L^p Spectral Independence of Elliptic Operators via Commutator Estimates

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Abstract

Let $\{T_p : q_1 \leq p \leq q_2\}$ be a family of consistent C_0 semigroups on $L^p(\Omega)$, with $q_1, q_2 \in [1, \infty)$ and $\Omega \subseteq \mathbb{R}^n$ open. We show that certain commutator conditions on T_p and on the resolvent of its generator A_p ensure the p independence of the spectrum of A_p for $p \in [q_1, q_2]$.

Applications include the case of Petrovskij correct systems with Hölder continuous coefficients, Schrödinger operators, and certain elliptic operators in divergence form with real, but not necessarily symmetric, or complex coefficients.

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0 Introduction

Under what circumstances is the L^p-spectrum of an elliptic operator acting in $L^p(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set, independent of $p \in [1, \infty)$? This question attracted new attention when B. Simon conjectured the p-independence of the spectrum of Schrödinger operators with Kato class potentials [25] in $L^p(\mathbb{R}^n)$. An affirmative answer was given by Hempel and Voigt in 1987 [18]. On the other hand, it is well-known that, in general, the L^p-spectra of elliptic operators depend on p. For various aspects in this direction see e.g. Amann [1], Arendt [3], Davies [11], [12], Davies, Simon, and Taylor [14], Hieber [20], Leopold and Schrohe [23], [24], Sturm [27].

Spectral properties of elliptic operators A are intimately related to questions about the proper definition of the domain of A in $L^p(\Omega)$. Here we adopt the view that A should be the generator of a strongly continuous semigroup T on $L^2(\Omega)$ which can be extended consistently to $L^p(\Omega)$. More precisely, we assume that there exists a C_0-semigroup T_p on $L^p(\Omega)$, $1 \leq p < \infty$, such that

$$T_p(t)f = T(t)f, \quad t \geq 0, f \in L^2 \cap L^p.$$

Denote by A_p the generator of T_p. In particular, A_p always is a closed operator.

The case which is best understood is when A is a second order selfadjoint differential operator in $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ open, with a heat-kernel bound for $T(t)$, i.e. $T(t)$ may be represented by an integral kernel $k(t, \cdot, \cdot)$ satisfying

$$|k(t,x,y)| \leq c\, t^{-n/2} e^{\omega t} e^{-a\frac{|x-y|^2}{t}}, \qquad t > 0,\ x, y \in \Omega, \tag{0.1}$$

for suitable $a, c > 0$, $\omega \geq 0$. It was shown independently by Davies [12] and Arendt [3] that then the spectrum of A_p is independent of $p \in [1, \infty)$. For elliptic operators of higher order less results are known. For an excellent and up-to-date summary of this subject we refer to [13].

For non selfadjoint operators acting in $L^2(\Omega)$, the approach of Arendt yields that the connected component of the resolvent set of A_p containing a right half plane is independent of $p \in [1, \infty)$, provided T satisfies a heat-kernel bound of the form (0.1). For an extension of this result to heat-kernel estimates of higher order, we refer to [21]. However, even when T is an analytic semigroup on $L^2(\Omega)$, it seems to be unknown whether an estimate of the form (0.1) implies p-invariance of the spectrum of A_p.

In this note we consider not necessarily self-adjoint operators of arbitrary order m. We prove that even a weaker estimate than (0.1) with only polynomial decay in $|x-y|$ of suitable order, yields L^p spectral independence, provided certain commutator relations are fulfilled. More explicitly: For λ in the L^p-resolvent, we additionally ask the boundedness of the iterated commutators $\mathrm{ad}^\alpha x(A-\lambda)^{-1}$ on $L^p(\Omega)$ for all $|\alpha| \leq n+1$, to guarantee that λ also belongs to the L^q-resolvent for arbitrary q. Heuristically, an iteration of the naive identity

$$[x_j, (\lambda - A_p)^{-1}] = (\lambda - A_p)^{-1}[x_j, A_p](\lambda - A_p)^{-1} \tag{0.2}$$

yields the required continuity whenever $D(A_p) \subseteq W_p^{m-1}(\Omega)$, for then $[x_j, A_p]$ is a differential operator of order $m-1$ and the right hand side is L^p-bounded.

In this light, the proof of L^p spectral invariance becomes very simple and transparent. Practically, however, one often lacks sufficient information on the domain, and (0.2) is more subtle than one might think at first glance.

Below, we first show the general theorem and then check the commutator conditions for several examples, namely Petrovskii correct elliptic systems of order m on $L^p(\mathbb{R}^n)^N$ with Hölder continuous coefficients, Schrödinger operators, and certain classes of elliptic operators in divergence form with real, but not necessarily symmetric, or complex coefficients.

Given a Schrödinger operator $A = -\frac{1}{2}\Delta + V$, the critical term $[x_j, A_p]$ on the right hand side of (0.2) is independent of the potential, and one should expect more general results than the known ones. Our assumption on the domain, however, forces us to impose a restriction on V that makes them even slightly weaker. We still include them because of the simplicity of the proof.

In the following, we denote by $D(A), \sigma(A), \rho(A)$ the domain, spectrum and resolvent set of a linear operator A acting in a Banach space X. Moreover, $\mathcal{L}(X, Y)$ denotes the space of all bounded linear operators from X into some Banach space Y.

1 Main Result

Let Ω be an open set in $\mathbb{R}^n$, $1 \leq p_0 < \infty$, and let A be the generator of a strongly continuous semigroup T_{p_0} on $L^{p_0}(\Omega)$. Given q_1, q_2 with $1 \leq q_1 \leq p_0 \leq q_2 < \infty$ we assume that there exists a family $\{T_p : q_1 \leq p \leq q_2\}$ of strongly continuous semigroups T_p acting on $L^p(\Omega)$, satisfying the consistency relation

$$T_p(t)f = T_{p_0}(t)f, \qquad f \in L^p(\Omega) \cap L^{p_0}(\Omega),\ t \geq 0.$$

By A_p denote the generator of T_p. Under certain assumptions on the iterated commutators of T_p and the resolvent of A_p with multiplication by x we shall show that the spectrum of A_p is independent of p. To this end we need the ad-notation.

1.1 Definition. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ be a multi-index. Given an operator $S : C_0^\infty(\Omega) \to \mathcal{D}'(\Omega)$ we set $\mathrm{ad}^0 x_j(S) = S$, $j = 1, \ldots, n$, and define the commutators of higher order recursively by

$$\mathrm{ad}^k x_j(S) = [x_j, \mathrm{ad}^{k-1} x_j(S)], \qquad k = 1, 2, \ldots;$$

x_j here denotes the operator of multiplication by the j-th coordinate function which acts both on $C_0^\infty(\Omega)$ and $\mathcal{D}'(\Omega)$. Finally we let

$$\mathrm{ad}^\alpha x(S) = \mathrm{ad}^{\alpha_1} x_1 \ldots \mathrm{ad}^{\alpha_n} x_n(S).$$

S might also be a quadratic sytem of operators. We then interpret x_j as multiplication by $x_j I$ and use the same notation.

We now state the main result of this note.

1.2 Theorem. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leq q_1, q_2 < \infty$, $q_1 \leq p_0 \leq q_2$, and let A be the generator of a strongly continuous semigroup T on $L^{p_0}(\Omega)$. Assume that $\{T_q : q_1 \leq q \leq q_2\}$ is a family of consistent C_0-semigroups on $L^q(\Omega)$ with generators A_q. Suppose that, for given $p \in [q_1, q_2]$, given $\lambda \in \rho(A_p)$, and all multi-indices α with $|\alpha| \leq n+1$,*

(A1) $\mathrm{ad}^\alpha x\, T_p(1) \in \mathcal{L}(L^1(\Omega), L^p(\Omega))$,

(A2) $\mathrm{ad}^\alpha x\, T_p(1) \in \mathcal{L}(L^p(\Omega), L^\infty(\Omega))$,

(A3) $\mathrm{ad}^\alpha x\, (\lambda - A_p)^{-1} \in \mathcal{L}(L^p(\Omega))$.

Then $\lambda \in \rho(A_q)$. In particular, the spectrum of A_q is independent of q for all $q \in [q_1, q_2]$, provided conditions (A1), (A2), *and* (A3) *hold for all $p \in [q_1, q_2]$.*

1.3 Corollary. *Let $\Omega \subseteq \mathbb{R}^n$ be open, and let A be the generator of a strongly continuous semigroup T on $L^2(\Omega)$ such that $T(t)f(x) = \int_\Omega k(t,x,y)f(y)dy$ for* a.a. $x \in \Omega$ *and*

$$|k(1,x,y)| \leq c\,(1+|x-y|)^{-2n-2} \tag{1.1}$$

for some $c \geq 0$. Let $\{T_p : 1 \leq p < \infty\}$ be the family of consistent C_0-semigroups on $L^p(\Omega)$ with generators A_p. Then (A1) *and* (A2) *result from the inequalities of Young and Hölder, and the spectrum of A_p is independent of $p \in [1,\infty)$ provided condition* (A3) *holds for all $p \in [1,\infty)$.*

The above estimate holds whenever we have the classical heat kernel bound

$$|k(t,x,y)| \leq c\,t^{-n/m} e^{\omega t} e^{-a\frac{|x-y|^{m/(m-1)}}{t^{1/(m-1)}}} \qquad \text{a.a. } x, y \in \Omega, t \geq 0 \tag{1.2}$$

with suitable $a > 0$, $c, \omega \geq 0$, and $m > 1$.

For the proof of the corollary we simply note that the integral kernel of $\mathrm{ad}^\alpha x T_p(1)$ is $(x-y)^\alpha k(1,x,y)$ which is $O((1+|x-y|)^{-n-1})$ for $|\alpha| \leq n+1$.

As a preparation for the proof of the theorem we need two lemmata.

1.4 Lemma. *Let $S \in \mathcal{L}(L^q(\Omega))$, and let $\lambda \in \rho(A_p), 1 \leq p, q < \infty$. If $Sf = (\lambda - A_p)^{-1}f$ for all $f \in L^p(\Omega) \cap L^q(\Omega)$, then $\lambda \in \rho(A_q)$ and $S = (\lambda - A_q)^{-1}$.*

Proof. Without loss of generality assume that $\lambda = 0$. Since A_p generates a C_0-semigroup T_p on $L^p(\Omega)$ we have $A_p \int_0^t T_p(s)fds = T_p(t)f - f$ for $t \geq 0$ and $f \in L^p(\Omega)$. Hence

$$\int_0^t T_p(s)fds = T_p(t)A_p^{-1}f - A_p^{-1}f. \tag{1.3}$$

for $f \in L^p(\Omega)$ and $t \geq 0$. Therefore,

$$\int_0^t T_q(s)fds = T_q(t)Sf - Sf \tag{1.4}$$

for all $f \in L^p(\Omega) \cap L^q(\Omega)$. Since $L^p(\Omega) \cap L^q(\Omega)$ is dense in $L^p(\Omega)$, equation (1.4) remains valid for $f \in L^q(\Omega)$. It follows that $Sf \in D(A_q)$ and $A_q Sf = f$ for $f \in L^q(\Omega)$. On the other hand, $ST_q(t)f = T_q(t)Sf$ for $f \in L^p(\Omega) \cap L^q(\Omega)$, so S and $T_q(t)$ commute for $t \geq 0$. Therefore, $SA_q f = A_q Sf = f$ for $f \in D(A_q)$. ◁

The following lemma is well-known. For a short proof see [4].

1.5 Lemma. *There is an isometric isomorphism*

$$\mathcal{L}(L^1(\Omega), L^\infty(\Omega)) \cong L^\infty(\Omega \times \Omega),$$

given by associating to an operator $S \in \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$ *its integral kernel.*

Proof of Theorem 1.2. Let $\lambda \in \rho(A_p)$. By rescaling it follows from (1.3) that

$$(\lambda - A_p)^{-1} = \int_0^2 e^{-\lambda s} T_p(s) ds + e^{-2\lambda} T_p(1)(\lambda - A_p)^{-1} T_p(1).$$

Employing the consistency relation and Lemma 1.4, it suffices to show that

$$S_p = T_p(1)(\lambda - A_p)^{-1} T_p(1)$$

induces a bounded operator on $L^q(\Omega)$. The assumptions (A1) and (A2) in connection with Lemma 1.5 imply that the operator S_p can be represented by an integral kernel $k \in L^\infty(\Omega \times \Omega)$. The iterated commutator $\mathrm{ad}^\alpha x(S_p)$ then has the integral kernel $k_\alpha(x, y) = (x-y)^\alpha k(x, y)$ for $x, y \in \Omega$. On the other hand,

$$\mathrm{ad}^\alpha x(S_p) = \sum_{\alpha_1 + \alpha_2 + \alpha_3 \leq \alpha} c_{\alpha_1 \alpha_2 \alpha_3} \, \mathrm{ad}^{\alpha_1} x \, T_p(1) \, \mathrm{ad}^{\alpha_2} x \, (\lambda - A_p)^{-1} \mathrm{ad}^{\alpha_3} x \, T_p(1)$$

with suitable constants by Leibniz' rule for derivations.

The assumptions (A1), (A2), and (A3) together with Lemma 1.5 imply that $k_\alpha \in L^\infty(\Omega \times \Omega)$, provided $|\alpha| \leq n+1$. Hence

$$(x, y) \mapsto k(x, y)|x_j - y_j|^k \in L^\infty(\Omega \times \Omega),$$

for all $k = 1, \ldots, n+1$ and $j = 1, \ldots, n$. Therefore

$$|k(x, y)| \leq C(1 + |x - y|)^{-n-1}$$

with a suitable constant C. The proof is complete. $\triangleleft$

2 Applications to Differential Operators

In this section we shall verify the assumptions of Theorem 1.2 for several classes of operators.

Petrovskij Correct Elliptic Systems

Let $\Omega = \mathbb{R}^n$, and let $\mathcal{A} = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be a system of differential operators with

(i) $a_\alpha \in BUC^\rho(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ for some $\rho \in (0, 1]$ and all $|\alpha| \leq m$,

(ii) $\sup_{x \in \mathbb{R}^n, |\xi|=1} \mathrm{Re}\, \sigma(\sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha) \leq -\delta$, for some $\delta > 0$.

Here, BUC^ρ is the space of bounded uniformly Hölder continuous functions, and σ denotes the spectrum in $\mathcal{L}(\mathbb{C}^N)$.

For $1 < p < \infty$ define the L^p realization A_p of $\mathcal{A}$ by

$$A_p f = \mathcal{A} f \qquad \text{for } \; f \in D(A_p) = W_p^m(\mathbb{R}^n)^N.$$

It follows from [2, Corollary 9.5] that A_p generates a strongly continuous (even analytic) semigroup T_p on $L^p(\mathbb{R}^n)^N$. It is well-known that $T_p(t)$ may be represented by a kernel $k(t,\cdot,\cdot)$ satisfying (1.2), see e.g. Friedman [16, Theorem 9.4.2].

This allows us to define a strongly continuous semigroup T_p on $L^p(\mathbb{R}^n)^N$, $1 \le p < \infty$, in a consistent way. Note that for $p = 1$, the strong continuity of T_1 follows from an argument due to Arendt [3]. According to Guidetti [17, Theorem 1.7], the domain of the generator A_1 of T_1 is

$$D(A_1) = \{u \in B^m_{1,\infty}(\mathbb{R}^n) : \mathcal{A}u \in L^1(\mathbb{R}^n)\}, \tag{2.1}$$

hence a subset of $W_1^{m-1}(\mathbb{R}^n)^N$ which includes $W_1^m(\mathbb{R}^n)^N$. We note a simple consequence:

2.1 Lemma. *Let $f \in D(A_1)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then $\varphi f \in D(A_1)$. Moreover, $C_0^\infty(\mathbb{R}^n)$ is dense in $D(A_1)$.*

Proof. For better legibility we may omit the argument $(\mathbb{R}^n)^N$ with all function spaces. From (2.1) we deduce that $\varphi f \in B^m_{1,\infty}$. Moreover, we see from Leibniz' rule that

$$\mathcal{A}(\varphi f) = \varphi \mathcal{A} f + \sum_{|\alpha| \le m} a_\alpha \sum_{0 < \beta \le \alpha} c_{\alpha\beta} (D^\beta \varphi)(D^{\alpha-\beta} f) \tag{2.2}$$

with the binomial coefficients $c_{\alpha\beta}$. This is an element in L^1, since $\mathcal{A}f \in L^1$ and $f \in W_1^{m-1}$. Next let $\varphi \in C_0^\infty$ with $\varphi(x) \equiv 1$ for small $|x|$. For $\varepsilon > 0$ define $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$. Given $f \in D(A_1)$ we have $\varphi_\varepsilon f \to f \in B^m_{1,\infty}$ and $\varphi_\varepsilon \mathcal{A} f \to \mathcal{A} f$ in L^1 as ε tends to zero. Now consider (2.2) with φ_ε instead of φ: Clearly, $(D^\beta \varphi_\varepsilon)(D^{\alpha-\beta} f) \to 0$ in L^1. So $\mathcal{A}(\varphi_\varepsilon f) \to \mathcal{A} f$ in L^1, and the argument is complete. $\triangleleft$

We can now show the following theorem:

2.2 Theorem. *$\sigma(A_p)$ is independent of p for $1 \le p < \infty$.*

Proof. In view of Corollary 1.3 it only remains to verify condition (A3), which follows by iteration from the lemma, below. $\triangleleft$

2.3 Lemma. *Let $1 \le p < \infty$, $1 \le j \le n$, and $\lambda \in \rho(A_p)$. Then*

$$[x_j, (\lambda - A_p)^{-1}] = (\lambda - A_p)^{-1} [x_j, A_p] (\lambda - A_p)^{-1} \in \mathcal{L}(L^p(\mathbb{R}^n)^N).$$

Proof. As before we omit the argument $(\mathbb{R}^n)^N$. C_0^∞ is dense in $D(A_p)$. This is obvious for $1 < p < \infty$ when $D(A_p) = W_p^m$; for $p = 1$ we saw it in Lemma 2.1. So $R_p := (\lambda - A_p) C_0^\infty$ is a dense subset of L^p consisting of compactly supported functions. In particular, $x_j u \in L^p$ for $u \in R_p$, and $[x_j, (\lambda - A_p)^{-1}]u$ defines an element of $(1 + |x|)L^p$. Next consider, for $u \in R_p$,

$$\begin{aligned} (\lambda - A_p)^{-1} [x_j, A_p] (\lambda - A_p)^{-1} u \;&=\; -(\lambda - A_p)^{-1} x_j (\lambda - A_p)(\lambda - A_p)^{-1} u \\ &\quad + (\lambda - A_p)^{-1} (\lambda - A_p) x_j (\lambda - A_p)^{-1} u. \end{aligned}$$

In view of the fact that $(\lambda - A_p)^{-1}u \in C_0^\infty$, all compositions make sense. We conclude that $[x_j, (\lambda - A_p)^{-1}]u = (\lambda - A_p)^{-1}[x_j, A_p](\lambda - A_p)^{-1}u$ in $(1+|x|)L^p$.

Now we note that $[x_j, A_p] = \sum a_\alpha [x_j, D^\alpha]$ is a differential operator of order $m-1$. Since $D(A_p) \subseteq W_p^{m-1}$, it extends to an operator in $\mathcal{L}(D(A_p), L^p)$. This gives the desired result. $\triangleleft$

Second Order Elliptic Operators on $L^p(\Omega)$ Satisfying the Lopatinskij-Shapiro Condition

Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\partial\Omega \in C^{2+\rho}$ for some $\rho \in (0,1)$. Consider a differential operator $\mathcal{A}$ of the form

$$\mathcal{A} := -\sum_{1 \le i,j \le n} a_{ij}(x)\partial_i\partial_j + \sum_{1 \le i \le n} a_i(x)\partial_i + a_0(x),$$

where $a_{ij}, a_i, a_0 \in BUC^\rho(\Omega)$ and

$$\sum_{1 \le i,j \le n} a_{ij}(x)\xi_i\xi_j \ge c|\xi|^2$$

for all $x \in \Omega$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and some constant $c > 0$. Let $B(x, \partial) := b(x)\cdot\nabla + b_0(x)$ be boundary operators. Writing $b = (b_1, \ldots, b_n)$ we suppose that $b_j \in C^\rho(\partial\Omega)$ for $j = 0, \ldots, n$, and $b(x)\cdot\nu(x) \ge c_0 > 0$; here $\nu(x)$ is the unit outward normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$. Given $p \in (1,\infty)$, the operator A_p^B defined by

$$D(A_p^B) := \{u \in W_p^2(\Omega); Bu = 0\} \qquad A_p^B u := \mathcal{A}u$$

generates an analytic semigroup T_p on $L^p(\Omega), 1 < p < \infty$. Furthermore, it is shown in [22] and [26] that the semigroup T_p generated by $-A_p^B - \omega$ for sufficiently large $\omega > 0$ satisfies an estimate of the form (0.1).

In view of the boundedness of Ω, multiplication by x_j, $j = 1, \ldots, n$, acts continuously on $L^p(\Omega)$ and (A3) holds. Hence we obtain the spectral independence:

2.4 Theorem. *$\sigma(A_p^B)$ is independent of p for $1 < p < \infty$.*

The Operator $b\Delta$

In general, there is no bound of the form (1.2) for the kernel of T_p if the coefficients a_α are only uniformly continuous, see [9]. Duong and Ouhabaz [15], however, recently established the estimate (0.1) for certain multiplicative perturbations of the Laplacian:

Let Δ be the Laplacian, and assume that $b \in BUC(\mathbb{R}^n, \mathbb{C})$ satisfies $\operatorname{Re} b \ge \delta$ for some constant $\delta > 0$ and $|\arg b| \le \theta < \pi/2$. For $1 < p < \infty$, the operator A_p defined by

$$A_p f = b\Delta f \qquad \text{for} \qquad f \in D(A_p) = W_p^2(\mathbb{R}^n)$$

generates a strongly continuous analytic semigroup T_p on $L^p(\mathbb{R}^n)$, cf. [2, Corollary 9.5]. According to [15] the kernel of T_p satisfies (0.1). The argument in the preceding section yields the following result.

2.5 Theorem. *$\sigma(A_p)$ is independent of p for $1 < p < \infty$.*

Schrödinger Operators

Let $\Omega = \mathbb{R}^n$, and let $A = -\frac{1}{2}\Delta + V$ be a Schrödinger operator. We assume, as usual, that

$$V = V_+ + V_-, \quad \text{where} \quad V_\pm \geq 0,\ V_- \in K_n, \quad \text{and } V_+ \in L^1_{\text{loc}}(G). \tag{2.3}$$

Here G is an open set in $\mathbb{R}^n$ whose complement is of measure zero, while K_n is the standard subset of $L^1_{\text{loc}}(\mathbb{R}^n)$ as defined in Simon [25, A.2] or Hempel-Voigt [19, Section 2]. In fact, we may replace the condition '$V_- \in K_n$' by '$V_- \in \hat{K}_n$ and $c(V_-) < 1$' in the language of Voigt [28, Section 5].

Via a truncation procedure the operator A generates a C_0-semigroup on $L^2(\mathbb{R}^n)$ which can be extended consistently to semigroups T_p on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, see e.g. [19, Theorem 2.2]. We denote the generator of T_p by A_p.

It was shown by Simon [25, Theorem B.6.7] that the integral kernel of $T_p(t)$ satisfies an estimate of the form (0.1). We therefore know that conditions (A1) and (A2) hold. Denote by $A_{0,p}$ the operator $-\frac{1}{2}\Delta$ with zero potential on $L^p(\mathbb{R}^n)$.

2.6 Remark. $A_2 = A_{0,2} + V$ is the form sum of operators according to [19, Theorem 2.2(b)]. Let $Q(\cdot,\cdot)$ be the quadratic form associated with A_2, and let $Q(A_2)$, $Q(A_{0,2})$ denote the form domains of the respective operators. Then $Q(A_2) \subseteq Q(A_{0,2}) = W^1_2(\mathbb{R}^n)$, and

$$D(A_2) = \{f \in Q(A_2) : \exists\, g \in L^2(\mathbb{R}^n) \text{ with } Q(f,h) = (g,h) \text{ for all } h \in Q(A_2)\}.$$

In that case, one lets $A_2 f = g$. We have:

2.7 Lemma. *Let $f \in D(A_2)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then $\varphi f \in D(A_2)$, and*

$$A_2(\varphi f) = \varphi A_2 f - \nabla\varphi\nabla f - \frac{1}{2} f\Delta\varphi. \tag{2.4}$$

Proof. Let $g \in L^2(\mathbb{R}^n)$ with $Q(f,h) = (g,h)$ for all $h \in Q(A_2)$. Set $g_\varphi = \varphi g - 2\nabla\varphi\nabla f - \frac{1}{2} f\Delta\varphi$. This is an element of $L^2(\mathbb{R}^n)$, since $f \in W^1_2(\mathbb{R}^n)$. Moreover, we deduce from Leibniz' rule that

$$\begin{aligned} Q(\varphi f, h) &= -\frac{1}{2}\int \varphi\nabla f\nabla h - \frac{1}{2}\int f\nabla\varphi\nabla h + \int V f\varphi h \\ &= -\frac{1}{2}\int \nabla f\nabla(\varphi h) + \int V f\varphi h + \frac{1}{2}\int h\nabla f\nabla\varphi - \frac{1}{2}\int f\nabla\varphi\nabla h \\ &= (g,\varphi h) + \frac{1}{2}\int h\nabla f\nabla\varphi - \frac{1}{2}\int f\nabla\varphi\nabla h. \end{aligned}$$

Now $\int f\nabla\varphi\nabla h = -\int h\nabla f\nabla\varphi - \int hf\Delta\varphi$, so $Q(\varphi f, h) = (g_\varphi, h)$. ◁

2.8 Corollary. *Let $f \in D(A_2)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \equiv 1$ near zero. Define $\varphi_\varepsilon = \varphi(\varepsilon\cdot)$. Then $A_2(\varphi_\varepsilon f) \to A_2 f$ in $L^2(\mathbb{R}^n)$ by (2.4), hence $\varphi_\varepsilon f \to f$ in $D(A_2)$ as $\varepsilon \to 0^+$, and $D_{2,c} := \{u \in D(A_2) : \operatorname{supp} u \text{ compact}\}$ is dense in $D(A_2)$ with respect to the graph norm.*

2.9 Lemma. *Suppose additionally that V_+ is T_1-regular in the sense of* [28, Definition 2.12]. *Then*
(a) *$A_1 = A_{0,1} + V$ as a sum of unbounded operators.*
(b) *$D(A_1) = \{u \in L^1(\mathbb{R}^n) : \Delta u, Vu \in L^1(\mathbb{R}^n)\}$. In particular, $D(A_1) \subseteq W^1_1(\mathbb{R}^n)$, multiplication by functions in $C_0^\infty(\mathbb{R}^n)$ acts continuously on $D(A_1)$, and $D_{1,c} := \{u \in D(A_1) : \operatorname{supp} u \text{ compact}\}$ is dense in $D(A_1)$ with respect to the graph norm.*

Proof. (a) is [28, Theorem 5.3]; (b) is immediate from (a). ◁

In the lemma, below, we employ the sets $D_{p,c}$ introduced in 2.8 and 2.9 for $p = 1, 2$.

2.10 Lemma. *Assume additionally that V_+ is T_1-regular. Let $p \in \{1, 2\}$, $\lambda \in \rho(A_p)$, and $u \in (\lambda - A_p)D_{p,c} \subseteq L^p(\mathbb{R}^n)$. Then*

$$[x_j, (\lambda - A_p)^{-1}]u = (\lambda - A_p)^{-1}[x_j, A_p](\lambda - A_p)^{-1}u,$$

and the commutators $[x_j, (\lambda - A_p)^{-1}]$ extend to bounded operators on $L^p(\mathbb{R}^n)$. In particular, (A3) *holds by iteration.*

Proof (cf. Lemma 2.3). Since A_p is a differential operator on its domain, $(\lambda - A_p)D_{p,c}$ is a dense subset of $L^p(\mathbb{R}^n)$, consisting of functions with compact support. The expression

$$[x_j, (\lambda - A_p)^{-1}]u = x_j(\lambda - A_p)^{-1}u - (\lambda - A_p)^{-1}x_j u$$

therefore makes sense and furnishes an element in $(1 + |x|)L^p(\mathbb{R}^n)$. Similarly, all compositions in

$$\begin{aligned}(\lambda - A_p)^{-1}[x_j, A_p](\lambda - A_p)^{-1}u \;&=\; (\lambda - A_p)^{-1}x_j(\lambda - A_p)(\lambda - A_p)^{-1}u \\ &\quad -(\lambda - A_p)^{-1}(\lambda - A_p)x_j(\lambda - A_p)^{-1}u\end{aligned} \tag{2.5}$$

are defined: Note that, on the right hand side, multiplication by x_j both times is applied to a function with compact support; in particular, it coincides with multiplication by a function in $C_0^\infty(\mathbb{R}^n)$. We thus get the desired identity. For $p = 1, 2$, the left hand side of (2.5) is defined on a dense subset of $L^p(\mathbb{R}^n)$ and extends to a bounded operator, for $[x_j, A_p] = \partial_{x_j} : D(A_p) \subset W_p^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous. ◁

2.11 Lemma. *Let V_+ be T_1-regular and $1 \le p \le 2$. Then*
(a) *$D(A_p) \subseteq W_p^1(\mathbb{R}^n)$.*
(b) *Multiplication by functions in $C_0^\infty(\mathbb{R}^n)$ acts continuously on $D(A_p)$.*
(c) *The space $D_{p,c} = \{u \in D(A_p) : \operatorname{supp} u \text{ compact}\}$ is dense in $D(A_p)$ with respect to the graph norm.*

Proof. Fix $\lambda_0 > \max\{\omega(T_1), \omega(T_2)\}$ where $\omega(T_2), \omega(T_2)$ denotes the growth bound of T_1, T_2, respectively. Since T_1 and T_2 are consistent semigroups we have

$$(\lambda_0 - A_p)^{-1}u = (\lambda_0 - A_1)^{-1}u, \;\; u \in L^1 \cap L^p, p \in [1, 2]. \tag{2.6}$$

(a) (cf. [19, Proof of Lemma 3.4]). For $p = 1, 2$, we deduce from 2.9 and 2.6 that the operators $\partial_{x_j}(\lambda_0 - A_p)^{-1}$ are bounded on L^p, $j = 1, \dots, n$. They agree on $L^1 \cap L^2$ by (2.6). Applying the theorem of Riesz-Thorin, they have bounded extensions to L^p, $1 \le p \le 2$. Those coincide with $\partial_{x_j}(\lambda_0 - A_p)^{-1}$ on $L^1 \cap L^p$. Hence $D(A_p) = (\lambda_0 - A_p)^{-1}L^p \subseteq W_p^1$.

(b) Let $\varphi \in C_0^\infty$. We have $[\varphi, A_2]f = \nabla\varphi\nabla f + \frac{1}{2}f\Delta\varphi$ for $f \in D(A_2)$ by (2.4); the same holds for A_1 and $f \in D(A_1)$ by Lemma 2.9. By (a), $[\varphi, A_p](\lambda_0 - A_p)^{-1}$ is bounded on L^p for $p = 1, 2$. On $L^1 \cap L^2$ both operators coincide. According to Riesz-Thorin we get bounded extensions $C_p \in \mathcal{L}(L^p)$ for $1 \le p \le 2$.

For $p = 1, 2$, and $u \in L^1 \cap L^2$, we see – just as in Lemma 2.10 – that

$$[\varphi, (\lambda_0 - A_p)^{-1}]u = (\lambda_0 - A_p)^{-1}[\varphi, A_p](\lambda_0 - A_p)^{-1}u = (\lambda_0 - A_p)^{-1}C_p u;$$

hence

$$\varphi(\lambda_0 - A_p)^{-1}u = (\lambda_0 - A_p)^{-1}(\varphi u + C_p u). \tag{2.7}$$

Due to (2.6), the above identity remains valid for $p \in [1,2]$ and $u \in L^1 \cap L^p$. By continuity, the identity extends also to $u \in L^p$. Hence multiplication by φ maps $D(A_p)$ to itself. It is continuous by the closed graph theorem.

(c) Let $\varphi \in C_0^\infty$, $\varphi \equiv 1$ near zero, and $\varphi_\varepsilon = \varphi(\varepsilon\cdot)$, $\varepsilon > 0$. For $u \in D(A_1) \cap D(A_2)$, we have $u \in D(A_p)$ by (2.6). Thus (2.4) and 2.9 imply that

$$A_p(\varphi_\varepsilon u) = \varphi_\varepsilon A_p u - \nabla\varphi_\varepsilon \nabla u - \frac{1}{2} u \Delta\varphi_\varepsilon. \tag{2.8}$$

It follows from (2.6) that $D(A_1) \cap D(A_2)$ is dense in $D(A_p)$. So (2.8) extends to $u \in D(A_p)$ by (a). Noting that $\nabla\varphi_\varepsilon = \varepsilon\nabla\varphi(\varepsilon\cdot)$ and $\Delta\varphi_\varepsilon = \varepsilon^2\Delta\varphi(\varepsilon\cdot)$, (2.8) implies the convergence of $\varphi_\varepsilon u$ to u in the graph norm of A_p. ◁

2.12 Theorem. *Let $A = -\frac{1}{2}\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with a potential V of type (2.3). We denote by T_p the family of consistent semigroups induced on $L^p(\mathbb{R}^n)$ via A, and we let A_p be the generator of T_p for $1 \le p < \infty$. We additionally ask V_+ to be T_1-regular. Then the spectrum of A_p on $L^p(\mathbb{R}^n)$ is independent of p for $1 \le p < \infty$.*

Proof. In view of the self-adjointness of the Schrödinger operator on $L^2(\mathbb{R}^n)$ it is sufficient to show p-independence for $1 \le p \le 2$, see [19, Remark 2.4]. In order to apply Theorem 1.2, we only have to check (A3). We did this for $p = 1, 2$ in Lemma 2.10. Having established Lemma 2.11 we can employ the same proof for $1 \le p \le 2$. ◁

2.13 Remark. (a) For V_+ to be T_1-regular it is sufficient that $C_0^\infty(G)$ is dense in $W_2^1(\mathbb{R}^n)$, see [28, Remark 5.9(a)].
(b) Voigt introduced another version of regularity in [29]. He informed us that A. Manavi recently showed the equivalence of both notions.

Elliptic Operators in Divergence Form: Real Coefficients

Let $\Omega = \mathbb{R}^n$, and let $a_{jk} \in L^\infty(\mathbb{R}^n, \mathbb{R})$ for all $j, k \in \{1, \ldots, n\}$. Consider the form $a : W_2^1(\mathbb{R}^n) \times W_2^1(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$a(u,v) = \sum_{j,k=1}^n \int a_{jk}\partial_j u \overline{\partial_k v} dx. \tag{2.9}$$

Suppose there is an $\alpha > 0$ such that

$$\sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \ge \alpha|\xi|^2 \tag{2.10}$$

for all $\xi \in \mathbb{R}^n$ and a.a. $x \in \mathbb{R}^n$. Let A be the operator associated with the form a, and let T be the C_0 semigroup on $L^2(\mathbb{R}^n)$ generated by A. Then $T(t)$ is an integral operator with a kernel satisfying (1.2), and there exists a family of consistent semigroups T_p on $L^p(\mathbb{R}^n)$ with generator A_p, $1 < p < \infty$. For a proof of these facts we refer to Aronson [6], Davies [11], Auscher [7], and Arendt–ter Elst [5]. Hence we only have to check (A3).

In order to modify the proof of Lemma 2.3 to our situation we impose additional conditions on the domain. Consider the spaces $D_{p,c} := \{u \in D(A_p) : \operatorname{supp} u \,\text{compact}\}$ and $R_{p,c} := (\lambda - A_p)D_{p,c}$ for $\lambda \in \mathbb{C}$. Assume that

(D1) $x_j f \in D(A_p)$ for $f \in D_{p,c}$ and $j = 1, \ldots, n$.

Since A_p acts as a differential operator on $D_{p,c}$ it follows that any $u \in R_{p,c}$ has compact support. Therefore

$$\begin{aligned}
&(\lambda - A_p)^{-1}[x_j, A_p](\lambda - A_p)^{-1}u \\
&= -(\lambda - A_p)^{-1}x_j(\lambda - A_p)(\lambda - A_p)^{-1}u + (\lambda - A_p)^{-1}(\lambda - A_p)x_j(\lambda - A_p)^{-1}u \\
&= [x_j, (\lambda - A_p)^{-1}]
\end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$ for $\lambda \in \rho(A_p)$ and $u \in R_{p,c}$. Assuming furthermore that

(D2) $D_{p,c}$ is dense in $D(A_p)$, and

(D3) $[x_j, A_p] \in \mathcal{L}(D(A_p), L^p(\mathbb{R}^n))$ for $j = 1, \ldots, n$,

the commutator $[x_j, (\lambda - A_p)^{-1}]$ extends to a bounded operator on $L^p(\mathbb{R}^n)$. By iteration we obtain (A3). Summing up we proved the following result.

2.14 Theorem. *Let $a_{jk} \in L^\infty(\mathbb{R}^n, \mathbb{R})$ for $j, k \in \{1, \ldots, n\}$, and suppose that (2.10) is satisfied. Let $a : W_2^1(\mathbb{R}^n) \times W_2^1(\mathbb{R}^n) \to \mathbb{C}$ be the form defined by (2.9). For $1 \le p < \infty$ denote by A_p the generator of the semigroup T_p on $L^p(\mathbb{R}^n)$ associated to a. Assume that* (D1), (D2), *and* (D3) *are satisfied for all $p \in [p_1, p_2]$, where $1 < p_1, p_2 < \infty$. Then $\sigma(A_p)$ is independent of p for $p \in [p_1, p_2]$.*

Some remarks concerning condition (D3) are in order. The operator $[x_l, A_p]$ is associated with the form b on $W_2^1(\mathbb{R}^n) \times W_2^1(\mathbb{R}^n)$ given by

$$b(u, v) = \sum_{j=1}^{n} \int a_{jl}(\partial_j u)v - a_{lj}u(\partial_j v)dx, \qquad l = 1, \ldots, n.$$

Suppose that

(D3a) $\partial_j a_{lj} \in L^\infty(\mathbb{R}^n)$ for all $j, l = 1, \ldots, n$.

Then integration by parts yields $-\int a_{lj}u\partial_j v = \int(\partial_j a_{lj})uv + \int a_{lj}(\partial_j u)v$ for $u, v \in W_2^1(\mathbb{R}^n)$. So (D3) follows, provided we additionally have $D(A_p) \subseteq W_p^1(\mathbb{R}^n)$.

2.15 Remark. (a) If $a_{jk} \in W_\infty^1(\mathbb{R}^n)$ for all $j, k \in \{1, \ldots, n\}$ then $D(A_p) = W_p^2(\mathbb{R}^n)$ for $1 < p < \infty$ by [2, Corollary 9.5], and conditions (D1), (D2), and (D3) are satisfied.
(b) A recent result of Coulhon and Duong [10] implies the existence of $\varepsilon > 0$ such that $D(A_p) \subseteq W_p^1(\mathbb{R}^n)$ for all $p \in (1, 2+\varepsilon)$. Hence (D3) is satisfied for $p \in (1, 2+\varepsilon)$ whenever (D3a) holds.

Elliptic Operators in Divergence Form: Complex Coefficients

Let $\Omega = \mathbb{R}^n$, where $n = 1, 2$, and let $a_{jk} \in L^\infty(\mathbb{R}^n, \mathbb{C})$. Suppose that there exists $\mu > 0$ such that, for almost all $x \in \Omega$,

$$\operatorname{Re} \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\overline{\xi_k} \ge \mu|\xi|^2, \qquad \xi \in \mathbb{C}^n \tag{2.11}$$

Let A be the operator associated with the form a given as above, and let T be the C_0-semigroup on $L^2(\mathbb{R}^n)$ generated by A. According to the results in [8] there exists a family of consistent semigroups T_p on $L^p(\mathbb{R}^n)$ with generator A_p such that $T_p(t)$ may be represented by a kernel $k(t, \cdot, \cdot)$ satisfying an estimate of the form (1.2). With the approach described in the previous section we then obtain:

2.16 Theorem. *Let $n = 1, 2$, and $a_{jk} \in L^\infty(\mathbb{R}^n, \mathbb{C})$ for $j, k \in \{1, \ldots, n\}$. Suppose that (2.11) is satisfied. Let A be the operator in $L^2(\mathbb{R}^n)$ associated with the form a defined as in (2.9). For $1 < p < \infty$ denote by A_p the generator of the semigroup T_p on $L^p(\mathbb{R}^n)$ associated with a.*

Then $\sigma(A_p)$ is independent of $p \in [p_1, p_2]$, $1 < p_1 < p_2 < \infty$, provided assumptions (D1), (D2), and (D3) hold for all $p \in [p_1, p_2]$

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