions $F(\tau_1,\omega)$ and $G(\tau_0,\tau_1,\omega)$ are smooth up to $\tau_0 = 0$, $\tau_1 = 0$, form a structure ring of a singularity of weight $\varphi(t)$.

The proof of this proposition is quite easy and we leave it to the reader.

Below, we shall consider different types of singularities with weight functions of depth at most one. Note, that both conical and cuspidal singularities are of this type.

Differential operators and Riemannian metrices

Note that the above defined structure rings allow to define such important notions on manifolds with singularities as the *(local) rings of differential operators* and the class of *Riemannian metrices* compatible with the structure of a manifold with singularities of a given type (and, hence, the corresponding geometrical operators). Naturally, differential operators on a manifold with singularities must have the following form near a singular point:

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha|\le m-j} a_{j\alpha}(t,\omega) \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial\omega}\right)^\alpha \tag{1.4}$$

with $a_{j\alpha}(t,\omega)$ from the corresponding local ring $\mathcal{R}^\varphi$ (here, as usual, m is an order of differential operator, and α is a multiindex). We shall give the another treatment of the set of operators of the form (1.4) connected with the representation of the singular point as a cone over the manifold Ω. To do this, we perform the variable change

$$r = \varphi(t). \tag{1.5}$$

Since

$$\frac{\partial}{\partial t} = \varphi'(t)\frac{\partial}{\partial r} = \phi(r)\frac{\partial}{\partial r}$$

with $\phi(r) = \varphi'(\varphi^{-1}(r))$, we arrive at the following form of the differential operators near the singular point under consideration:

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha| \le m-j} b_{j\alpha}(r,\omega) \left(\phi(r) \frac{\partial}{\partial r}\right)^j \left(\frac{\partial}{\partial \omega}\right)^\alpha, \tag{1.6}$$

where the regularity of coefficients $b_{j\alpha}(r,\omega)$ at the point $r = 0$ is determined by the corresponding local ring $\mathcal{R}^\varphi$. Note that since variable change (1.5) transforms the half-line $\overline{\mathbf{R}}_+$ into the segment $[0,1]$ (the infinite point being taken to the origin), the infinite cylinder C is transformed into the cone

$$K = ([0,1] \times \Omega) / (\{0\} \times \Omega). \tag{1.7}$$

We shall refer representation (1.4) as *"cylindrical"* representation of differential operators, and (1.7) as *"conical"* representation.

Let us focus now on the consideration of the Riemannian metric compatible with the given type of a singular point. It is clear that the general form of such a metric have to be

$$ds^2 = c(t) \left[g_{00}(t,\omega)\, dt^2 + \sum_{j=1}^{n} g_{0j}(t,\omega)\, dt d\omega^j + \sum_{j,l=1}^{n} g_{jl}(t,\omega)\, d\omega^j d\omega^l \right],$$

where the coefficients g_{jl} are taken from the corresponding local ring, and $c(t)$ is a conformal factor chosen in such a way that the distance from any point of the manifold M to the considered singular point is finite. This latter can be chosen in the following way.

Performing variable change (1.5), we transform the metric to the "conical form"

$$\begin{aligned} ds^2 \;\; = \;\; & c(t) \Bigg[(\varphi'(t))^{-2} g_{00}(t,\omega)\, dr^2 + (\varphi'(t))^{-1} \sum_{j=1}^{n} g_{0j}(t,\omega)\, dt d\omega^j \\ & + \sum_{j,l=1}^{n} g_{jl}(t,\omega)\, d\omega^j d\omega^l \Bigg] \Bigg|_{t=\varphi^{-1}(r)} \end{aligned}$$

and the required generalization can be achieved by the choice

$$c(t) = (\varphi'(t))^2 = \phi^2(r).$$

So, in the "conical form" we arrive at the following class of typical metrices:

$$ds^2 = \left[g_{00}(r,\omega)\, dr^2 + \phi(r) \sum_{j=1}^{n} g_{0j}(r,\omega)\, dt d\omega^j + \phi^2(r) \sum_{j,l=1}^{n} g_{jl}(r,\omega)\, d\omega^j d\omega^l \right],$$

where the regularity of coefficients $g_{jl}(r,\omega)$ is determined by the corresponding local ring $\mathcal{R}^{\varphi}$. The most important case of the metrices to be used near conical point are direct products of the standard metric dr^2 on $[0,1]$ and a metric g_Ω on the base of the cone possibly dependent on r:

$$ds^2 = dr^2 + \phi^2(r)\, g_\Omega(r), \tag{1.8}$$

where the behavior of the metric $g_\Omega(r)$ near $r=0$ is determined by the corresponding local ring.

Let us consider the Beltrami-Laplace operator corresponding to metric (1.8). It is easy to see that this operator has the form

$$\Delta_g = \frac{1}{\phi^2(r)}\left[\left(\phi(r)\frac{\partial}{\partial r}\right)^2 + \Delta_{g_\Omega}\right],$$

where Δ_{g_Ω} is the Beltrami-Laplace operator corresponding to the metric g_Ω on the base Ω of the corresponding cylinder (cone). Note that this operator differs from operators of the type (1.6) by the "conformal" factor $\phi^{-2}(r)$. Hence, this is also natural to consider the ring of differential operators of the form

$$\hat{H} = \phi^{-m}(r)\sum_{j=0}^{m}\sum_{|\alpha|\le m-j} b_{j\alpha}(r,\omega)\left(\phi(r)\frac{\partial}{\partial r}\right)^j\left(\frac{\partial}{\partial\omega}\right)^\alpha, \tag{1.9}$$

where m is an order of the operator $\hat{H}$. The operators of the type (1.9) also form a ring which can be viewed as a structure ring of the corresponding singularity.

Remark that in the investigation of solutions to an equation of the form $\hat{H}u = f$ the factor $\phi^{-m}(r)$ is inessential. However, in the construction of structure rings for non-point singularities this factor leads to the essential modifications of the obtained rings.

Note also that the operators of the form (1.9) can be written down as the operators on the half-axis $\mathbf{R}_+$

$$\hat{H} = \phi^{-m}(r)\,\hat{H}\left(r, \phi(r)\frac{d}{dr}\right) \tag{1.10}$$

with symbols $\hat{H}(r,p)$ having their values in the ring of differential operators on Ω. Clearly, the dependence of the symbols on the variable r is governed by the corresponding local ring of functions.

In the following subsections we shall consider some important particular cases of singular points of a manifold with singularities.

1.1.2 Power stabilization. Cuspidal points

Local ring

Let us consider the case of power stabilization, that is

$$\varphi(t) = t^{-\gamma}$$

for some positive value of γ (this value is fixed for each given type of singularity). For such choice of φ one can see that the logarithmic derivative

$$\varphi^{-1}(t)\,\varphi'(t) = -\gamma t^{-1}$$

is a regular function of t^{-1} and, hence, we are considering the weight function of depth zero. In accordance to the results of Subsection 1.1.1, the elements of the corresponding local ring (we denote it by $\mathcal{R}^{\gamma}_{\mathrm{loc}}$) have the form

$$f(t,\omega) = F\left(t^{-\gamma},\omega\right) + t^{-\gamma}G\left(t^{-1},t^{-\gamma},\omega\right),$$

where the functions $F(\tau_1,\omega)$ and $G(\tau_0,\tau_1,\omega)$ are smooth up to $\tau_0 = 0$, $\tau_1 = 0$. This is exactly the "cylindrical representation" of the local ring corresponding to the power stabilization. To obtain the "conical representation" of this ring, we have to perform the variable change

$$r = t^{-\gamma},$$

thus obtaining the following expression for elements of the local ring:

$$f(r,\omega) = F(r,\omega) + rG\left(r^{1/\gamma},r,\omega\right) \tag{1.11}$$

with regular at the origin functions F and G.

Note that in the particular case when the number $k = \gamma^{-1}$ is an integer, the expression (1.11) for the typical element from the local ring $\mathcal{R}^{\gamma}_{\mathrm{loc}}$ can be simplified in the following way:

$$f(r,\omega) = F(r,\omega)$$

with $F(r,\omega)$ being a C^∞-function up to the point $r = 0$.

Differential operators and Riemannian metrices

Specifying general forms (1.4), (1.6) of a differential operator on a manifold with singular points of general type to the case of power stabilization, we obtain the following form of a typical differential operator near a point of the considered type

$$\hat{H} = \sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(a_{j\alpha}\left(t^{-\gamma},\omega\right) + t^{-\gamma}b_{j\alpha}\left(t^{-1},t^{-\gamma},\omega\right)\right)\left(\frac{\partial}{\partial t}\right)^{j}\left(\frac{\partial}{\partial\omega}\right)^{\alpha}$$

in the "cylindrical" form, or

$$\hat{H}=\sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(a_{j\alpha}(r,\omega)+rb_{j\alpha}\left(r^{1/\gamma},r,\omega\right)\right)\left(r^{1+1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(\frac{\partial}{\partial\omega}\right)^{\alpha}$$

in the "conical" one. Note also that in the case of integer $\gamma^{-1}=k$ the latter expression can be simplified:

$$\hat{H}=\sum_{j=0}^{m}\sum_{|\alpha|\le m-j}a_{j\alpha}(r,\omega)\left(r^{1+k}\frac{\partial}{\partial r}\right)^{j}\left(\frac{\partial}{\partial\omega}\right)^{\alpha}.$$

This is exactly the form of the typical differential operator near the *cusp-type* singularity. So, we see that the case of power stabilization corresponds to the case of cusp-type singularities.

Let us pass to the consideration of the form of Riemannian metrices corresponding to singular points of power stabilization. We write down here only the form of the product metric in the "conical" representation; all the rest can be written down by the reader with the help of the results of Subsection 1.1.1. In accordance to the formula (1.8), we have

$$ds^2=dr^2+r^{2+2/\gamma}g_\Omega(r), \tag{1.12}$$

where

$$g_\Omega(r)=\sum_{j,l=1}^{n}\left[g'_{jl}(r,\omega)+rg''_{jl}\left(r^{1/\gamma},r,\omega\right)\right]d\omega^j d\omega^l.$$

In the case when $k=\gamma^{-1}$ is an integer, we obtain

$$ds^2=dr^2+r^{2+2k}\sum_{j,l=1}^{n}g_{jl}(r,\omega)\,d\omega^j d\omega^l$$

with $g_{jl}(r,\omega)$ regular up to $r=0$.

In accordance to what was told in the end of the previous subsection, the corresponding Beltrami-Laplace operator has the form

$$\Delta_g=r^{-2(1+1/\gamma)}\left[\left(r^{1+1/\gamma}\frac{\partial}{\partial r}\right)^2+\Delta_{g_\Omega}(r)\right],$$

where the dependence of $\Delta_{g_\Omega}(r)$ on r is governed by the corresponding function ring. So, the other possibility of the definition of the local ring of differential operators corresponding to the point of the considered type is

$$\hat{H}=r^{-m(1+1/\gamma)}\sum_{j=0}^{m}\sum_{|\alpha|\le m-j}\left(a_{j\alpha}(r,\omega)+rb_{j\alpha}\left(r^{1/\gamma},r,\omega\right)\right)\left(r^{1+1/\gamma}\frac{\partial}{\partial r}\right)^{j}\left(\frac{\partial}{\partial\omega}\right)^{\alpha},$$

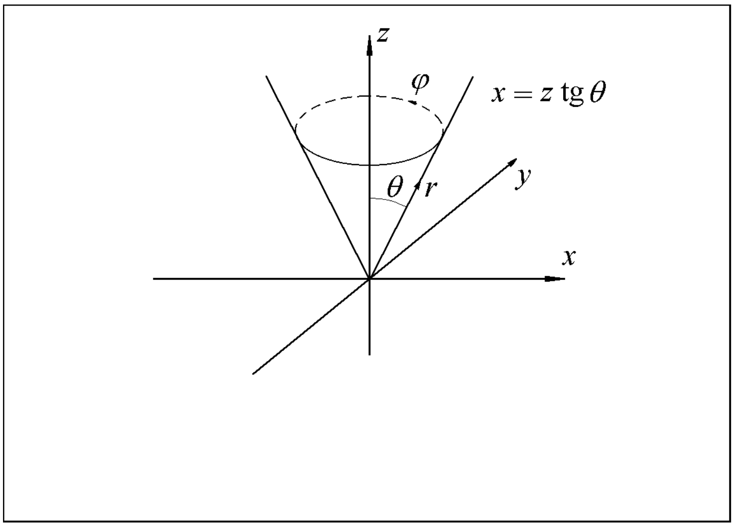

Figure 1.1. Conical singularity.

where m is the order of the considered operator.

The Riemannian metric of the form (1.12) can be obtained as an induced metric with the help of the embedding shown on Figure 1.2.

1.1.3 Exponential stabilization. Conical points

Local ring

In this subsection, we shall consider the case of exponential stabilization, that is,

$$\varphi(t) = e^{-t}.$$

Since in this case the logarithmic derivative

$$\varphi^{-1}(t)\,\varphi'(t) = -1,$$

the considered weight function again has the depth zero. Moreover, since this derivative does not depend on the variable t^{-1} at all, we can define the ring $\mathcal{R}_{\mathrm{loc}}^{\exp}$ as a ring of functions having the form

$$f(t,\omega) = F\left(e^{-t},\omega\right) \tag{1.13}$$

with a function $F(\tau,\omega)$ regular at $\tau = 0$.

Relation (1.13) describes the ring $\mathcal{R}_{\mathrm{loc}}^{\exp}$ in the "cylindrical" representation. To present the description of this ring in "conical" representation, we have to perform the variable change

$$r = e^{-t},$$

thus arriving at the following representation of functions from $\mathcal{R}_{\mathrm{loc}}^{\exp}$ in terms of the variables (r, ω): this ring consists of functions $f(r, \omega)$ regular up to the point $r = 0$.

Clearly, we can define the ring $\mathcal{R}_{\mathrm{loc}}^{\exp}$ as it is prescribed by the general theory. In this case the typical function from this ring will be

$$f(t, \omega) = F\left(e^{-t}, \omega\right) + e^{-t} G\left(t^{-1}, e^{-t}, \omega\right)$$

with regular at the origin functions F and G. However, since this choice of the local ring does not lead to any additional effects, we use below the ring $\mathcal{R}_{\mathrm{loc}}^{\exp}$ given by (1.13).

Differential equations and Riemannian metrices

Specifying general forms (1.4), (1.6) of a differential operator on a manifold with singular points of general type to the case of exponential stabilization, we obtain a typical differential operator near a point of the type with exponential stabilization in the form

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha| \leq m-j} a_{j\alpha}\left(e^{-t}, \omega\right) \left(\frac{\partial}{\partial t}\right)^{j} \left(\frac{\partial}{\partial \omega}\right)^{\alpha}$$

in the "cylindrical" form, or

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha| \leq m-j} a_{j\alpha}(r, \omega) \left(r \frac{\partial}{\partial r}\right)^{j} \left(\frac{\partial}{\partial \omega}\right)^{\alpha}$$

in the "conical" one. It is clear that this form is exactly the form of a typical differential operator near a singular point of the conical type, so we can conclude that the exponential stabilization describes *conical singularities* of the manifold M.

Let us pass to the description of typical Riemannian metrices near a point of the conical type. The general expression for such a metric is (we write down here only the form of the product metric in the "conical" representation). In accordance to the formula (1.8), we have

$$ds^2 = dr^2 + r^2 g_{\Omega}(r),$$

where $g_{\Omega}(r)$ is a metric on Ω depending on r in a regular way up to the point $r = 0$.

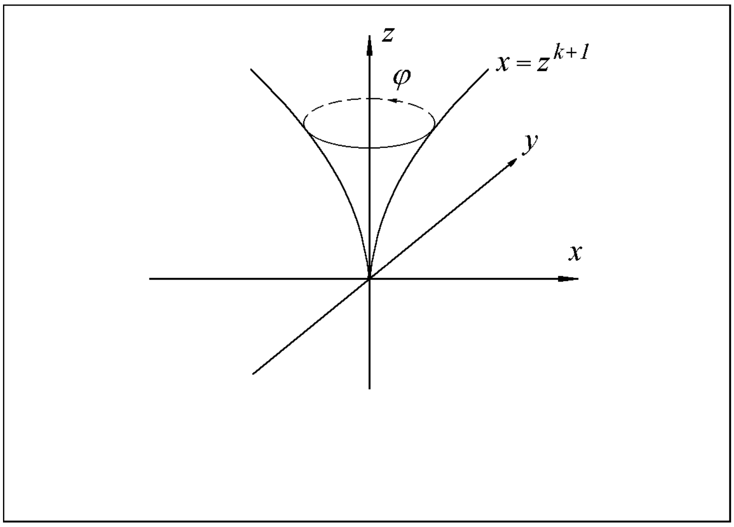

Figure 1.2. Cuspidal singularity.

The corresponding Beltrami-Laplace operator is

$$\Delta_g = r^{-2}\left[\left(r\frac{\partial}{\partial r}\right)^2 + \Delta_{g_\Omega}(r)\right],$$

where the dependence of the coefficients of the operator $\Delta_{g_\Omega}(r)$ on r is regular up to $r = 0$. So, the other definition of the local ring of a conical point is

$$\hat{H} = r^{-m}\sum_{j=0}^{m}\sum_{|\alpha|\le m-j} a_{j\alpha}(r,\omega)\left(r\frac{\partial}{\partial r}\right)^j\left(\frac{\partial}{\partial\omega}\right)^\alpha.$$

In what follows we shall omit such remarks.

A Riemannian metric of the considered type can be obtained as an induced with the help of the embedding of the circular cone into the space $\mathbf{R}^3$ (see Figure 1.1).

1.1.4 Exponential stabilization of arbitrary degree

Local ring

Here we shall consider a structure ring of a singularity defined by the function

$$\varphi(t) = e^{-t^\gamma} \tag{1.14}$$

with some positive real value of γ. Note, that the speed of stabilization defined by this function is faster than the speed of simple exponential stabilization considered in the previous subsection for $\gamma > 1$ and slower for $\gamma < 1$ (for $\gamma = 1$ this function coincides with that considered in the previous subsection).

On the contrary to the two preceding subsections, the depth of the weight function (1.14) does not equal zero. Actually, the logarithmic derivative of the function (1.14)

$$\varphi_1(t) = \varphi^{-1}(t)\,\varphi'(t) = -\gamma t^{\gamma-1}$$

is not a regular function of t^{-1} for any value of γ except for $\gamma = 1$ (for $\gamma > 1$ this function even is not bounded as $t \to \infty$). However, the second logarithmic derivative

$$\varphi_2(t) = \varphi_1^{-1}(t)\,\varphi_1'(t) = (\gamma - 1)\,t^{-1}$$

is a regular function in t^{-1}, and, hence, function (1.14) is a weight function of depth 1. So, the typical form of a function from the corresponding structure ring $R_{\mathrm{loc}}^{\gamma,\exp}$ is

$$f(t,\omega) = F\left(e^{-t^\gamma},\omega\right) + e^{-t^\gamma} P_N\left[t^{\gamma-1}\right] G\left(t^{-1}, e^{-t^\gamma},\omega\right), \tag{1.15}$$

where, as above, the functions $F(\tau_1,\omega)$ and $G(\tau_0,\tau_1,\omega)$ are regular up to $\tau_0 = 0$, $\tau_1 = 0$, and P_N is a polynomial with smooth in ω coefficients (the degree of this polynomial depends on the function $f(t,\omega)$.

Relation (1.15) expresses the form of elements from $R_{\mathrm{loc}}^{\gamma,\exp}$ in "cylindrical" form. To describe these elements in the "conical" form we use the variable change

$$r = e^{-t^\gamma}$$

which leads us to the expression

$$f(t,\omega) = F(r,\omega) + rP_N\left[\left(\ln\frac{1}{r}\right)^{1-1/\gamma}\right] G\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma}, r, \omega\right)$$

with similar functions F, G, and P_N. The latter expression can be rewritten in more simple form

$$f(t,\omega) = F(r,\omega) + rP_N\left[\ln\frac{1}{r}\right] G\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma}, r, \omega\right)$$

since the dependence on $(\ln 1/r)^{-1/\gamma}$ taking place under the sigh of the polynomial P_N can be included into the function G.

Differential equations and Riemannian metrices

The form of differential equations in a neighborhood of a singular point of the considered type is:

$$\hat{H} = \sum_{j=0}^{m} \sum_{|\alpha|\leq m-j} \left[a'_{j\alpha}\left(e^{-t^{\gamma}},\omega\right) + e^{-t^{\gamma}} P_{j\alpha}\left[t^{\gamma-1}\right] a''_{j\alpha}\left(t^{-1}, e^{-t^{\gamma}},\omega\right)\right] \left(\frac{\partial}{\partial t}\right)^{j} \left(\frac{\partial}{\partial \omega}\right)^{\alpha},$$

where $P_{j\alpha}$ are polynomials of different degrees $N_{j\alpha}$. To write down the form of a typical differential operator in the "conical" form, we perform the variable change

$$r = e^{-t^{\gamma}}.$$

In view of the relation

$$\frac{\partial}{\partial t} = -\gamma r \left(\ln\frac{1}{r}\right)^{1-1/\gamma} \frac{\partial}{\partial r},$$

so that

$$\phi\left(r\right) = -\gamma r \left(\ln\frac{1}{r}\right)^{1-1/\gamma},$$

we thus arrive at the expression

$$\begin{aligned} \hat{H} &= \sum_{j=0}^{m} \sum_{|\alpha|\leq m-j} \left[a'_{j\alpha}\left(r,\omega\right) + rP_{j\alpha}\left(\ln\frac{1}{r}\right) a''_{j\alpha}\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma}, r, \omega\right)\right] \\ &\quad \times \left(r\left(\ln\frac{1}{r}\right)^{1-1/\gamma} \frac{\partial}{\partial r}\right)^{j} \left(\frac{\partial}{\partial\omega}\right)^{\alpha}. \end{aligned}$$

The latter expression describes exactly the ring of differential operators near a singular point of exponential stabilization of degree γ.

Let us pass to the description of Riemannian metrices typical for singular points of the considered type. We again write down only the metric of the product type in the "conical" representation:

$$ds^2 = dr^2 + r^2 \left(\ln\frac{1}{r}\right)^{2-2/\gamma} g_{\Omega}\left(r\right),$$

where

$$g_{\Omega}\left(r\right) = \sum_{j,l=1}^{n} \left[g'_{jl}\left(r,\omega\right) + rP_{jl}\left(\ln\frac{1}{r}\right) g''_{jl}\left(\left(\ln\frac{1}{r}\right)^{-1/\gamma}, r, \omega\right)\right] d\omega^{j} d\omega^{l},$$

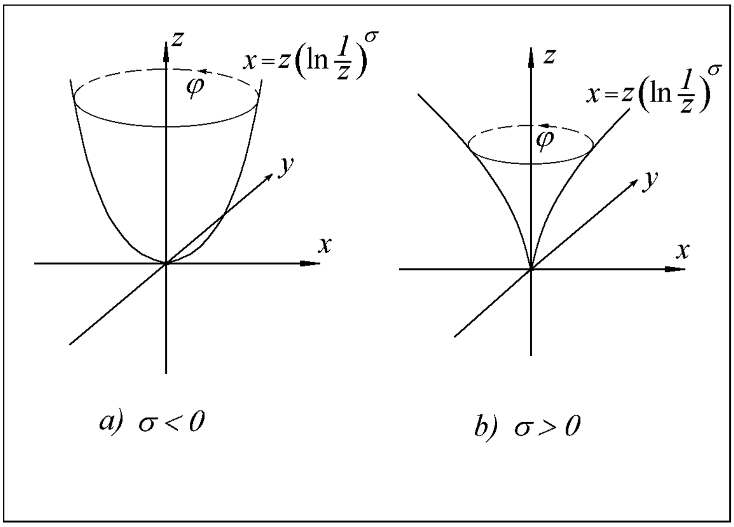

Figure 1.3. Week smoothness and week cusp.

functions $g'_{jl}(r,\omega)$ and $g''_{jl}\left((\ln 1/r)^{-1/\gamma}, r, \omega\right)$ are regular at $r = 0$, and P_{jl} is a polynomial with coefficients smooth in ω.

The considered types of singularities can be described by embeddings of some surfaces into the space $\mathbf{R}^3$. To this end, we consider the surface S in $\mathbf{R}^3$ obtained by rotation of the curve

$$x = z\left(\ln\frac{1}{z}\right)^{\delta}$$

with some real δ around Oz axis (see Figure 1.3). One can easily compute the expression for the Riemannian metric on S induced from $\mathbf{R}^3$:

$$ds^2 = \left\{1 + \left[\left(\ln\frac{1}{z}\right)^{\delta} - \delta\left(\ln\frac{1}{z}\right)^{\delta-1}\right]\right\} dz^2 + z^2\left(\ln\frac{1}{z}\right)^{2\delta} d\varphi^2,$$

which essentially coincides with the above constructed metric for $\delta = 1 - 1/\gamma$. So, the considered type of singularity corresponds to a surface with week smoothness at its singular point for $\delta < 0$ (see Figure 1.3 a)), and with week cusp for $\delta > 0$ (see Figure 1.3 b)); in the last case the latter expression needs some renormalization.

1.1.5 Strong exponential stabilization

Local ring

In this subsection, we consider the speed of stabilization determined by the function

$$\varphi(t) = \exp\left(-e^{t}\right).$$

Again, this function is a weight function of depth 1 since

$$\begin{aligned}
\varphi_1(t) &= \varphi^{-1}(t)\,\varphi'(t) = -e^{t}, \\
\varphi_2(t) &= \varphi_1^{-1}(t)\,\varphi_1'(t) = 1,
\end{aligned}$$

and the latter function is regular in t^{-1}.

Since the derivation of the corresponding formulas is quite similar to that in the previous subsections, we shall simply write down the corresponding expressions for a typical function from the corresponding local ring in both "cylindrical" and "conical" representations. These expressions are:

$$f(t,\omega) = F\left(\exp\left(-e^{t}\right),\omega\right) + \exp\left(-e^{t}\right) P_N\left[e^{t}\right] G\left(\exp\left(-e^{t}\right),\omega\right)$$

in the "cylindrical" representation, and

$$f(r,\omega) = F(r,\omega) + rP_N\left[\ln\frac{1}{r}\right] G(r,\omega)$$

in the "conical" one.

Differential equations and Riemannian metrices

Here, we present local expressions for elements of the ring of differential operators on a manifold with singular points with strong exponential stabilization, and for typical Riemannian metrices in a neighborhood of such a point. We shall use here the "conical" representation of the above objects; the reader can write down their "cylindrical" equivalents by herself or himself.

The typical differential operator near a singular point of the considered type is

$$\begin{aligned}
\hat{H} &= \sum_{j=0}^{m}\sum_{|\alpha|\le m}\left(a'_{j\alpha}(r,\omega) + rP_{j\alpha}\left(\ln\frac{1}{r}\right)a''_{j\alpha}(r,\omega)\right) \\
&\quad \times\left(r\left(\ln\frac{1}{r}\right)\frac{\partial}{\partial r}\right)^{j}\left(\frac{\partial}{\partial\omega}\right)^{\alpha}.
\end{aligned}$$

The metric of the product type in the "conical" representation is

$$ds^2 = dr^2 + r^2 \left(\ln \frac{1}{r}\right)^2 g_\Omega (r),$$

where

$$g_\Omega (r) = \sum_{j,l=1}^{n} \left[g'_{jl} (r, \omega) + r P_{jl} \left(\ln \frac{1}{r}\right) g''_{jl} \left(\left(\ln \frac{1}{r}\right), r, \omega\right)\right] d\omega^j d\omega^l,$$

the functions $g'_{jl} (r, \omega)$ and $g''_{jl} ((\ln 1/r), r, \omega)$ are regular at $r = 0$, and P_{jl} is a polynomial with coefficients smooth in ω.

To conclude this section we remark that, as far as we know, the singularities of the two last types have not been considered in the literature earlier.

1.2 General types of singularities

In this section, we consider the operations of constructing more complicated singularities from the more simple ones. As it was already mentioned, these operations are direct multiplication of a manifold with singularities by a smooth compact manifold without boundary and constructing the suspension (cone) over a manifold with singularities. Throughout this section we shall consider only the local models, and will use the second version of the local ring of differential operators near considered singular points (see (1.9)).

1.2.1 Conification operation

Let M be a manifold with singularities of the type $\varphi (t)$. Then (see Subsubsection 1.1.1, formula (1.10)), the elements of the local structure ring of differential operators corresponding to any singular point of this manifold have locally the form

$$\hat{H} = \phi^{-m} (r) \hat{H} \left(r, \phi (r) \frac{d}{dr}\right) \tag{1.16}$$

where $\phi (r) = \varphi' (\varphi^{-1} (r))$. Here the symbol $\hat{H} (r, p)$ is a function with values in differential operators on the base Ω of the corresponding cone.

Consider the cone K_M over this manifold:

$$K_M = \{[0, 1) \times M\} / \{\{0\} \times M\} \tag{1.17}$$

(the suspension over this manifold). The manifold K_M can be considered as a local topological model of a singularity with the corner from which the wedges of the type $\varphi(t)$ are emanated (the amount of these wedges coincide with the number of singular point on the manifold M).

To describe the corresponding ring of differential operators, we introduce the variable $\rho \in [0, 1)$ corresponding to representation (1.17), and the weight function of the finite depth $\varphi_1(\tau)$, where the variables τ and ρ are connected with the help of the relation

$$\rho = \varphi_1(\tau).$$

Let $\phi_1(\rho) = \varphi_1'\left(\varphi_1^{-1}(\rho)\right)$. Consider the set of operators of the form

$$\hat{H} = \phi_1^{-m}(\rho)\,\hat{H}\left(\rho, \phi_1(\rho)\frac{d}{d\rho}\right) \tag{1.18}$$

with symbols $\hat{H}(\rho, \xi)$ taking its values in the ring of differential operators on manifold M with singularities (having the form (1.16) near each singular point of the manifold M) and with dependence on ρ determined by the ring of functions with stabilization of speed $\varphi_1(\tau)$.

It is easy to check that the set of operators (1.18) forms an algebra. This is exactly a local algebra of the new corner point obtained by the conification operation.

To conclude this subsection, we shall write down the form of the operators from the constructed algebra in a neighborhood of the most interesting points of K_M, where $|r|$ and $|\rho|$ are small. This form is

$$\hat{H} = \phi_1^{-m}(\rho)\,\phi^{-m}(r)\,\hat{H}\left(\rho, r, \phi(r)\frac{\partial}{\partial r}, \phi(r)\,\phi(\rho)\frac{\partial}{\partial \rho}\right),$$

where, as usual, m is an order of the operator $\hat{H}$, and the function $\hat{H}(\rho, r, p, \xi)$ takes its values in the ring of differential operators on the base Ω of a cone corresponding to a singular point of the initial manifold M. The dependence of the symbol on the coordinates r and ρ is determined by the corresponding function rings. We remark also, that in the regions where $|r| \geq \varepsilon > 0$ the behavior of the operators from the considered rings are of the wedge type which is described in the next subsection.

1.2.2 Edgification operation

Let us consider now the operation of a direct product with a smooth manifold. Let M_{loc} be a local model of a manifold M with singularities

$$M_{\text{loc}} = \{[0, 1) \times \Omega\} \,/\, \{\{0\} \times \Omega\}$$

(to be definite we consider the "conical" representation of the considered local model, and denote the corresponding variable in $[0, 1)$ by r. Suppose that the type of the corresponding singular point on this topological model is defined by the weight function $\varphi(t)$, so that the expressions of elements of the corresponding structure ring of differential operators are given by relation (1.16).

Let X be a smooth compact manifold without boundary (in consideration of the local models we can consider even open smooth manifolds), and let x be local coordinates on X.

Consider the direct product

$$M_{\mathrm{loc}} \times X$$

as a new (topological) local model, and define the structure ring of differential operators of the new singularity as a tensor product of the ring of C^∞-differential operators on X and the local ring of the singularity of the manifold M_{loc}.

Then one can derive that the local expressions of the elements of the new structure ring of differential operators have the form

$$\phi^{-m}(r)\,\hat{H}\left(x, r, \phi(r)\frac{\partial}{\partial r}, \phi(r)\frac{\partial}{\partial x}\right),$$

where, as above, m is the order of the operator, and the symbol $\hat{H}(x, r, p, \xi)$ takes its values in the ring of differential operators on the base Ω of the cone corresponding the initial singularity. The described type of degeneration of differential operators we call *edge degeneration*.

In conclusion, we remark that with the help of the two operations described above, one can construct the whole hierarchy of singularities having different types with respect to different levels. We shall not describe here all this hierarchy since the reader can do it by herself or himself.

Chapter II

Semi-classical expansions of solutions to differential equations on manifolds with singularities

In this chapter[1], we shall consider asymptotic expansions of solutions to differential equations on manifolds with singularities. The first aim of this investigation is to examine asymptotic behavior of solutions to homogeneous differential equations on manifolds with point-type singularities. We shall first consider the class of differential equations on manifolds with conical and cuspidal points, that is, equations having the form

$$\hat{H}\left(r, r^{1+\alpha}\frac{d}{dr}\right)u = 0, \tag{2.1}$$

where, as usual, r is a coordinate along the axis of the corresponding model (topological) cone, and $\hat{H}(r,p)$ is a polynomial in p with coefficients in the ring of C^∞-differential operators on the base Ω of this cone smoothly dependent on the variable r up to $r = 0$. We consider here arbitrary real nonnegative values of α.

The case $\alpha = 0$, corresponding to the conical singularities is well-known (see [7]). In this case the asymptotic expansions of solutions to the considered equation are of conormal type

$$u(r,\omega) = \sum_j r^{p_j} \sum_{l=0}^{m_j-1} a_{jl}(\omega) \ln^l r,$$

where p_j form a sequence with $\operatorname{Re} p_j \to +\infty$, $a_{jl}(\omega)$ are smooth functions on Ω, and m_j are the so-called multiplicities of the points p_j. Therefore, we shall fix our attention at the investigation of the cuspidal points, that is, at the case $\alpha > 0$.

[1]The results presented below the reader can find in [8, 12].

2.1 Asymptotics of solutions at cuspidal points

2.1.1 Formal asymptotic expansions

In this subsection, we construct the formal solutions to equation (2.1). Here we present only the formal computations; all the verifications of the constructions here will be done in the next subsection. The computations here show the structure of the future asymptotic expansions and serve a good guide for the constructions of the following subsection.

First, to be able to apply the WKB-technique to equation (2.1), we modify this equation with the help of a small parameter h in the following way

$$\hat{H}\left(r, h\hat{p}\right) u = 0, \tag{2.2}$$

here by $\hat{p}$ we denote the derivative $r^{1+\alpha} d/dr$; the solutions to the initial equation will be obtained at $h = 1$. Then we shall construct asymptotic expansions with respect to this parameter. It occurs that the terms of the obtained asymptotic expansions are also graduated with respect to the increasing powers of r, and, hence, putting $h = 1$ we obtain the needed asymptotic expansion.

So, let us search for the solution to equation (2.2) in the form

$$u = \exp\left\{\frac{1}{h} S\left(r\right)\right\} \sum_{j=0}^{\infty} h^j A_j\left(r\right). \tag{2.3}$$

Since we have

$$\begin{aligned} h\hat{p}u &= \exp\left\{\frac{1}{h} S\left(r\right)\right\} \left\{S'\left(r\right)\hat{p}\left(r\right) + h\hat{p}\right\} \sum_{j=0}^{\infty} h^j A_j\left(r\right) \\ &= \exp\left\{\frac{1}{h} S\left(r\right)\right\} \left\{r^{1+\alpha} S'\left(r\right) + h\hat{p}\right\} \sum_{j=0}^{\infty} h^j A_j\left(r\right), \end{aligned}$$

the substitution of (2.3) into (2.2) gives

$$\exp\left\{\frac{1}{h} S\left(r\right)\right\} \hat{H}\left(r, r^{1+\alpha} S'\left(r\right) + h\hat{p}\right) \sum_{j=0}^{\infty} h^j A_j\left(r\right) = 0.$$

Cancelling out the exponential, we arrive at the equation for the amplitude functions $A_j\left(r\right)$

$$\hat{H}\left(r, r^{1+\alpha} S'\left(r\right) + h r^{1+\alpha}\frac{d}{dr}\right) \sum_{j=0}^{\infty} h^j A_j\left(r\right) = 0.$$

Now, separating powers of h in the usual way, we arrive at the following recurrent relations:

$$\begin{aligned}
\hat{H}(r,P(r))\,A_0(r) &= 0, \\
\hat{H}(r,P(r))\,A_1(r) &= -\left[\frac{\partial\hat{H}}{\partial p}(r,P(r))\left(r^{1+\alpha}\frac{d}{dr}\right)\right. \\
&\qquad \left. +\frac{1}{2}\frac{\partial^2\hat{H}}{\partial p^2}(r,P(r))\,\varphi(r)\,\frac{dP(r)}{dr}\right]A_0(r), \qquad (2.4)\\
\hat{H}(r,P(r))\,A_j(r) &= -\sum_{i=1}^{j}\hat{\mathcal{P}}_i A_{j-i}(r),\ j=2,3,\dots,
\end{aligned}$$

where $P(r)=r^{1+\alpha}S'(r)$, and the operator $\hat{\mathcal{P}}_1$ equals

$$\hat{\mathcal{P}}_1=\frac{\partial\hat{H}}{\partial p}(r,P(r))\left(r^{1+\alpha}\frac{d}{dr}\right)+\frac{1}{2}\frac{\partial^2\hat{H}}{\partial p^2}(r,P(r))\,r^{1+\alpha}\frac{dP(r)}{dr}. \qquad (2.5)$$

To begin with, let us consider the first equation from (2.4). Since we are interested in the construction of a *nontrivial* asymptotic solution of the form (2.3), we have to require $A_0(r)\neq 0$. Hence, the operator

$$\hat{H}(r,P(r))$$

must have a nontrivial kernel.

To analyze the condition of degeneracy of the latter operator, let us consider the analytic family

$$\hat{H}(r,p) \qquad (2.6)$$

of operators on the manifold Ω as an analytic family in p for each (sufficiently small) fixed r. We require that this family is an *elliptic family of differential operators on Ω with the parameter p*. From here, it follows that:

i) The operator $\hat{H}(r,p)$ is a Fredholm one for each values of (r,p).

ii) For each r there exists a p_0 such that the operator $\hat{H}(r,p_0)$ is invertible.

It is known (see, e. g. [7]) that under the above conditions there exists an inverse $\hat{H}^{-1}(r,p)$ for operator (2.6), meromorphically dependent on p. Denote by

$$p=p(r)$$

the equation of singularity set of the operator $\hat{H}^{-1}(r,p)$ (the spectrum of the operator pencil (2.6)). In general, the function $p(r)$ is a multivalued one.

Suppose, for simplicity, that all poles of the family $\hat{H}^{-1}(r,p)$ are simple up to the point $r=0$. Then it is evident that the function $p(x)$ splits to the univalent branches

$$p = p_i(r), \ i = 1, 2, \dots,$$

such that the functions $p_i(r)$ are regular functions in x (they are analytic for analytic $\hat{H}(r,p)$ and smooth for smooth $\hat{H}(r,p)$.

Remark 1 The latter assumption means that we are working in the situation of *nondegenerate Lagrangian manifolds [1, 5]*. The set of the degeneracy points of the operator $\hat{H}(r,p)$ determines a germ $p=p(r)$ at $r=0$ of Lagrangian manifolds (possible, with singularities) in the phase space (r,p). The assumption that the function $p=p(r)$ splits into regular branches means exactly that this germ splits into a union of nondegenerate Lagrangian manifolds.

Now we have

$$S(r) = S_i(r), \ A_0(r) = A_{0i}(r),$$

where $S_i(r)$ and $A_{0i}(r)$ are determined by some branch $p=p_i(r)$ with the help of the relations

$$r^{1+\alpha} S_i'(r) = p_i(r), \ A_{0i}(r) \in \operatorname{Ker} \hat{H}(r, p_i(r)).$$

In what follows we fix some branch $p_i(r)$ and omit the subscript i.

Later on, we remark that, under the above assumptions, the operator

$$\hat{H}(r, p(r))$$

has zero index, so that the dimensions of the kernel and the cokernel of this operator coincide with each other. To simplify the presentation below, we suppose that

$$\dim \operatorname{Ker} \hat{H}(r,p(r)) = \dim \operatorname{Coker} \hat{H}(r,p(r)) = 1,$$

and denote by $U(r)$ and $V(r)$ the generators in the spaces $\operatorname{Ker}\hat{H}(r,p(r))$ and $\operatorname{Coker}\hat{H}(r,p(r))$, respectively.

So, we have

$$A_0(r) = a_0(r)\, U(r),$$

where the function $a_0(r)$ is, up to the moment, unknown.

Let us proceed with the investigation of the subsequent equations in (2.4). The first transport equation from (2.4) has the form

$$\begin{aligned} \hat{H}(r,p(r))\, A_1(r) \ &= \ -\left[\frac{\partial \hat{H}}{\partial p}(r,p(r)) \left(r^{1+\alpha} \frac{d}{dr} \right) \right. \\ &\qquad \left. + r^{1+\alpha} \frac{1}{2} \frac{\partial^2 \hat{H}}{\partial p^2}(r,p(r)) \frac{dp(r)}{dr} \right] A_0(r). \end{aligned}$$

Since the operator $\hat{H}(r, p(r))$ on the left in the latter relation is not invertible, for this equation to have a solution $A_1(r)$ it is necessary that its right-hand part satisfy the compatibility conditions (we have cancelled out the factor $x^{1+\alpha}$)

$$\left\langle V(r), \left[\frac{\partial \hat{H}}{\partial p}(r, p(r)) \frac{d}{dr} + \frac{1}{2} \frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r)) \frac{dp(r)}{dr}\right] U(r) \right\rangle a_0(r) = 0,$$

or

$$\left[\left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r)) U(r) \right\rangle \frac{d}{dr} + \left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r)) \frac{dU(r)}{dr} \right\rangle \right.$$
$$\left. + \frac{1}{2} \left\langle V(r), \frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r)) \frac{dp(r)}{dr} U(r) \right\rangle \right] a_0(r) = 0,$$

which is solvable in regular functions under the assumption that

$$\left\langle V(r), \frac{\partial \hat{H}}{\partial p}(r, p(r)) U(r) \right\rangle \neq 0. \tag{2.7}$$

The (nonvanishing) value $a_0(0)$ remains undefined and determines an (arbitrary) multiplicative function naturally involved into a solution to the homogeneous equation in question.

Let us pass to the examination of the rest transport equations in (2.4). To do this we need examine orders at zero of all operators $\hat{\mathcal{P}}_j$ involved into these equations. Let us give the exact definitions.

A function $f(r)$ will be called a function of (power) order γ at $r = 0$ (simply the order in the sequel), if it is representable in the form

$$f(r) = r^\gamma g(r), \; g(0) \neq 0.$$

An operator $\hat{P}$ will be called an operator of (power) order l if the order of the function $\hat{P}f$ is not less than $l + \gamma$ for any function f of order γ. We denote the power order of the operator by $\mathrm{ord}_p \hat{P}$ (p stands for "power").

The following affirmation takes place.

Lemma 1 *The operators $\hat{\mathcal{P}}_j$ involved into relation* (2.4) *are operators of power order $j\alpha$. Besides, all these operators beginning from $j = 1$ can be represented in the form $\hat{\mathcal{P}}_j = r^{1+\alpha} \hat{P}_j$.*

Proof. It is clear that the operators $\hat{\mathcal{P}}_j$ are determined from the relation (we use the Feynman indices (see [2, 6]

$$\hat{H}\left(\overset{2}{r}, \overline{r^{1+\alpha}S'(r) + hr^{1+\alpha}\frac{d}{dr}}^{1}\right) = \sum_{j=1}^{m} h^j \hat{\mathcal{P}}_j.$$

Besides,

$$\hat{\mathcal{P}}_0 = \hat{H}(r, p(r)),$$

and the operator $\hat{\mathcal{P}}_1$ is given by relation (2.5). From here, it is clear that the orders of these two operators are 0 and α, respectively. Later on, it suffices to prove the assertion for operators of order $m+1$ with the symbol having the form $\hat{H}_1(r,p)\,p$ provided that it is already proved for all operators of order m. If

$$\hat{H}_1\left(\overset{2}{r}, \overline{r^{1+\alpha}S'(r) + hr^{1+\alpha}\frac{d}{dr}}^{1}\right) = \sum_{j=1}^{m} h^j \hat{\mathcal{P}}_j^{(1)},$$

then, due to the inductive hypothesis we have $\operatorname{ord}_p \hat{\mathcal{P}}_j^{(1)} = j\alpha$. Then we have

$$\hat{H}_1\left(\overset{2}{r}, \overline{r^{1+\alpha}S'(r) + h\varphi(x)\frac{d}{dx}}^{1}\right)\left(r^{1+\alpha}S'(r) + hr^{1+\alpha}\frac{d}{dr}\right) = \sum_{j=1}^{m} h^j \hat{\mathcal{P}}_j,$$

where

$$\hat{\mathcal{P}}_j = \hat{\mathcal{P}}_j^{(1)}\left(r^{1+\alpha}S'(r)\right) + \hat{\mathcal{P}}_{j-1}^{(1)}\left(r^{1+\alpha}\frac{d}{dr}\right). \tag{2.8}$$

Since

$$\operatorname{ord}_p\left(r^{1+\alpha}S'(r)\right) = \operatorname{ord}_p\left(p(x)\right) = 0,\ \operatorname{ord}_p\left(r^{1+\alpha}\frac{d}{dr}\right) = \alpha,$$

then, clearly,

$$\operatorname{ord}_p \hat{\mathcal{P}}_j = \min\left\{\operatorname{ord}_p \hat{\mathcal{P}}_j^{(1)}, \operatorname{ord}_p \hat{\mathcal{P}}_{j-1}^{(1)} + \alpha\right\} = j\alpha,$$

as required. The last affirmation of the Lemma is a direct consequence of formula (2.8).

Let us consider the second transport equation (the rest can be examined in a quite similar manner). The third equation in (2.4) reads

$$\hat{H}(r, p(r))\,A_2(r) = -\left[\hat{\mathcal{P}}_1 A_1(r) + \hat{\mathcal{P}}_2 A_0(r)\right]. \tag{2.9}$$

From the above consideration of the first transport equation it is clear that the general solution to the second equation from (2.4) has the form

$$A_1(r) = a_1(r)\,U(r) + A_1^*(r)\,, \tag{2.10}$$

where $A_1^*(r)$ is a particular solution to this equation, having power order $\alpha + 1$. Hence, the equation for the function $A_2(r)$ reads

$$\hat{H}(r, p(r))\,A_2(r) = -\left[\hat{\mathcal{P}}_1[a_1(r)\,U(r)] + \hat{\mathcal{P}}_1[A_1^*(r)] + \hat{\mathcal{P}}_2[A_0(r)]\right].$$

The compatibility condition for the last equation is

$$\begin{aligned}
&\left[\left\langle V(r)\,, \frac{\partial \hat{H}}{\partial p}(r, p(r))\,U(r)\right\rangle \frac{d}{dr} + \left\langle V(r)\,, \frac{\partial \hat{H}}{\partial p}(r, p(r))\,\frac{dU(r)}{dr}\right\rangle\right. \\
&\left.+\frac{1}{2}\left\langle V(r)\,, \frac{\partial^2 \hat{H}}{\partial p^2}(r, p(r))\,\frac{dp(r)}{dr}U(r)\right\rangle\right] a_1(r) \\
&= -\left[\hat{P}_1[A_1^*(r)] + \hat{P}_2[A_0(r)]\right],
\end{aligned}$$

where $\hat{P}_j$ are operators introduced in Lemma 1. Clearly, the last equation is solvable with respect to $a_1(r)$ in regular functions (see inequality (2.7) above). Later on, since the power orders of operators $\hat{P}_1$ and $\hat{P}_2$ equal -1 and $k-1$, respectively, the function on the right in the last equation has power order $k-1$. Hence, there exists a unique solution to the obtained equation with power order k; this solution is determined by the Cauchy data $a_1(0) = 0$. Now the general solution to equation (2.9) is written down in the form

$$A_2(r) = a_2(r)\,U(r) + A_2^*(r)\,,$$

where $A_2^*(r)$ is of power order 2α. We remark also that the power order of function (2.10) is α.

The continuation of the above described procedure leads us to the following statement:

Proposition 2 *There exists a formal solution u to equation* (2.2) *of the form* (2.3) *such that all the functions $A_j(r)$ have power order $j\alpha$ in the variable r. This solution is defined uniquely up to a multiplicative constant.*

From this Proposition it follows that expansion (2.3) (for $h = 1$) is an asymptotic expansion in r near $r = 0$ only for $\alpha > 0$ since only in this case the terms of this asymptotic expansion are graduated by descent as $r \to 0$. Moreover,

$$A_j(r) = r^{j\alpha} a(r) \tag{2.11}$$

with $a(r)$ possessing a standard Taylor expansion in powers of r.

2.1.2 Analysis of the obtained expansion

In this subsection, we detalize the obtained asymptotic expansion for different values of the parameter α. It is evident that the only thing important is the range $1+\alpha$ of degeneration of equation (2.1).

Let us consider first the Hamilton-Jacobi equation for the function $S(x)$. It has the form

$$r^{1+\alpha}S'(r)=p(r), \tag{2.12}$$

where $p(r)$ is one of the branches of the equation of the spectrum of the operator family $\hat{H}(r,p)$. Note that, by the above assumptions, the function $p(r)$ is regular at $r=0$. Expanding the function $p(r)$ into the Taylor series

$$p(r)=\sum_{j=0}^{\infty}p_j r^j,$$

we reduce equation (2.12) to the form

$$S'(r)=\sum_{j=0}^{\infty}p_j r^{j-1-\alpha}.$$

The result of integration of the latter equation differs for integer and noninteger values of α. Namely, if α is not an integer, we obtain, up to an additive constant,

$$S(r)=\sum_{j=0}^{\infty}\frac{p_j}{j-\alpha}r^{j-\alpha}. \tag{2.13}$$

On the opposite, if $\alpha=k>0$ is an integer, we get

$$S(r)=\sum_{j=0}^{k-1}\frac{p_j}{j-k}r^{j-k}+p_k\ln r+\sum_{j=k+1}^{\infty}\frac{p_j}{j-k}r^{j-k}. \tag{2.14}$$

Let us consider first the case of integer $\alpha=k$. Substituting formulas (2.14) and (2.11) into formula (2.3) with $h=1$, we arrive at the relation

$$\begin{aligned} u &= \exp\left\{\sum_{j=0}^{k-1}\frac{p_j}{j-k}r^{j-k}+p_k\ln r+\sum_{j=k+1}^{\infty}\frac{p_j}{j-k}r^{j-k}\right\}\sum_{j=0}^{\infty}r^{jk}a_j(r) \\ &= \exp\left\{\sum_{j=0}^{k-1}\frac{p_j}{j-k}r^{j-k}\right\}r^{p_k}\exp\left\{\sum_{j=k+1}^{\infty}\frac{p_j}{j-k}r^{j-k}\right\}\sum_{j=0}^{\infty}r^{jk}a_j(r)\end{aligned}$$

for the asymptotic expansion of solution to equation (2.1). Expanding the second exponential term and the functions $a_j(r)$ into the Taylor series (all these functions are regular at $r=0$), we obtain finally

$$u = \exp\left\{\sum_{j=0}^{k-1} \frac{p_j}{j-k} r^{j-k}\right\} r^{p_k} \sum_{j=0}^{\infty} r^j b_j, \tag{2.15}$$

where b_j are some constants (more precisely, $b_j = b_j(\omega)$; we recall that we consider the function u as a function of r with values in functions of ω).

In the case of noninteger α we use formula (2.13) instead of (2.14) and obtain

$$\begin{aligned} u &= \exp\left\{\sum_{j=0}^{\infty} \frac{p_j}{j-\alpha} r^{j-\alpha}\right\} \sum_{j=0}^{\infty} r^{j\alpha} a_j(r) \\ &= \exp\left\{\sum_{j=0}^{[\alpha]} \frac{p_j}{j-\alpha} r^{j-\alpha}\right\} \exp\left\{\sum_{j=[\alpha]+1}^{\infty} \frac{p_j}{j-\alpha} r^{j-\alpha}\right\} \sum_{j=0}^{\infty} r^{j\alpha} a_j(r). \end{aligned}$$

Again using the Taylor expansions for bounded components in the latter formula, we obtain finally

$$u = \exp\left\{\sum_{j=0}^{[\alpha]} \frac{p_j}{j-\alpha} r^{j-\alpha}\right\} \sum_{j=0}^{\infty} r^{q_j} u_j, \tag{2.16}$$

where u_j are some constants, and q_j are numbers of the form

$$\{q_j\} = \{l(i-\alpha) + s\alpha + k \mid l \geq 0,\ i > [\alpha],\ k \geq 0;\ l, i, k \in \mathbf{Z}\}$$

ordered in the increasing direction. It is easy to see that the set of powers q_j is discrete in $\mathbf{R}_+$, but the closer α is to an integer, the closer the points of this power is to one another; so to say, *this set tends to a dense set as α tends to an integer.* Remark that the series on the right in (2.15) and (2.16) are, as a rule, divergent.

So, we can formulate the following features of asymptotic expansions of solutions to homogeneous equations near cuspidal points of the manifold M:

1. Asymptotic expansions of solutions to homogeneous equations near singularity points of the cuspidal type are represented by *divergent* series (in contrast to the conical case) and, hence, require resurgent analysis for their definition.

2. Since expressions under the exponent signs in (2.15), (2.16) are not homogeneous in r, at least for $\alpha > 1$, the *resurgent representation* [11] is required to *resummate* the corresponding series in terms of functions with simple singularities.

3. The integer points $\alpha = k \in \mathbf{Z}_+$ are *"catastrophe points"* of the obtained asymptotic expansions since these expansions are not continuous as $\alpha \to k \in \mathbf{Z}_+$ (the supports of the corresponding resurgent functions tend to a dense set in the complex plane $\mathbf{C}$.

2.2 Construction of resurgent asymptotics

In this section, we shall prove the existence of *resurgent solutions* to homogeneous equations on manifolds having point-type singularities. Since we want to clarify the nature of "catastrophes" taking place for such solutions and since for such an examination we have to use rather general transformations (as it will be seen later), it is convenient to carry out all the considerations in the framework of general algebras [12].

2.2.1 Statement of the problem

Let us fix some generating group $G(p)$ of integral representation in the algebra $\mathcal{U}$. Consider the equation

$$\hat{H}\left(\overset{2}{a}, \overset{1}{\hat{p}}\right) u = 0, \tag{2.17}$$

where $\hat{H}(x, p)$ is a polynomial in p of order m with holomorphic coefficients taking its values in the algebra of differential operators on Ω, and a is some element of the algebra $\mathcal{U}^G$. We suppose that the operator of multiplication by a is an operator of negative order in the double filtration defined by the asymptotic scales

$$\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\} \text{ and } \left\{\mathcal{U}\mathcal{F}_\alpha,\ \alpha \in \mathbf{R}\right\}.$$

Since the first scale is stronger than the second, the last requirement on the operator a means that

- either this operator has negative order with respect to the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$,
- or this operator has zero order with respect to $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$ and has the negative order with respect to $\left\{\mathcal{U}\mathcal{F}_\alpha,\ \alpha \in \mathbf{R}\right\}$.

Let us interpret these two requirements in terms of the operator $\hat{a}$ in the dual space.

In the first case the fact that the operator a has negative order in the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$ means that the corresponding operator $\hat{a}$ shifts supports of functions $U(p)$ to the right in the p-plane to some positive value (which is interpreted as the negative order of this operator). For example, this operator can be just the shift in the p-plane by a complex number p_0 with positive real part. Such an operator can be represented as the convolution with the function

$$a(p) = \langle G^*(p), a \rangle ,$$

namely

$$\hat{a}U(p) = a(p) * U(p),$$

with the support contained as a whole in the half-plane $\{\mathrm{Re} p > 0\}$. To be short, we suppose that the support of the function $a(p)$ consists of a single point σ, $\mathrm{Re}\,\sigma > 0$.

In the second case, we again represent the operator $\hat{a}$ as the convolution with the corresponding function $a(p)$. Since this operator has zero order with respect to the scale $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$, the support of the element a is contained as a whole in the half-plane $\{\mathrm{Re} p > 0\}$, and it intersects the line $\{\mathrm{Re} p > 0\}$. We shall restrict ourselves with the most simple but the most interesting case when the support of a consists of a single point $p = 0$. In this case the function $a(p)$ has the only singularity point $p = 0$, and, since $\hat{a}$ is an operator of negative order with respect to the scale $\{\mathcal{UF}_\alpha,\ \alpha \in \mathbf{R}\}$, the singularity at this point is weak, that is

$$|a(p)| \le C\,|p|^\beta$$

with some $C > 0$ and $\beta > -1$.

The process of solving equation (2.17) in these two cases is quite different, and we consider the constructing of an asymptotic expansion in each of these two cases distinctly.

2.2.2 Asymptotic expansion (first case)

Since the element a involved into equation (2.17) is in some sense small (it has the negative order), it is natural to expand this equation in powers of this element:

$$\hat{H}(a,\hat{p})\,u = \left[\hat{H}_0(\hat{p}) + a\hat{H}_1(a,\hat{p})\right] u = 0.$$

Passing to the dual equation with the help of the representation

$$u = \mathcal{R}_G\left[U(p)\right],$$

one arrives at the following equation for the function $U(p)$:

$$\left[\hat{H}_0(p)+\overset{3}{\hat{a}}\ \hat{H}_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)\right]U(p)=0. \tag{2.18}$$

Let us search for the solution of the latter equation with the help of the successive approximation method. Neglecting the term $\hat{a}\hat{H}_1(\hat{a},p)\,U(p)$, we determine the zeroth iteration $U_0(p)$ as a solution to the equation

$$\hat{H}_0(p)\,U_0(p)=0. \tag{2.19}$$

As above, we shall assume that the family $\hat{H}_0(p)$ of operators on the manifold Ω is meromorphically invertible. Moreover, to be short, we also suppose that all poles p_j of the inverse family $\hat{H}_0^{-1}(p)$ are simple, and the dimensions of the kernel and the cokernel of the operator $\hat{H}_0(p_j)$ equal one. Then we can write down a solution to equation (2.19) in the form

$$U_0^{(j)}(p)=\frac{A_0^{(j)}}{2\pi i\,(p-p_j)},$$

where $A_0^{(j)}$ belongs to the kernel of the operator $\hat{H}_0(p_j)$ (we recall that equation (2.18) have to be solved in the space of hyperfunctions).

Now, searching for the next iteration in the form

$$U_1^{(j)}(p)=U_0^{(j)}(p)+U_1^{(j)}(p)\,,$$

where the function $U_1^{(j)}(p)$ is a resurgent image of a resurgent function with the support at the point $p_j+\sigma$, we obtain the equation for this function in the form

$$H_0(p)\,U_1^{(j)}(p)+\overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)U_1^{(j)}(p)+H_0(p)\,U_0^{(j)}(p)+\overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)U_0^{(j)}(p)=0.$$

Again neglecting the term $\hat{a}H_1(\hat{a},p)\,U_1^{(j)}(p)$, we arrive at the expression for $U_1^{(j)}(p)$:

$$U_1^{(j)}(p)=-U_0^{(j)}(p)-[H_0(p)]^{-1}\,\overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)U_0^{(j)}(p)\,.$$

Note, that since the function $U_0^{(j)}(p)$ has no singularities at point $p_j+\sigma$, the first term on the right in the latter relation can be omitted, and we arrive at the final expression for the function $U_1^{(j)}(p)$:

$$U_1^{(j)}(p)=-\,[H_0(p)]^{-1}\,\overset{3}{\hat{a}}\ H_1\left(\overset{2}{\hat{a}},\overset{1}{p}\right)U_0^{(j)}(p)\,.$$

Continuing this process, we shall construct a solution to equation (2.17) in the form

$$U(p) = U_0^{(j)}(p) + U_1^{(j)}(p) + U_2^{(j)}(p) + \ldots. \tag{2.20}$$

Taking into account that

- the supports of only a finite number of terms of series (2.20) have a nonempty intersection with any left half-plane $\{\mathrm{Re} p < R\}$, and
- the filtration $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$ is complete,

one obtains that series (2.20) converge in the space of resurgent elements thus determining a solution to equation (2.17).

So, we have arrived at the following statement:

Theorem 1 *If the operator $\hat{a}$ has a negative order with respect to the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$, there exist a full system of resurgent solutions to equation* (2.17)*. Each solution of this system is determined by some spectral point p_j of the family $\hat{H}_0(p)$, and the support of this solution consists of the point p_j itself together with shifts of this point by a point σ from the support of the element a. The set of singularities of the function $U(p)$ is the union of the spectral set $\{p_j\}$ of the family $\hat{H}_0(p)$, and all above described shifts of this set.*

2.2.3 Asymptotic expansion (second case)

The case when the operator a has zero order with respect to the asymptotic scale $\left\{\mathcal{U}_R^{G,r},\ R \in \mathbf{R}\right\}$ is a little bit more difficult[2]. However, the initial steps in the constructing an asymptotic solution go quite similar to the above considered case. Namely, we expand the equation in powers of the operator a and transform the obtained equation into the equation for the function

$$U(p) = \langle G^*(p), u\rangle$$

with the help of the representation $\mathcal{R}_G$. Similar to the above case, we get

$$\left[\hat{H}_0(p) + \overset{2}{\hat{a}}\ \hat{H}_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right)\right] U(p) = 0. \tag{2.21}$$

[2] We recall that we consider the case when the element a involved into equation (2.17) is supposed to have the support consisting of the single point $p = 0$.

Using again the successive approximation method, we obtain a formal solutions to this equation in the form of the series

$$U^{(j)}(p) = \sum_{k=0}^{\infty} U_k^{(j)}(p), \tag{2.22}$$

where $U_0^{(j)}(p)$ is any entire function (for example, one can put $U_0^{(j)}(p) = 1$), and the functions $U_k^{(j)}(p)$ are solutions of the following recurrent system

$$\begin{aligned}
\hat{H}_0(p)\, U_1^{(j)}(p) &= -\overset{2}{\hat{a}}\, \hat{H}_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) U_0^{(j)}(p), \\
\hat{H}_0(p)\, U_k^{(j)}(p) &= -\overset{2}{\hat{a}}\, \hat{H}_1\left(\overset{2}{\hat{a}}, \overset{1}{p}\right) U_{k-1}^{(j)}(p), \ k = 2, 3, \ldots.
\end{aligned}$$

The difference between the above considered case and this one is that all the functions $U_k^{(j)}(p)$ are supported at one and the same point $p = p_j$, and, hence, we must prove the convergence of series (2.22) in the space of endlessly continuable functions.

Such an affirmation can be proved, if one takes into account the fact that the operator $\hat{a}$ can be represented as the convolution

$$\hat{a}U(p) = a(p) * U(p)$$

with the function $a(p) = \langle G^*(p), a \rangle$ which has a weak singularity at the origin (and the origin is the only singular point of this function). Due to this fact, equation (2.21) is an equation of the Volterra type, and this allows one to obtain the needed estimates (we omit here the details).

So, we arrive at the following statement:

Theorem 2 *Under the above formulated conditions, for any p_j there exists a resurgent solution to equation (2.17) with support at $p = p_j$. The singularity set of the corresponding function $U(p)$ is the spectrum of the operator family $\hat{H}_0(p)$.*

Note that Theorems 1 and 2 establish only the existence of resurgent solutions to the corresponding equation but provide no information about the character of singularities of the corresponding resurgent images. The investigation of the structure of singularities for these images has to be done in every particular case.

2.3 Examples and applications

In the previous section we have constructed formal asymptotic expansions for solutions to homogeneous differential equations on manifolds having cusp-type singularities and proved that these expansions are of the resurgent character. However, as it was mentioned in the previous subsection, the investigation of the particular types of singularities of the resurgent images of the above obtained expansion is different for different equations and can be done only for the concrete equations.

To illustrate such an investigation, we consider here two cases of the simple cusp ($k = 1$) and that of the second order ($k = 2$). It occurs that in the case $k = 1$ the standard resummation procedure based on the Borel-Laplace transform supplies us with the representation of solutions which gives the full picture of asymptotic behavior of this solution near the considered cusp point. At the same time, for cusps of higher order (in fact, even for $k = 2$) the application of the Borel-Laplace transform occurs to be inappropriate. The matter is that under the action, say, 2-Borel transform the corresponding function in the dual space is not more a function with simple singularities, and the information about the asymptotic behavior of solutions occurs to be "hidden" in such a representation. In this case one should use the general resurgent representation [11, 10].

2.3.1 Case of a simple cusp

The first example we consider here, is a differential equation on a manifold with a simple cusp. Such an equation has the form

$$\hat{H}\left(r, \omega, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) u = 0, \tag{2.23}$$

where, as above, r is a coordinate along the axis of a cone, and ω are local coordinates in the base Ω of this cone. First, we expand this equation into the Taylor series in r up to the second order:

$$\left[\hat{H}\left(0, \omega, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) + r\frac{\partial \hat{H}}{\partial r}\left(0, \omega, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right) + r^2\hat{H}_1\left(r, \omega, r^2\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega}\right)\right] u = 0.$$

Applying the Borel transform to the latter equation, we obtain the equation for the Borel image $U(p, \omega)$ of the function $u(r, \omega)$:

$$\left[\hat{H}\left(0, \omega, p, \frac{\partial}{\partial \omega}\right) + \left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial \hat{H}}{\partial r}\left(0, \omega, p, \frac{\partial}{\partial \omega}\right)\right.$$

$$+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_1\left(\left(\frac{\partial}{\partial p}\right)^{-1}, \omega, p, \frac{\partial}{\partial \omega}\right)\Bigg]\,U(p,\omega)=0.$$

It is convenient to rewrite the latter equation in the operator form[3]

$$\left[\hat{H}(0,p)+\left(\frac{\partial}{\partial p}\right)^{-1}\frac{\partial \hat{H}}{\partial r}(0,p)+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_1\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)\right]U(p)=0, \qquad (2.24)$$

where all the operators considered are acting on functions determined on the manifold Ω. To investigate the asymptotic behavior of solutions to this equation one should:

- Prove the existence of an *endlessly-continuable* solution to equation (2.24) and investigate the position of singularities of this solution.
- To investigate the em asymptotics in smoothness of the constructed solution at any point of its singularity.

For simplicity, we suppose that

The inverse $\hat{R}(p)$ of the operator $\hat{H}(0,p)$ has a simple poles only and that the kernel (and, hence, the cokernel) of the operator $\hat{H}(0,p_j)$ is one-dimensional for any spectral value p_j of the family $\hat{H}(0,p)$.

Later on, we consider the case when the point $p=0$ is a spectral point of the family $\hat{H}(0,p)$. The latter condition does not anyway lead to the loss of generality since in the opposite case it is sufficient to multiply the solution to the initial equation by the function $\exp(-S/r)$ with appropriate value of S. We suppose also that the quantity

$$\left\langle V, \frac{\partial \hat{H}}{\partial p}(0,0)\,U\right\rangle$$

does not vanish, where U and V are generators of the kernel and cokernel of the operator $\hat{H}(0,0)$, respectively.

The following affirmation takes place.

[3] More precisely, we consider the operator

$$\hat{H}\left(r,\omega,r^2\frac{\partial}{\partial r},\frac{\partial}{\partial \omega}\right)$$

as the operator $\hat{H}\left(r, r^2 d/dr\right)$, acting on functions with values in the Sobolev space scale on the manifold Ω. The same concernes all other operators involved into the considered equation.

Theorem 3 *Under the above formulated conditions there exist endlessly-continuable solutions to equation* (2.24)*, with the singularity of the form*

$$U(p) = p^{\gamma} \sum_{j=0}^{\infty} c_j p^j$$

at the origin. The number γ *in the latter relation is determined from*

$$\left\langle V, \frac{\partial \hat{H}}{\partial r}(0,0)\, U \right\rangle + \gamma \left\langle V, \frac{\partial \hat{H}}{\partial p}(0,0)\, U \right\rangle = 0.$$

Now the resurgent analysis method [11] shows that the above constructed formal asymptotic expansions are summable.

The rest part of this subsection is aimed at the proof of Theorem 3.

Proof. It is convenient to write down the equation for the Borel transform of the function $v(r,\omega)$ which is obtained from the solution $u(r,\omega)$ of the initial problem by the change of the unknown

$$u(r,\omega) = r^{\gamma} v(r,\omega).$$

(Below we shall show that the value of γ must be determined from the last relation of the theorem, but temporarily we rest this value undefined.) In doing so, we arrive at the equation

$$\left\{ \hat{H}(0,p) + \left(\frac{\partial}{\partial p}\right)^{-1} \left[\frac{\partial \hat{H}}{\partial r}(0,p) + \gamma \frac{\partial \hat{H}}{\partial p}(0,p) \right] \right.$$
$$\left. + \left(\frac{\partial}{\partial p}\right)^{-2} \hat{H}_2\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p \right) \right\} V(p) = 0.$$

for the function $V(p,\omega) = \mathcal{B}[v(r,\omega)]$. Let us perform one more modification of the latter equation putting

$$v(r,\omega) = \left(1 + r\hat{B}\right) w(r,\omega),$$

or

$$V(p) = \left(1 + \left(\frac{\partial}{\partial p}\right)^{-1} \hat{B}\right) W(p),$$

where the operator $\hat{B}$ will be determined later. The resulting equation for W is

$$\begin{aligned}&\left\{\hat{H}(0,p)+\left(\frac{\partial}{\partial p}\right)^{-1}\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)+\hat{H}(0,p)\,\hat{B}\right]\right.\\&\left.+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1},p\right)\right\}W(p)=0.\end{aligned}\tag{2.25}$$

We notice that due to the above assumptions the following relation takes place:

$$\left[\hat{H}(0,p)\right]^{-1}=\frac{R_0}{p}+R_1(p)\,,\tag{2.26}$$

where R_0 is a finite-dimensional operator, and $R_1(p)$ is regular at $p=0$. Let us search for the solution of equation (2.25) in the form

$$W(p)=\left[\hat{H}(0,p)\right]^{-1}\tilde{W}(p)+\left[\hat{H}(0,p)\right]^{-1}0.$$

(We remark that the last summand does not vanish since equation (2.25) is considered as an equation in hyperfunctions. This summand equals $p^{-1}U_0(p)+U_1(p)$, where $U_1(p)$ is a regular function, and $U_0(p)$ is an element of the kernel of the operator $\hat{H}(0,p)$). We arrive at the following equation for $\tilde{W}(p)$:

$$\begin{aligned}&\left\{1+\left(\frac{\partial}{\partial p}\right)^{-1}\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)+\hat{H}(0,p)\,\hat{B}\right]\left[\hat{H}(0,p)\right]^{-1}\right.\\&\left.+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1},p\right)\left[\hat{H}(0,p)\right]^{-1}\right\}\tilde{W}(p)=F(p)\,,\end{aligned}\tag{2.27}$$

where the function $F(p)$ has simple singularities at $p=0$:

$$F(p)=\ln p\sum_{j=0}^{\infty}F_jp^j.$$

To investigate the form of singularities of the function $U(p)$, it is sufficient to notice that the operator

$$\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1},p\right)$$

involved into equation (2.25) has the form

$$\left(\frac{\partial}{\partial p}\right)^{-2}\hat{H}_3\left(\left(\frac{\partial}{\partial p}\right)^{-1}, p\right)U(p)=\left(\frac{\partial}{\partial p}\right)^{-1}\hat{H}_4(p,p')*U(p)\Big|_{p=p'},$$

where the convolution is taken over the variable p, and the function $\hat{H}_4(p,p')$ equals

$$\hat{H}_4(p,p')=\sum_{i=0}^{\infty}\frac{p^j}{j!}\hat{h}_j(p'),\tag{2.28}$$

if the expansion of the function $\hat{H}_3(r,p)$ in powers of r is

$$\hat{H}_3(r,p')=\sum_{i=0}^{\infty}r^j\hat{h}_j(p').$$

We note that function (2.28) is an entire function in p for any fixed p.

Let us try now to define the number γ and the operator $\hat{B}$ in such a way that the operator

$$\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)+\hat{H}(0,p)\hat{B}\right]\left[\hat{H}(0,p)\right]^{-1},\tag{2.29}$$

involved in the left-hand part of (2.27), is regular at $p=0$. Due to (2.26), the singular part of this operator is

$$p^{-1}\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)+\hat{H}(0,p)B\right]R_0.$$

Let P be a projector to the image of the operator $\hat{H}(0,0)$. The operator

$$\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)$$

can be rewritten in the following form:

$$\begin{aligned}\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p) \quad &= \quad P\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)\right]\\ &\quad +(1-P)\left[\frac{\partial\hat{H}}{\partial r}(0,p)+\gamma\frac{\partial\hat{H}}{\partial p}(0,p)\right].\end{aligned}$$

Choose the number γ from the condition

$$(1-P)\left[\frac{\partial \hat{H}}{\partial r}(0,p)+\gamma\frac{\partial \hat{H}}{\partial p}(0,p)\right]R_0=0.$$

For this choice to be possible, it is necessary that the inequality

$$\left\langle V,\frac{\partial \hat{H}}{\partial p}(0,0)\,U\right\rangle\neq 0,$$

holds (see the considerations before Theorem 3).Here, as above, U and V are generators of the kernel and cokernel of the operator $\hat{H}(0,0)$, consequently. Now we put[4]

$$\hat{B}=\left[\hat{H}(0,0)\right]^{-1}P\left[\frac{\partial \hat{H}}{\partial r}(0,0)+\gamma\frac{\partial \hat{H}}{\partial p}(0,0)\right].$$

The reader will easily verify that with such choice of γ and B operator (2.29) is regular.

So, the equation (2.27) for $\tilde{W}$ takes the form of a Volterra equation

$$\left[1+\left(\frac{\partial}{\partial p}\right)^{-1}\hat{C}_1(p)+\left(\frac{\partial}{\partial p}\right)^{-2}\hat{C}_2(p)\right]\tilde{W}(p)=F(p),$$

where the operator $\hat{C}_1(p)$ is regular for $p=0$, and the operator $\hat{C}_2(p)$ has a simple polar singularity at this point. The end of the proof of the theorem can be carried out with the help of a standard method of successive approximations.

2.3.2 Case of a cusp of higher multiplicity

As we have already mentioned, the case $k=2$ is, in essence, a general one, and we restrict ourselves by the consideration of this case.

We recall (see formula (2.15) above) that formal solutions to equation (2.1) for $\alpha=k=2$ have the form

$$u(r)=\exp\left(-\frac{p_0}{2r^2}-\frac{p_1}{r}\right)r^{p_2}\sum_{l=0}^{\infty}r^l u_l. \tag{2.30}$$

[4]The operator $\hat{H}(0,0)$ is understood here as an operator from the subspace complementary to the image R_0 to the image of $\hat{H}(0,0)$. Then, the composition $\left[\hat{H}(0,0)\right]^{-1}P$ (and, hence, the operator $\hat{B}$) is well-defined.

The main difference between this case and the case considered in the previous subsection is that if we represent the solution $u(r)$ in the form of the 2-Borel transform

$$u(r) = \int_{\Gamma} \exp\left\{-\frac{p}{2r^2}\right\} U(p)\, dp$$

where the contour Γ is chosen as a standard contour of resurgent representation, then the function $U(p)$ fails to have simple singularities. The reason for this phenomenon is that, as we have mentioned above, that the phase function (action) of asymptotic expansion (2.30) *is not a homogeneous function in* r. So, we use the change of the unknown which reduces our function to the case of homogeneous action, then investigate this function with the help of the 2-Laplace transform, and, finally, show that the initial function can be written down as a resurgent representation with the function $U(r,s)$ having simple singularities.

So, to have the possibility to use 2-Laplace transform for the construction of a function with simple singularities, one should perform the change of the unknown

$$u(r) = \exp\left(-\frac{p_1}{r}\right) r^{p_2} v(r). \tag{2.31}$$

Similar to the formal theory, we shall investigate a solution corresponding to some fixed value of p_0, which is chosen as

$$p = p(0),$$

where $p = p(r)$ is some branch of the spectrum of family (2.6).

Let us represent equation (2.1) for $\alpha = k = 2$ in the form

$$\left[\hat{H}_0\left(r^3\frac{d}{dr}\right) + r\hat{H}_1\left(r^3\frac{d}{dr}\right) + r^2\hat{H}_2\left(r^3\frac{d}{dr}\right) + r^3\hat{H}_3\left(r, r^3\frac{d}{dr}\right)\right] u(r) = 0$$

and substitute the function $u(r)$ in the form (2.31) in it. We have

$$\left(r^3\frac{d}{dr}\right) u(r) = e^{-\frac{p_1}{r}} r^{p_2}\left\{rp_1 + r^2 p_2 + \left(r^3\frac{d}{dr}\right)\right\} v(r).$$

Therefore, after the substitution the equation is reduced to the form

$$\begin{aligned} &\hat{H}_0\left(rp_1 + r^2p_2 + r^3\frac{d}{dr}\right) + r\hat{H}_1\left(rp_1 + r^2p_2 + r^3\frac{d}{dr}\right) \\ +r^2\hat{H}_2\left(rp_1 + r^2p_2 + r^3\frac{d}{dr}\right) &+ r^3\hat{H}_3\left(r, rp_1 + r^2p_2 + r^3\frac{d}{dr}\right) v(r) \quad = \quad 0. \end{aligned}$$

Expanding operators

$$\hat{H}_j\left(rp_1 + r^2p_2 + r^3\frac{d}{dr}\right), \quad j = 0, 1, 2$$

into the Taylor series in r up to the third power, we arrive at the equation

$$\begin{aligned}&\left\{\hat{H}_0\left(r^3\frac{d}{dr}\right) + r\left[\hat{H}_1\left(r^3\frac{d}{dr}\right) + p_1\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\right.\\ &+r^2\left[\hat{H}_2'\left(p_1, r^3\frac{d}{dr}\right) + p_2\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\\ &\left.+r^3\hat{H}_3'\left(r, r^3\frac{d}{dr}\right)\right\} v(r) = 0,\end{aligned} \tag{2.32}$$

where

$$\hat{H}_2'\left(p_1, r^3\frac{d}{dr}\right) = \hat{H}_2\left(r^3\frac{d}{dr}\right) + p_1\frac{\partial\hat{H}_1}{\partial p}\left(r^3\frac{d}{dr}\right) + \frac{1}{2}\frac{\partial^2\hat{H}_0}{\partial p^2}\left(r^3\frac{d}{dr}\right).$$

We choose the numbers p_1 and p_2 in such a way that the operators

$$\left[\hat{H}_1(p) + p_1\frac{\partial\hat{H}_0}{\partial p}(p)\right]\hat{H}_0^{-1}(p)$$

and

$$\left[\hat{H}_2'(p_1, p) + p_2\frac{\partial\hat{H}_0}{\partial p}(p)\right]\hat{H}_0^{-1}(p)$$

have no singularity at the point considered. As we have already seen below, to do this one have to use one more change of the unknown. For $k = 2$ such a change has the form

$$v(r) = \left(1 + r\hat{B}_1 + r^2\hat{B}_2\right) w(r), \tag{2.33}$$

where $\hat{B}_1$ and $\hat{B}_2$ are some (unknown, up to the moment) operators in the functional space E. Substituting (2.33) into (2.32), we derive the equation for $w(r)$ in the form

$$\begin{aligned}&\left\{\hat{H}_0\left(r^3\frac{d}{dr}\right) + r\left[\hat{H}_1\left(r^3\frac{d}{dr}\right) + p_0\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right) + \hat{H}_0\left(r^3\frac{d}{dr}\right)\hat{B}_1\right]\right.\\ &+r^2\left[\hat{H}_2''\left(p_1, B_1, r^3\frac{d}{dr}\right) + p_2\frac{\partial\hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right) + \hat{H}_0\left(r^3\frac{d}{dr}\right)\hat{B}_2\right]\\ &\left.+r^3\hat{H}_3''\left(r, r^3\frac{d}{dr}\right)\right\} w(r) = 0,\end{aligned}$$

where

$$\hat{H}_2''\left(p_1, \hat{B}_1, r^3\frac{d}{dr}\right) = \hat{H}_2'\left(p_1, r^3\frac{d}{dr}\right) + \left[\hat{H}_1\left(r^3\frac{d}{dr}\right) + p_1\frac{\partial \hat{H}_0}{\partial p}\left(r^3\frac{d}{dr}\right)\right]\hat{B}_1.$$

The last equation is already prepared for the application of the 2-Borel transform. Applying this transform, we arrive at the following equation for the Borel image $W(p)$ of the function $w(r)$:

$$\begin{aligned} &\left\{\hat{H}_0(p) + \left(\frac{\partial}{\partial p}\right)^{-1/2}\left[\hat{H}_1(p) + p_1\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_1\right]\right. \\ &+ \left(\frac{\partial}{\partial p}\right)^{-1}\left[\hat{H}_2''\left(p_1, \hat{B}_1, p\right) + p_2\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_2\right] \\ &\left.+ \left(\frac{\partial}{\partial p}\right)^{-3/2}\hat{H}_3''\left(\left(\frac{\partial}{\partial p}\right)^{-1/2}, p\right)\right\} W(p) = 0. \end{aligned} \tag{2.34}$$

We reduce the latter equation to an equation of the Volterra type by putting

$$W(p) = \hat{H}_0^{-1}(p)\, W_1(p) + \hat{H}_0^{-1}(p)\, 0$$

(the last summand on the right does not vanish since the equation is considered in hyperfunctions). For $W_1(p)$ we obtain

$$\begin{aligned} &\left\{1 + \left(\frac{\partial}{\partial p}\right)^{-1/2}\left[\hat{H}_1(p) + p_1\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_1\right]\hat{H}_0^{-1}(p)\right. \\ &+ \left(\frac{\partial}{\partial p}\right)^{-1}\left[\hat{H}_2''(p_1, B_1, p) + p_2\frac{\partial \hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\hat{B}_2\right]\hat{H}_0^{-1}(p) \\ &\left.+ \left(\frac{\partial}{\partial p}\right)^{-3/2}\hat{H}_3''\left(\left(\frac{\partial}{\partial p}\right)^{-1/2}, p\right)\hat{H}_0^{-1}(p)\right\} W_1(p) = F(p) \end{aligned} \tag{2.35}$$

with some right-hand part $F(p)$ having a polar singularity of the first order at $p = p_0$. Equation (2.35) is an equation of the Volterra type. The coefficients of this equation have polar singularities of the first order at $p = p_0$. As above, for the solution to this equation obtained by the successive approximation method to have simple singularities, it is necessary to require that the operator coefficients of $(\partial/\partial p)^{-1/2}$ and $(\partial/\partial p)^{-1}$ are regular at $p = p_0$. Since, under the above assumptions,

$$\hat{H}_0^{-1}(p) = \frac{R_0}{p - p_0} + R_1(p)$$

with $R_1(p)$ regular at $p = p_0$, the coefficient of $(\partial/\partial p)^{-1/2}$ can be written down in the form

$$\frac{1}{p-p_0}\left[\hat{H}_1(p_0) + p_1\frac{\partial\hat{H}_0}{\partial p}(p_0) + \hat{H}_0(p_0)\,\hat{B}_1\right] R_0 + \{\text{regular operator}\}\,.$$

We have to choose the number p_1 and the operator $\hat{B}_1$ in such a way that the operator in the square brackets on the left in the latter relation vanishes.

Since the image of the operator R_0 coincides with the kernel of the operator $\hat{H}_0(p)$, this image is a one-dimensional subspace generated by the vector $U = U(0)$. Hence, if we choose p_1 from the relation

$$\left\langle V, \left[\hat{H}_1(p_0) + p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right] U\right\rangle = 0$$

(which is possible since $\left\langle V, \partial\hat{H}_0/\partial p\,(p_0)\,U\right\rangle \neq 0$, see relation (2.7) for $r = 0$), then the image of the operator

$$\left[\hat{H}_1(p_0) + p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right] R_0$$

will be contained in the image of the operator $\hat{H}_0(p_0)$ and, hence, we can determine the operator $\hat{B}_1$ by the relation

$$\hat{B}_1 = -\hat{H}_0^{-1}(p_0)\left[\hat{H}_1(p_0) + p_1\frac{\partial\hat{H}_0}{\partial p}(p_0)\right] R_0.$$

It is clear that for such choice of the operator $\hat{B}_1$ the coefficient of $(\partial/\partial p)^{-1/2}$ in equation (2.35) is regular at $p = p_0$.

Let us consider now the coefficient of the operator $(\partial/\partial p)^{-1}$ in equation (2.35). It equals

$$\left[\hat{H}_2''\left(p_1, \hat{B}_1, p\right) + p_2\frac{\partial\hat{H}_0}{\partial p}(p) + \hat{H}_0(p)\,\hat{B}_2\right]\hat{H}_0^{-1}(p)\,, \qquad (2.36)$$

where p_1 and $\hat{B}_1$ are already fixed. The procedure of the choice of the number p_2 and the operator $\hat{B}_2$ from the condition of regularity of operator (2.36) is literally the same as the procedure of the choice of p_1 and $\hat{B}_1$ above. We leave the description of this procedure to the reader.

Now equation (2.35) can be solved with the help of the successive approximation method. As a result, we obtain the following affirmation:

Proposition 3 *There exist a unique solution to equation* (2.35) *which is analytic everywhere except for poles of the operator family* $\hat{H}_0(p)$ *and which has simple singularities at the point* p_0.

We emphasize that the latter proposition *does not imply* that the corresponding solution $u(r)$ to equation (2.1) for $\alpha = k = 2$ occurs to be a resurgent function with simple singularities in the sense of 2-Laplace transform. From the other hand, the following statement takes place:

Theorem 4 *For each pole* p_0 *of the operator family* $\hat{H}_0(p)$ *there exists a solution* $u(r)$ *to equation* (2.1) *with* $\alpha = k = 2$ *of the form* (2.30)*, which is a resurgent function with simple singularities in the sense of the resurgent representation*

$$u(r) = \int_{\Gamma} e^{-s} V_1(s, r)\, ds. \tag{2.37}$$

Proof. As it follows from Proposition 3, the function $w(r)$, which is 2-Laplace transform of the solution $W(p)$ to equation (2.34) is a resurgent function with simple singularities in the sense of 2-Laplace transform. Clearly, the same is true for the function $v(r)$ defined via $w(r)$ by relation (2.33). This means that the function $v(r)$ admits a representation of the form

$$v(r) = \int_{\Gamma} e^{-\frac{p}{r^2}} V(p)\, dp, \tag{2.38}$$

where the function $V(p)$ is a function with simple singularities. Representation (2.38) can be easily rewritten in the form of resurgent representation (2.37). To complete the proof of the theorem, it remains to note that the operators of multiplication by $\exp(-p_1/r)$ and r^{p_2} involved into representation (2.31) of the function $u(r)$ preserve the class of resurgent functions with simple singularities (the first of these operators realizes the shift in the s-plane, and the second just changes the powers of r involved into the considered expansion).

To conclude this subsection, we remark that the constructed resurgent solutions clearly coincide with results of resummation of formal solutions obtained in Subsection 2.1. This follows from the fact that the computational procedure for coefficients of formal expansion is unique.

2.3.3 Investigation of the asymptotic catastrophe

Here we investigate ([12]) the equation

$$\hat{H}\left(x, x^{1+\alpha}\frac{d}{dx}\right) u = 0 \tag{2.39}$$

for $\alpha \in \mathbf{C}$, $\mathrm{Re}\alpha \geq 0$. The most interesting is the investigation of the behavior of the solutions as $\alpha \to 0$, so we shall consider α to be sufficiently small in module.

The interest for the investigation of the limit $\alpha \to 0$ is due to the fact that solutions to equation (2.39) has quite different structure in cases $\alpha \neq 0$ and $\alpha = 0$. In the first case each solution is a function with support at a single point p_j, where $p_1, p_2, \ldots$ are spectral points of the family $\hat{H}(0,p)$. The asymptotics of these solutions have the form

$$u_j \simeq e^{-\frac{p_j}{\alpha x^\alpha}} \sum_{k=0}^{\infty} c_k x^{\gamma_k}, \tag{2.40}$$

where γ_k is some increasing sequence of reals such that $\gamma_k \to +\infty$ as $k \to \infty$. The series on the right in the latter expansion are, as a rule, divergent, and one must use the resurgent analysis for resummating these series.

On the opposite, in the second case (that is, for $\alpha = 0$), equation (2.39) is an equation of Fuchs type, and the solution to this equation has the well-known conormal form

$$u_j \simeq \sum_{k=0}^{\infty} x^{p_{kj}} \sum_{l=0}^{m_{kj}-1} c_{jkl} \ln^l x, \tag{2.41}$$

where

$$\{p_{kj} = p_j + k\} \tag{2.42}$$

are lattices originated from points p_j with step 1, and m_{kj} are corresponding multiplicities. As it can shown, this functions are resurgent with respect to the Borel-Mellin representation; the supports of these functions are given by (2.42), and there are finite number of terms in the expansions corresponding to each point of support, so that the resummation procedure is not needed for the interpretation of this expansion.

One can show, that the question how expansion (2.40) transforms into expansion (2.41) can be solved with the above introduced technique.

Let us compute the corresponding family of groups. Due to the above theory, the corresponding function $G(p,x,\alpha)$ must be a solution to the equation

$$x^{1+\alpha}\frac{d}{dx}G(p,x,\alpha) = pG(p,x,\alpha).$$

The general solution to this equation is

$$G(p,x,\alpha) = C(p,\alpha)\, e^{-\frac{p}{\alpha x^\alpha}},$$

where $C(p,\alpha)$ is an arbitrary constant. Since we are interested in investigation of the limit $\alpha \to 0$, we must choose this constant in such a way that the function

$G(p,x,\alpha)$ is regular at $\alpha=0$. Note that we cannot use here $C(p,\alpha)=1$ since the function $\exp[-p/(\alpha x^{\alpha})]$ has an essential singularity at the point $\alpha=0$. It is easy to see that to obtain a function $G(p,x,\alpha)$ regular at $\alpha=0$ it is sufficient to put

$$C(p,\alpha)=e^{\frac{p}{\alpha}}.$$

So, finally we have

$$G(p,x,\alpha)=e^{-p\left(\frac{1}{\alpha x^{\alpha}}-\frac{1}{\alpha}\right)},$$

and the corresponding representation has the form

$$\mathcal{R}_{G,\alpha}(U)=\int_{\Gamma}e^{-p\left(\frac{1}{\alpha x^{\alpha}}-\frac{1}{\alpha}\right)}U(p)\;dp.$$

The limit of this representation as $\alpha\to 0$ is

$$\lim_{\alpha\to 0}\mathcal{R}_{G,\alpha}(U)=\mathcal{R}_{G,0}(U)=\int_{\Gamma}x^{p}U(p)\;dp,$$

which, naturally, coincides with the Borel-Mellin representation. So, we have constructed a deformation of the representations connecting Borel-Laplace representations of different orders α with the Borel-Mellin representation.

Now, one has to investigate the corresponding operator $\hat{x}_{\alpha}$. Denote by $\mathcal{L}_{\alpha}$ the classical Laplace representation of order α:

$$\mathcal{L}_{\alpha}(U)=\int_{\Gamma}e^{-\frac{p}{\alpha x^{\alpha}}}U(p)\;dp.$$

Clearly, we have

$$\mathcal{R}_{G,\alpha}(U)=\mathcal{L}_{\alpha}\left(e^{\frac{p}{\alpha}}U\right).$$

Now, taking into account the relation

$$x\mathcal{L}_{\alpha}(U)=\mathcal{L}_{\alpha}\left(\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}}U\right),$$

one has

$$\begin{aligned}x\mathcal{R}_{G,\alpha}(U)&=x\mathcal{L}_{\alpha}\left(e^{\frac{p}{\alpha}}U\right)=\mathcal{L}_{\alpha}\left(\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}}e^{\frac{p}{\alpha}}U\right)\\&=\mathcal{L}_{\alpha}\left(e^{\frac{p}{\alpha}}\left[e^{-\frac{p}{\alpha}}\alpha^{-\frac{1}{\alpha}}\left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}}e^{\frac{p}{\alpha}}\right]U\right)=\mathcal{R}_{G,\alpha}(\hat{x}_{\alpha}U),\end{aligned}$$

where the operator $\hat{x}_\alpha$ is given by

$$\hat{x}_\alpha = e^{-\frac{p}{\alpha}} \alpha^{-\frac{1}{\alpha}} \left(\frac{d}{dp}\right)^{-\frac{1}{\alpha}} e^{\frac{p}{\alpha}}. \tag{2.43}$$

The latter operator can be rewritten as the convolution with some function $X(p,\alpha)$. To do this, we use the relation

$$\left(\frac{d}{dp}\right)^{-\beta} U(p) = U(p) * \frac{p^{\beta-1}}{\Gamma(\beta)}$$

valid for $\beta > 0$ (here $\Gamma(\beta)$ is the Gamma function). Using this relation, operator (2.43) can be rewritten in the form

$$\hat{x}_\alpha U(p) = U(p) * X(p,\alpha),$$

where

$$X(p,\alpha) = \frac{\alpha^{-\frac{1}{\alpha}}}{\Gamma\left(\frac{1}{\alpha}\right)} p^{\frac{1}{\alpha}-1} e^{-\frac{p}{\alpha}}.$$

One can see from the latter relation that the less in module the number α is, the more negative order with respect to the asymptotic scale $\{\mathcal{UF}_\alpha,\ \alpha \in \mathbf{R}\}$ the operator $\hat{x}_\alpha$ has. From the other hand, this operator has zero order with respect to the scale $\left\{\mathcal{U}^{G,r}_{R,\alpha},\ R \in \mathbf{R}\right\}$ for any α with $\mathrm{Re}\alpha > 0$.

The following affirmation is almost evident.

Lemma 2 *The following relation takes place*

$$\lim_{\alpha \to 0} \hat{x}_\alpha = T_1,$$

where T_1 is the shift by 1 in the plane $\mathbf{C}$:

$$T_1 U(p) = U(p-1).$$

Proof. Since

$$\lim_{\alpha \to 0} \mathcal{R}_{G,\alpha}(\hat{x}_\alpha U(p)) = \lim_{\alpha \to 0} x \mathcal{R}_{G,\alpha}(U(p)) = x\mathcal{R}_{G,0}(U(p)) = \mathcal{R}_{G,0}(T_1 U(p)),$$

the required statement follows from the invertibility of the representation $\mathcal{R}_{G,\alpha}$ up to $\alpha = 0$.

Now one can see that the terms of asymptotic expansion (2.40) being concentrated at one and the same point of support p_j of the resurgent solution for $\mathrm{Re}\alpha > 0$, "jump" from this point to points $p_j + k$, $k \in \mathbf{Z}_+$ when α becomes to be equal to zero.

Bibliography

[1] V.P. Maslov. *Théorie des Perturbations et Méthods Asymptotiques.* Dunod, Paris, 1972. French transl. from Russian.

[2] V.P. Maslov. *Operator Methods.* Nauka, Moscow, 1973. English transl.: "*Operational Methods*", Mir, Moscow, 1976.

[3] V. G. Maz'ya and B. A. Plamenevskii. On the asymptotic of solutions of differential equations with operator coefficients. *DAN SSSR*, **196**, 1971, 512 –515. English transl. in: *Soviet Math. Dokl.* **12** (1971).

[4] V. G. Maz'ya and B. A. Plamenevskii. On the asymptotic solution of the Dirichlet problem near the isolated singularity of the boundary. *Vestnik Leningrad Univ. Math.*, , No. 13, 1977. English transl. in: *Vestnik Leningrad Univ. Math.* **10** (1982), 295 – 302.

[5] A. Mishchenko, V. Shatalov, and B. Sternin. *Lagrangian Manifolds and the Maslov Operator.* Springer-Verlag, Berlin – Heidelberg, 1990.

[6] V. Nazaikinskii, B. Sternin, and V. Shatalov. *Methods of Noncommutative Analysis. Theory and Applications.* Mathematical Studies. Walter de Gruyter Publishers, Berlin – New York, 1995.

[7] B.-W. Schulze. *Pseudodifferential Operators on Manifolds with Singularities.* North-Holland, Amsterdam, 1991.

[8] B.-W. Schulze, B. Sternin, and V. Shatalov. *Asymptotic Solutions to Differential Equations on Manifolds with Cusps.* Max-Planck-Institut für Mathematik, Bonn, 1996. Preprint MPI/96-89.

[9] B.-W. Schulze, B. Sternin, and V. Shatalov. Structure Rings of Singularities and Differential Equations. In *Differential Equations, Asymptotic Analysis,*

and Mathematical Physics, number 100 in Mathematical Research, 1996, pages 325–347, Berlin. Akademie Verlag.

[10] B. Sternin and V. Shatalov. *Asymptotic Theory and Resurgent Functions.* Max-Planck-Institut für Mathematik, Bonn, 1996. Preprint MPI 96-29.

[11] B. Sternin and V. Shatalov. *Borel-Laplace Transform and Asymptotic Theory.* CRC-Press, Boca Raton, Florida, USA, 1996.

[12] B. Sternin and V. Shatalov. *Deformations of Resurgent Representations and Exact Asymptotics of Solutions to Differential Equations.* Max-Planck-Institut für Mathematik, Bonn, 1996. Preprint MPI 96-132.