ISOMETRIC PROPERTIES OF THE HANKEL TRANSFORMATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. It is shown that the Hankel transformation H_ν acts in a class of weighted Sobolev spaces. Especially, the isometric mapping property of H_ν which holds on $L^2(\mathbb{R}_+, rdr)$ is extended to spaces of arbitrary Sobolev order. The novelty in the approach consists in using techniques developed by B.-W. Schulze and others to treat the half-line $\mathbb{R}_+$ as a manifold with a conical singularity at $r = 0$. This is achieved by pointing out a connection between the Hankel transformation and the Mellin transformation. The procedure proposed leads at the same time to a short proof of the Hankel inversion formula. An application to the existence and higher regularity of solutions, including their asymptotics, to the 1-1-dimensional edge-degenerated wave equation is given.

1. INTRODUCTION

For $\nu \geq 0$, we consider the Hankel transformation of order ν,

$$\boxed{H_\nu u(\xi) = \int_0^\infty J_\nu(\xi r) u(r) r\, dr} \tag{1}$$

where $J_\nu(r)$ is the Bessel function of the first kind. A classical result says that H_ν induces an isometry

$$H_\nu \colon L^2(\mathbb{R}_+, rdr) \to L^2(\mathbb{R}_+, rdr).$$

Moreover, H_ν is self-inverse on $L^2(\mathbb{R}_+, rdr)$, by the Hankel inversion theorem. The major goal of this contribution is to extend the Hankel transformation H_ν by continuity to a class of weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, defined in (38), where $\gamma > -3/2 - \nu$, and to study the properties of H_ν on these spaces. Especially, $L^2(\mathbb{R}_+, rdr) = \mathcal{H}^{0,-1/2}(\mathbb{R}_+)$, and H_ν induces a self-inverse isometry from $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ onto itself, for any $s \in \mathbb{R}$.

The results mentioned are obtained by utilizing a connection between the Hankel transformation and the Mellin transformation, M. The starting point is the observation that, for $u \in C_0^\infty(\mathbb{R}_+)$, we have

$$|MH_\nu u(z)| \leq C_\delta \langle z \rangle^\delta |Mu(z)| \text{ if } \operatorname{Re} z = 1 - \delta \tag{2}$$

for a certain constant $C_\delta > 0$, and any $\delta \in \mathbb{R}$, $\delta > -1 - \nu$. Here $\langle z \rangle = (1 + |z|^2)^{1/2}$. In particular, $C_\delta = 1$ for $\delta = 0$ such that

$$|MH_\nu u(z)| = |Mu(z)| \text{ if } \operatorname{Re} z = 1,$$

by replacing u by $H_\nu u$ in (2). Investigating these phenomena more closely, it turns out that the Hankel transformation H_ν as operator acting from $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ to $\mathcal{H}^{s-\gamma-1/2,-1-\gamma}(\mathbb{R}_+)$ with $\gamma > -3/2 - \nu$ can be rewritten as

$$\boxed{H_\nu = I \operatorname{op}_M^{1/2-\gamma}(a_\nu) = \operatorname{op}_M^{3/2+\gamma}(b_\nu) I} \tag{3}$$

Date: June 6, 1997.

1991 *Mathematics Subject Classification.* Primary: 44A15, 46F12; Secondary: 35L80.

Key words and phrases. Hankel transformation, weighted Sobolev spaces, discrete asymptotics; edge-degenerate wave equation.

This research was supported by the Deutsche Forschungsgemeinschaft and by the Volkswagenstiftung (RiP-program at Oberwolfach).

i.e., as composition of the Mellin operator $\mathrm{op}_M^{1/2-\gamma}(a_\nu)$ with symbol

$$a_\nu(z) = 2^{1-z}\,\frac{\Gamma((\nu-z)/2+1)}{\Gamma((\nu+z)/2)} \tag{4}$$

and the inversion I defined by

$$Iu(r) = r^{-2}u(r^{-1}) \tag{5}$$

and likewise as composition of the inversion I and the Mellin operator $\mathrm{op}_M^{3/2+\gamma}(b_\nu)$ with symbol

$$b_\nu(z) = 2^{z-1}\,\frac{\Gamma((\nu+z)/2)}{\Gamma((\nu-z)/2+1)}. \tag{6}$$

Notice the fundamental relation

$$a_\nu(z)^{-1} = b_\nu(z) = a_\nu(2-z) \tag{7}$$

which in its consequence, e.g., yields the Hankel inversion theorem. Notice further that $a_\nu(z)$ is meromorphic on $\mathbb{C}$ with simple poles at $z = \nu+2j+2$, $j\in\mathbb{N}$, while $b_\nu(z)$ is meromorphic on $\mathbb{C}$ with simple poles at $z=-\nu-2j$, $j\in\mathbb{N}$. The first pole for $a_\nu(z)$ appearing at $z=\nu+2$ (or the first pole for $b_\nu(z)$ appearing at $z=-\nu$) is responsible for the impossibility to explain the Hankel transformation H_ν on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, $\gamma\le -3/2-\nu$.

It is seen that the situation under consideration is quite symmetric with respect to the conormal order $\gamma=-1/2$. This corresponds to the weight line $\Gamma_1=\{z\in\mathbb{C};\mathrm{Re}\,z=1\}$ in the Mellin image. For that reason, we shall henceforth write $1/2+\delta$ instead of γ. Then $1/2-\gamma$ and $3/2+\gamma$ in (3) become resp. $1-\delta$ and $1+\delta$.

Let us discuss an illuminating example.

Example 1. Let $p\in\mathbb{C}$, $|\mathrm{Re}\,p|<1+\nu$. Then, for $\delta,\delta'\in\mathbb{R}$, $\delta<\delta'$, we have $r^{p-1}\in\mathcal{H}^{\infty,-1/2+\delta}(\mathbb{R}_+)+\mathcal{H}^{\infty,-1/2+\delta'}(\mathbb{R}_+)$ if and only if $\delta<\mathrm{Re}\,p<\delta'$. Now choosing δ,δ' in such a way that $\delta>-1-\nu$, $-\delta'>-1-\nu$, we see that the Hankel transformation of order ν is applicable and, using a table, e.g., [3, 6.8(1), 8.5(7)], [6, 2.12.2.2], we find that

$$H_\nu\left\{r^{p-1}\right\}(\xi) = b_\nu(p+1)\,\xi^{-(p+1)}. \tag{8}$$

This can be refined as follows. Choose a cut-off function $\omega\in C^\infty(\overline{\mathbb{R}}_+)$, i.e., $\omega=1$ near $r=0$ and $\omega=0$ for large r, and suppose that $\mathrm{Re}\,p>-1-\nu$ instead of $|\mathrm{Re}\,p|<1+\nu$. Then

$$H_\nu\left\{\omega(r)r^{p-1}\right\}\in\begin{cases}\mathcal{Z}_\nu(\mathbb{R}_+)\oplus I\mathcal{E}_P(\mathbb{R}_+) & \text{if } p\notin 1+\nu+2\mathbb{N},\\ \mathcal{Z}_\nu(\mathbb{R}_+) & \text{if } p\in 1+\nu+2\mathbb{N}.\end{cases}$$

Here $\mathcal{Z}_\nu(\mathbb{R}_+)$ is Zemanian's space which provides the asymptotic behaviour near $r=0$, cf. Lemma 6, and $I\mathcal{E}_P(\mathbb{R}_+)$ with $P=\{(-p+1,0)\}$ gives the asymptotics at $r=\infty$; $\mathcal{E}_P(\mathbb{R}_+)$ abbreviates $\mathrm{span}\{\omega(r)r^{p-1}\}$. Moreover, $I\mathcal{E}_P(\mathbb{R}_+)\subset\bigcap_{\delta>-\mathrm{Re}\,p}\mathcal{H}^{\infty,-1/2+\delta}(\mathbb{R}_+)$ such that the Hankel transformation is applicable to $I\mathcal{E}_P(\mathbb{R}_+)$ if we choose $\delta>-1-\nu$. Compared with the Hankel inversion theorem, however, H_ν is displaced by certain $H_\nu^{(j)}$ provided that $-1-\nu-2j<\delta<1-\nu-2j$ for $j\in\mathbb{N}$, $j\ge 1$, as we will see soon.

A basic property of the Hankel transformation H_ν is that it diagonalizes the differential operator

$$L_\nu = -r^{-2}\left(r\frac{\partial}{\partial r}\right)^2 + r^{-2}\nu^2, \tag{9}$$

i.e., we have

$$H_\nu L_\nu = \xi^2 H_\nu, \tag{10}$$

where ξ^2 means multiplication in the image space of H_ν. Making use of (3), this property is easily verified, e.g., on the space $\mathcal{H}^{-\infty,1/2}(\mathbb{R}_+)$:

$$H_\nu L_\nu = I\,\mathrm{op}_M^2(a_\nu) r^{-2}\,\mathrm{op}_M^0(l_\nu) = I\xi^{-2}\,\mathrm{op}_M^0(T^2 a_\nu)\,\mathrm{op}_M^0(l_\nu) = \\ = \xi^2 I\,\mathrm{op}_M^0((T^2 a_\nu) l_\nu) = \xi^2 I\,\mathrm{op}_M^0(a_\nu) = \xi^2 H_\nu,$$

because of $(T^2 a_\nu)(z)\, l_\nu(z) = a_\nu(z)$ for

$$2^{-1-z}\frac{\Gamma((\nu-z)/2)}{\Gamma((\nu+z)/2+1)}(-z^2+\nu^2) = 2^{1-z}\frac{\Gamma((\nu-z)/2)(\nu-z)/2}{\Gamma((\nu+z)/2+1)/(\nu+z/2)},$$

where $(T^2 a_\nu)(z) = a_\nu(z+2)$ and $l_\nu(z) = -z^2+\nu^2$ is the Mellin symbol of $r^2 L_\nu$.

We shall employ property (10) to discuss the solvability of the edge-degenerate wave equation in 1-1 dimensions,

$$\partial_t^2 u + L_\nu u = f(t) \ \text{ on } \mathbb{R}\times\mathbb{R}_+, \tag{11}$$

where the given right-hand side f fulfils $f = 0$ for $t < 0$. The wave operator $\partial_t^2 + L_\nu$ mainly becomes an constant coefficient operator having the symbol $-\tau^2+\xi^2$ when viewed with respect to the Fourier transformation in t and the Hankel transformation in r. Thus it is possible to investigate (11) without decomposing its solution into differently behaved pieces, near resp. $r = 0$ and $r = \infty$. For a heuristic justification, see (32). Moreover, the operator $\partial_t^2 + L_\nu$ serves as model for an operator on a configuration having conical singularities in the spatial variables that is strictly hyperbolic away from the set of singularities. In our example, the conical singularity is at $r = 0$ on $\mathbb{R}_+$; then $\{(t,r)\in\mathbb{R}\times\mathbb{R}_+; r = 0\}$ becomes the set of singularities. This set is identified as an *edge* by the *typical degeneracy* of the differential operator $\partial_t^2 + L_\nu$ near it. This is referred to as edge-degenerated, cf. [7, Chapter 3]. We shall prove that Eq. (11) possesses a unique solution u satisfying $u = 0$ for $t < 0$ in the wedge Sobolev space $\mathcal{W}_\eta^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)$ provided that $f\in\mathcal{W}_\eta^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)$. Further we shall verify that, for $s\geq 0$ and up to certain asymptotic terms of a specific constant discrete asymptotics type, the conormal order of this solution u increases if the conormal order of the right-hand side f increases. For the sake of completeness, we also provide the corresponding result in the case that for f branching discrete asymptotics are allowed. But we are very brief in the proof of this result, since the functional-analytic arguments necessary for it are beyond the scope of this paper.

The main references for us concerning the Hankel transformation are E. C. Titchmarsh [11] and A. H. Zemanian [13]. Let us remark that while for these authors the Hankel transformation was given by the integral in (1) we look at it mainly as being the operator (3). The techniques utilized to treat the point $r = 0$ as a conical singularity for the half-line $\mathbb{R}_+$ are borrowed from B.-W. Schulze [7], [9].

The paper is organized as follows. In Section 2 we state our main results concerning the properties of the Hankel transformation. These results are proved in Section 3. In this section we further recapitulate some results of A. H. Zemanian looked at from our point of view on the Hankel transformation. Finally, in Section 4, we apply the results previously obtained to study the one-dimensional edge-degenerate wave equation. In an appendix, the reader finds various definitions of functional spaces used throughout this paper, including spaces with asymptotics, and the convention of Mellin operators.

2. Statement of the Results

In this section we state our main results concerning the properties of the Hankel transformation. These results are then proved in Section 3. The Hankel transformation H_ν is extended to a class of weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, the definition of which can be found in the appendix, cf. (38). The various extensions that coincide on intersections are again denoted by H_ν. Especially, it makes sense to give a meaning to H_ν on sums of such spaces, which becomes important when defining the Hankel transformation on $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, see Lemma 9.

We introduce the following partition of the real line $\mathbb{R}$: for $j \in \mathbb{Z}$, we put

$$I_j = \begin{cases} (-1+2j+\nu, 1+2j+\nu) & \text{if } j \geq 1, \\ (-1-\nu, 1+\nu) & \text{if } j = 0, \\ (-1+2j-\nu, 1+2j-\nu) & \text{if } j \leq -1. \end{cases}$$

If we wish to express the dependence on ν explicitely, then we write $I_j = I_j(\nu)$. We have $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} \overline{I_j}$, whereas, for j, $l \in \mathbb{Z}$, $j \neq l$, $\overline{I_j} \cap \overline{I_l}$ consists of one point, if $|j-l| = 1$, and is empty otherwise.

Theorem 1 (Main Theorem). *Let $s \in \mathbb{R}$, $\delta \in I_0 \cup \bigcup_{j\geq 1} \overline{I_j}$, i.e., $\delta > -1-\nu$.*

(a) *The Hankel transformation H_ν, defined on $C_0^\infty(\mathbb{R}_+)$ by* (1), *extends by continuity to*

$$\boxed{H_\nu \colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)} \tag{12}$$

(b) *For $\delta \in I_0 \cup \bigcup_{j\geq 1} I_j$, i.e., $\delta \notin 1+\nu+2\mathbb{N}$, H_ν given by* (12) *is an isomorphism.*

(c) *For $\delta \in \overline{I_j} \cap \overline{I_{j+1}}$ for some $j \in \mathbb{N}$, i.e., $\delta \in 1+\nu+2\mathbb{N}$, H_ν given by* (12) *is injective, but not surjective, with dense range.*

(d) *For $\delta \in I_0$, i.e., $|\delta| < 1+\nu$, the inverse to H_ν is again H_ν.*

(e) *For $\delta = 0$, H_ν is even an isometry.*

Remark 1. The statements about continuity in (12) extend to $H_\nu \colon \mathcal{H}^{\pm\infty,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{\pm\infty,-1/2-\delta}(\mathbb{R}_+)$. This also concerns the statements in (b), (c), and (d).

Remark 2. There exists no reasonable way to define the Hankel transformation H_ν on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ in case $\delta \leq -1-\nu$.

For $\delta \in I_j$, where $j \in \mathbb{N}$, we denote the inverse to H_ν as operator defined by (12) by $H_\nu^{(j)}$. Then $H_\nu^{(j)}$ is actually independent of $\delta \in I_j$ in the sense explained above for H_ν. For $j = 0$, we have $H_\nu^{(0)} = H_\nu$, by Theorem 1(d).

Theorem 2. *Let j, $l \in \mathbb{N}$, $j < l$. Then, for $\delta \in I_j$, $\delta' \in I_l$, and any $u \in \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2-\delta'}(\mathbb{R}_+)$, we have that*

$$\boxed{H_\nu^{(j)}u - H_\nu^{(l)}u = \sum_{k=j}^{l-1} c_k(u)\, \xi^{\nu+2k}} \tag{13}$$

where the functionals c_k are given by

$$c_k(u) = \frac{(-1)^k}{k!} \frac{2^{-\nu-2k}}{\Gamma(\nu+k+1)}\, \tilde{u}(\nu+2k+2). \tag{14}$$

These functionals are continuous on $\big(\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2-\delta'}(\mathbb{R}_+)\big)'$.

Introducing, for $r \in \mathbb{R}_+$, the "modified" Bessel function

$$J_\nu^{(l)}(r) = \left(\frac{r}{2}\right)^\nu \sum_{k=l}^\infty \frac{(-1)^k}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{r}{2}\right)^{2k},$$

i.e., $J_\nu^{(l)}(r)$ is obtained via subtracting off the first l terms in the series expansion of $J_\nu(r)$, and using (13), (14) with $j = 0$, a brief calculation shows that, for $l \in \mathbb{N}$, we have, supplementary to (1),

$$\boxed{H_\nu^{(l)}u(\xi) = \int_0^\infty J_\nu^{(l)}(\xi r) u(r) r\, dr, \quad u \in C_0^\infty(\mathbb{R}_+)} \tag{15}$$

Closing we determine the inverse to H_ν that corresponds to the situation described in Theorem 1(c). The densely defined operator thus obtained is denoted by $H_\nu^{(j\to j+1)}$.

Theorem 3. *Let $s\in\mathbb{R}$, $j\in\mathbb{N}$, and $\delta=1+\nu+2j$. Then the range of H_ν as operator from $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ to $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ consists of all functions $u\in\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ for which $\tilde{u}(z)/(z-(\nu+2j+2))$ is square-integrable on $\{z\in\mathbb{C};\operatorname{Re}z=1+\delta\}$ close to $z=\nu+2j+2$.*

If $u\in C_0^\infty(\mathbb{R}_+)$, then u belongs to this range if and only if $\tilde{u}(\nu+2j+2)=0$. In this case the action of $H_\nu^{(j\to j+1)}$ on u is given by (15) with l replaced by either j or $j+1$.

3. Proofs

We prove Theorems 1–3 in a series of lemmas. The basic estimate is provided in Lemma 1. Then, in Lemma 2 and Lemma 3, it is shown that the operators $I\operatorname{op}_M^{1-\delta}(a_\nu)=\operatorname{op}_M^{1+\delta}(b_\nu)I$ replacing H_ν are resp. inverse to each other and an isometry when $\delta=0$. Finally, in Lemma 4, the operators H_ν and $I\operatorname{op}_M^{1}(a_\nu)=\operatorname{op}_M^{1}(b_\nu)I$ are identified. This immediately extends to values of $\delta>-1-\nu$ by holomorphy of a_ν in the corresponding region. All together, Theorem 1 is implied. Theorem 2 and Theorem 3 equally follow by a simple calculation. For the functional spaces used and notation of Mellin operators, see the appendix.

Lemma 1 (The basic estimate). *Let $\chi\in C^\infty(\mathbb{C})$ be an excision function for the poles of $a_\nu(z)$. Then, for any $\delta\in\mathbb{R}$, we have the estimate*

$$\sup_{\operatorname{Re}z=1-\delta}(1+|z|^2)^{-\delta/2}|\chi(z)\,a_\nu(z)|<\infty. \tag{16}$$

Similarly, for $b_\nu(z)$ we have the estimate

$$\sup_{\operatorname{Re}z=1+\delta}(1+|z|^2)^{-\delta/2}|\chi(2-z)\,b_\nu(z)|<\infty. \tag{17}$$

Moreover, these estimates hold uniformly for $\delta\in[\delta_0,\delta_1]$, $\delta_0,\delta_1\in\mathbb{R}$, $\delta_0<\delta_1$. In addition,

$$|a_\nu(z)|=|b_\nu(z)|=1 \text{ for } \operatorname{Re}z=1. \tag{18}$$

Thereby, being an excision function for the poles of $a_\nu(z)$ means

$$\chi(z)=\begin{cases}1 & \text{if } |z-\nu-2j-2|\geq 2\epsilon \text{ for all } j\in\mathbb{N}\\ 0 & \text{if } |z-\nu-2j-2|\leq\epsilon \text{ for some } j\in\mathbb{N}\end{cases}$$

for certain $\epsilon>0$. Notice that then $\chi(2-z)$ is an excision function for the poles of $b_\nu(z)$.

The estimates in Lemma 1 basically mean that outside the poles of $a_\nu(z)$

$$|a_\nu(z)|\asymp(1+|z|^2)^{\delta/2} \text{ for } \operatorname{Re}z=1-\delta,$$

and similarly for $b_\nu(z)$. Repeat the notation $\langle z\rangle=(1+|z|^2)^{1/2}$.

Proof of Lemma 1. We prove the estimate for $a_\nu(z)$, $b_\nu(z)$ simultaneously. Fix $\delta_0,\delta_1\in\mathbb{R}$, $\delta_0<\delta_1$. Then

$$C_0\langle z\rangle^\delta\leq\left|\frac{\Gamma((\nu+z)/2+\delta)}{\Gamma((\nu+z)/2)}\right|\leq C_1\langle z\rangle^\delta \tag{19}$$

for $z\in\mathbb{C}$, $\operatorname{Re}z=1-\delta$, $|\operatorname{Im}z|\geq\epsilon$, and $\delta\in[\delta_0,\delta_1]$, with some constants $C_0,C_1>0$ only depending on δ_0, δ_1, and $\epsilon>0$. Indeed, $\Gamma((\nu+z)/2+\delta)/\Gamma((\nu+z)/2)$ behaves like a classical pseudo-differential symbol of order δ on $\operatorname{Re}z=1-\delta$, since there we have an asymptotic expansion

$$\frac{\Gamma((\nu+z)/2+\delta)}{\Gamma((\nu+z)/2)}\sim z^\delta\sum_{s=0}^{\infty}\alpha_s z^{-s} \text{ as } \operatorname{Im}z\to\pm\infty$$

with coefficients α_s depending on δ, cf. [5, Chapter IV, §5.1].

Now let $z \in \mathbb{C}$, $\operatorname{Re} z = 1 - \delta$. Then

$$\frac{\nu - \bar{z}}{2} + 1 = \frac{\nu + z}{2} + \delta$$

such that

$$\left|\frac{\Gamma((\nu - z)/2 + 1)}{\Gamma((\nu + z)/2)}\right| = \left|\frac{\Gamma((\nu - \bar{z})/2 + 1)}{\Gamma((\nu + z)/2)}\right| = \left|\frac{\Gamma((\nu + z)/2 + \delta)}{\Gamma((\nu + z)/2)}\right|$$

for $\Gamma(z) = \overline{\Gamma(\bar{z})}$ and, by (19),

$$C_0' \langle z \rangle^\delta \le |a_\nu(z)| = |b_\nu(z)^{-1}| \le C_1' \langle z \rangle^\delta$$

for $\operatorname{Re} z = 1 - \delta$, $|\operatorname{Im} z| \ge \epsilon$, with certain constants C_0', $C_1' > 0$. □

The involution I defined in (5) satisfies $M(Iu)(z) = Mu(2 - z)$ for $\operatorname{Re} z = 1 - \delta$, $u \in \mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$. Hence, for any s, $\delta \in \mathbb{R}$, it realizes an isometry

$$I \colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{s,-1/2-\delta}(\mathbb{R}_+).$$

Moreover, we have the relation $I \operatorname{op}_M^{1-\delta}(h) = \operatorname{op}_M^{1+\delta}(h_I) I$ on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$, where h is any Mellin symbol given on $\Gamma_{1-\delta}$, cf. (39), and $h_I(z) = h(2 - z)$. The estimates (16), (17) allow us to define the Mellin operators

$$\operatorname{op}_M^{1-\delta}(a_\nu) \colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{s-\delta,-1/2+\delta}(\mathbb{R}_+),$$
$$\operatorname{op}_M^{1+\delta}(b_\nu) \colon \mathcal{H}^{s,-1/2-\delta}(\mathbb{R}_+) \to \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$$

for s, $\delta \in \mathbb{R}$, $\delta \notin -1 - \nu - 2\mathbb{N}$. Thus we can consider the compositions

$$I \operatorname{op}_M^{1-\delta}(a_\nu) = \operatorname{op}_M^{1+\delta}(b_\nu) I \colon \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+) \to \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+) \tag{20}$$

for $\delta \notin -1 - \nu - 2\mathbb{N}$.

Lemma 2. *The operators $I \operatorname{op}_M^{1-\delta}(a_\nu)$, $I \operatorname{op}_M^{1+\delta}(a_\nu)$ are inverse to each other provided that $\pm\delta \notin 1 + \nu + 2\mathbb{N}$.*

Proof. For $\pm\delta \notin 1 + \nu + 2\mathbb{N}$, we simply have

$$I \operatorname{op}_M^{1-\delta}(a_\nu) \operatorname{op}_M^{1-\delta}(b_\nu) I = I \operatorname{op}_M^{1-\delta}(a_\nu b_\nu) I = \mathrm{id}_{\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)},$$
$$\operatorname{op}_M^{1-\delta}(b_\nu) I^2 \operatorname{op}_M^{1-\delta}(a_\nu) = \operatorname{op}_M^{1-\delta}(b_\nu) \operatorname{op}_M^{1-\delta}(a_\nu) = \mathrm{id}_{\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)}.$$

□

Lemma 3. *For every $s \in \mathbb{R}$, the operator $I \operatorname{op}_M^1(a_\nu)$ is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$.*

Proof. From Lemma 2 we know that $I \operatorname{op}_M^1(a_\nu)$ is an isomorphism from $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ onto itself. That $I \operatorname{op}_M^1(a_\nu)$ maps isometrically on this space follows from the facts that I is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ and $\operatorname{op}_M^1(a_\nu)$ maps $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ isometrically into itself by the relation (18). □

Lemma 4. *On the space $C_0^\infty(\mathbb{R}_+)$, we have*

$$H_\nu = I \operatorname{op}_M^1(a_\nu) = \operatorname{op}_M^1(b_\nu) I. \tag{21}$$

Proof. Let $u \in C_0^\infty(\mathbb{R}_+)$. Then, for $\operatorname{Re} z = 1$, we have

$$M(H_\nu u)(z) = M_{\xi \to z}\Big(\int_0^\infty J_\nu(\xi r) u(r) r\, dr\Big)$$
$$= \int_0^\infty r^{-z} b_\nu(z) u(r) r\, dr,$$

since the improper integral $\int_0^\infty \xi^{z-1} J_\nu(\xi r)\, d\xi = r^{-z} b_\nu(z)$ converges uniformly in $r \in [a_0, a_1]$ for $0 < a_0 < a_1 < \infty$, cf. [3, 8.5(7)], such that the change of order of integration is justified. Thus we obtain that

$$M(H_\nu u)(z) = b_\nu(z) Mu(2 - z) = b_\nu(z) M(Iu)(z) \text{ if } \operatorname{Re} z = 1,$$

i.e., actually $H_\nu u = \mathrm{op}_M^1(b_\nu) I u = I\, \mathrm{op}_M^1(a_\nu) u$. □

As a simple consequence of Lemma 4 we get that (21) extends to hold on $\mathcal{H}^{-\infty,-1/2}(\mathbb{R}_+)$, and then, for $\delta > -1-\nu$, even on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ by taking into account that $a_\nu(z)$ is holomorphic in the region $\operatorname{Re} z < \nu + 2$. We have now collected enough material to conclude the proof of Theorem 1 besides (c) that should be finished together with the proof of Theorem 3.

Proof of Theorem 1(a), (b), (d), (e). We have mentioned that, for $\delta > -1-\nu$, $H_\nu = I\, \mathrm{op}_M^{1-\delta}(a_\nu) = \mathrm{op}_M^{1+\delta}(b_\nu) I$ holds as mappings from $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ to $\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+)$. The mapping property (12) is then concluded by the estimates in Lemma 1. This gives us (a). The assertions (b), (d) are implied by Lemma 2, and (e) follows from Lemma 3. □

Proof of Theorem 2. Let $u \in C_0^\infty(\mathbb{R}_+)$. Then

$$\begin{aligned}\mathrm{op}_M^{1-\delta}(a_\nu)u - \mathrm{op}_M^{1-\delta'}(a_\nu)u &= \frac{1}{2\pi i}\int_{\Gamma_{1-\delta}} r^{-z} a_\nu(z)\tilde{u}(z)\,dz - \frac{1}{2\pi i}\int_{\Gamma_{1-\delta'}} r^{-z} a_\nu(z)\tilde{u}(z)\,dz \\ &= -\sum_{k=j}^{l-1} \operatorname{Res}_{z=\nu+2k+2}\ r^{-z} a_\nu(z)\tilde{u}(z) \\ &= \sum_{k=j}^{l-1} \frac{(-1)^k\, 2^{-\nu-2k}}{k!\,\Gamma(\nu+k+1)}\, r^{-\nu-2k-2}\,\tilde{u}(\nu+2k+2),\end{aligned}$$

thus $H_\nu^{(j)}u - H_\nu^{(k)}u = I\,\mathrm{op}_M^{1-\delta}(a_\nu)u - I\,\mathrm{op}_M^{1-\delta'}(a_\nu)u = \sum_{k=j}^{l-1} c_k(u)\,\xi^{\nu+2k}$, where the functionals c_k are given by (14). The proof is finished. □

Proof of Theorems 1(c), 3. Notice that, for $u \in \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, the condition stated in Theorem 3 is equivalent to

$$\langle z\rangle^s\, a_\nu(z)\tilde{u}(z) \in L^2(\Gamma_{1+\delta}). \tag{22}$$

First let $u = H_\nu v$ for $v \in \mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$. Then $\tilde{u}(z) = b_\nu(z)\tilde{v}(2-z)$ for $\operatorname{Re} z = 1+\delta$ such that $\langle z\rangle^s a_\nu(z)\tilde{u}(z) = \langle z\rangle^s \tilde{v}(2-z) \in L^2(\Gamma_{1+\delta})$. Vice versa, if $u \in \mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$ and (22) is satisfied, then v defined by $\tilde{v}(z) = a_\nu(2-z)\tilde{u}(2-z)$ for $\operatorname{Re} z = 1-\delta$ belongs to $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ and satisfies $u = H_\nu v$. The operator $H_\nu^{(j\to j+1)}$ is densely defined on $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, since the condition (22) is clearly fulfilled if $\tilde{u}(z) = 0$ on $\Gamma_{1+\delta}$ close to $z = \nu + 2j + 2$. □

Proof of Remark 2. If it were possible to give a meaning to H_ν on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$, then the functionals c_k, defined in (14), would extend to continuous functionals on this space which is not the case when $\delta \le -1-\nu$. □

Remark 3. From the above it is seen that, for δ, $\delta' \in \mathbb{R}$, $\delta > -1-\nu$, and $\delta' \le \delta$, we have

$$\begin{aligned}&H_\nu\colon \mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+) \cap \mathcal{H}^{-\infty,-1/2+\delta'}(\mathbb{R}_+) \to \\ &\to \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+) \cap \Big(\mathcal{H}^{-\infty,-1/2-\delta'}(\mathbb{R}_+) \oplus \mathcal{E}_{Q_\nu}^{-1/2-\delta}(\mathbb{R}_+)_{[\delta'-\delta,0]}\Big) \subseteq \mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+).\end{aligned}$$

Hence, by intersecting $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ with $\mathcal{H}^{-\infty,-1/2+\delta'}(\mathbb{R}_+)$, up to asymptotic terms of type Q_ν, cf. Lemma 6, the behaviour near $\xi = 0$ in the image of the Hankel transformation is improved up to the conormal order $-1/2-\delta'$. Notice, in particular, that $\mathcal{E}_{Q_\nu}^{-1/2-\delta}(\mathbb{R}_+)_{[\delta'-\delta,0]} = \emptyset$ if $\delta' > -1-\nu$.

We want to indicate still another possibility to verify Theorem 3 that shall be of use in defining H_ν for $\nu < 0$: as in [13], we introduce the differential operator

$$N_\nu = r^{-1}\big(-r\frac{d}{dr} + \nu - 1\big), \tag{23}$$

i.e., we have $N_\nu = r^{-1}\operatorname{op}_M^\gamma(n_\nu)$ for all $\gamma \in \mathbb{R}$ with the Mellin symbol $n_\nu(z) = z + \nu - 1$. Obviously, $N_\nu\colon \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-1,\gamma-1}(\mathbb{R}_+)$ for s, $\gamma \in \mathbb{R}$. Its crucial property is

$$H_{\nu+1}N_{\nu+1} = \xi H_\nu,$$

verified on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$, for $\delta > -1-\nu$, by

$$\begin{aligned} H_{\nu+1}N_{\nu+1} = I op_M^{2-\delta}(a_{\nu+1}) r^{-1}\operatorname{op}_M^{1-\delta}(n_{\nu+1}) = I\xi^{-1}\operatorname{op}_M^{1-\delta}(T^1 a_{\nu+1})\operatorname{op}_M^{1-\delta}(n_{\nu+1}) = \\ = \xi I\operatorname{op}_M^{1-\delta}((T^1 a_{\nu+1})\, n_{\nu+1}) = \xi I\operatorname{op}_M^{1-\delta}(a_\nu) = \xi H_\nu, \end{aligned}$$

since $(T^1 a_{\nu+1})(z)\, n_{\nu+1}(z) = a_\nu(z)$.

Lemma 5. *Let $s, \gamma \in \mathbb{R}$. Then, for $\gamma \neq -1/2 + \nu$, N_ν as operator acting from $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ to $\mathcal{H}^{s,\gamma-1}(\mathbb{R}_+)$ is a bijection. For $\gamma < -1/2+\nu$, its inverse denoted by $N_\nu^{(0)}$ is given by*

$$N_\nu^{(0)}u(r) = -r^{-\nu+1}\int_0^r r'^{\nu-1}u(r')\,dr', \tag{24}$$

while for $\gamma > -1/2+\nu$ its inverse then denoted by $N_\nu^{(1)}$ is given by

$$N_\nu^{(1)}u(r) = r^{-\nu+1}\int_r^\infty r'^{\nu-1}u(r')\,dr'. \tag{25}$$

Moreover, for $\gamma = -1/2-\nu$, the operator N_ν is still injective, but not surjective, with dense range. For $s \geq 1$, a function $u \in \mathcal{H}^{s-1,-3/2-\nu}(\mathbb{R}_+)$ belongs to its range if and only if the integral $\int_0^\infty r^{\nu-1}u(r)\,dr$ exists as an improper one and its value is zero. In that case the action of the inverse denoted by $N_\nu^{(0\to1)}$ on u is given by either (24) or (25).

Proof. It is in fact a matter of calculation. □

Now starting from $H_{\nu+1}N_{\nu+1} = \xi H_\nu$, for j, $l \in \mathbb{N}$, $j \leq l$, $\delta \in I_j(\nu)$, we arrive at the fundamental relation

$$\boxed{H_\nu^{(j)} = N_{\nu+1}^{(1)}\dots N_{\nu+j}^{(1)}N_{\nu+j+1}^{(0)}\dots N_{\nu+l}^{(0)}H_{\nu+l}\,r^l} \tag{26}$$

where $H_\nu^{(j)}$ as well as the composition on the right-hand side are regarded as acting from $\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+)$ to $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$. For j, $l \in \mathbb{N}$, $j < l$, and $\delta \in \overline{I_j(\nu)} \cap \overline{I_{j+1}(\nu)}$, the following variant of (26) is valid,

$$H_\nu^{(j\to j+1)} = N_{\nu+1}^{(1)}\dots N_{\nu+j}^{(1)}\,N_{\nu+j+1}^{(0\to1)}\,N_{\nu+j+2}^{(0)}\dots N_{\nu+l}^{(0)}H_{\nu+l}\,r^l,$$

where it is easily seen that $N_{\nu+j+2}^{(0)}\dots N_{\nu+l}^{(0)}H_{\nu+l}\,r^l$ maps $\mathcal{H}^{-\infty,-1/2-\delta}(\mathbb{R}_+)$ into the domain of definition of $N_{\nu+j+1}^{(0\to1)}$. Further relations can be derived from (26), e.g., choosing $k \in \mathbb{N}$, $k \leq j$, we get

$$\begin{aligned} H_\nu^{(j)} &= N_{\nu+1}^{(1)}\dots N_{\nu+k}^{(1)}\left(N_{\nu+k+1}^{(1)}\dots N_{\nu+j}^{(1)}N_{\nu+j+1}^{(0)}\dots N_{\nu+l}^{(0)}H_{\nu+l}\,r^{l-k}\right)r^k \\ &= N_{\nu+1}^{(1)}\dots N_{\nu+k}^{(1)}H_{\nu+k}^{(j-k)}\,r^k. \end{aligned}$$

It is obvious how to gain Theorem 3 from the according variant for $H_\nu^{(j\to j+1)}$ of the latter equality and Lemma 5.

Further results. We discuss and extend some results of A. H. Zemanian from [13] looked at from our point of view. The discussion in [13] is centered around the test function space $\mathcal{Z}_\nu(\mathbb{R}_+)$, on which the Hankel transformation acts as a self-inverse isomorphism, and its dual.

1. *Space of test functions.* Zemanian proposed in [13] as space of test functions for the Hankel transformation the space $\mathcal{Z}_\nu(\mathbb{R}_+)$, defined as the space of all $u \in C^\infty(\mathbb{R}_+)$ for which, for all $j,\, l \in \mathbb{Z}_+$, the seminorm

$$u \mapsto \sup_{r\in\mathbb{R}_+} \left| r^j \Big(r^{-1}\frac{d}{dr}\Big)^l \big(r^{-\nu}u(r)\big) \right|$$

is finite. $\mathcal{Z}_\nu(\mathbb{R}_+)$ equipped with this fundamental system of seminorms is a Fréchet space. It was shown in [13] that H_ν induces a topological isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto itself.

Recall that $\mathcal{S}(\mathbb{R}_+)$ is the space of all restrictions of functions in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$, while $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ is the space of all functions in $\mathcal{S}(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$. $\mathcal{S}'(\mathbb{R}_+)$ and $\mathcal{S}_0'(\overline{\mathbb{R}}_+)$ are similarly defined. Note that $\mathcal{S}'(\mathbb{R}_+)$ is the dual to $\mathcal{S}_0(\overline{\mathbb{R}}_+)$, $\mathcal{S}_0'(\overline{\mathbb{R}}_+)$ is the dual to $\mathcal{S}(\mathbb{R}_+)$. $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ is a closed subspace of $\mathcal{S}(\mathbb{R}_+)$, $\mathcal{S}'(\mathbb{R}_+)$ is a quotient of $\mathcal{S}_0'(\overline{\mathbb{R}}_+)$.

Lemma 6. *For $\nu \in \mathbb{R}$, we have*

$$\mathcal{Z}_\nu(\mathbb{R}_+) = \big\{ r^\nu \phi(r^2);\, \phi \in \mathcal{S}(\mathbb{R}_+) \big\}. \tag{27}$$

In particular, for $\nu > 0$,

$$\mathcal{Z}_\nu(\mathbb{R}_+) = \mathcal{S}^{1/2}_{Q_\nu}(\mathbb{R}_+), \tag{28}$$

with Q_ν being the asymptotic type $\{(-\nu-2j, 0)\}_{j\in\mathbb{N}}$.

Proof. It is easy to verify that all functions of the form

$$r^\nu \phi_0(r^2) + \phi_1(r),$$

where $\phi_0 \in \mathcal{S}(\mathbb{R}_+)$, $\phi_1 \in \mathcal{S}_0(\overline{\mathbb{R}}_+)$, belong to $\mathcal{Z}_\nu(\mathbb{R}_+)$. Vice versa, that all functions in $\mathcal{Z}_\nu(\mathbb{R}_+)$ are of this form is stated in a somewhat different form in [13, Lemma 5.2-1]. Now the mapping

$$\phi(r) \mapsto r^\nu \phi(r^2)$$

realizes an isomorphism from $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ onto $\mathcal{S}_0(\overline{\mathbb{R}}_+)$. Hence (27) follows. An immediate consequence is that, for any $\gamma < 1/2 + \nu$, it holds that $\mathcal{Z}_\nu(\mathbb{R}_+) = \mathcal{S}^\gamma_{Q_\nu}(\mathbb{R}_+)$. □

We especially obtain that

$$\mathcal{S}(\mathbb{R}_+) \to \mathcal{Z}_\nu(\mathbb{R}_+),\ \ \phi(r) \mapsto r^\nu \phi(r^2) \tag{29}$$

realizes an isomorphism from $\mathcal{S}(\mathbb{R}_+)$ onto $\mathcal{Z}_\nu(\mathbb{R}_+)$.

Lemma 7. *Let $\nu \in \mathbb{R}$, $\nu > 0$. Further let $u \in \mathcal{H}^{-\infty,1/2}(\mathbb{R}_+)$. Then $u \in \mathcal{S}^{1/2}_{Q_\nu}(\mathbb{R}_+)$ if and only if its Mellin transform $\tilde{u}(z)$ is meromorphic on $\mathbb{C}$ with at most simple poles at $-\nu-2j$, $j \in \mathbb{N}$, and additionally, for every excision function $\chi \in C^\infty(\mathbb{C})$ for these poles, we have that*

$$\chi(z)\tilde{u}(z) \in \mathcal{S}(\Gamma_\beta)$$

holds uniformly for $\beta \in [\beta_0, \beta_1]$, β_0, $\beta_1 \in \mathbb{R}$, $\beta_0 < \beta_1$, where $\mathcal{S}(\Gamma_\beta)$ is the Schwartz space on the weight line Γ_β.

Proof. This kind of characterization of the Mellin image of $\mathcal{S}^\gamma_P(\mathbb{R}_+)$ for $\gamma \in \mathbb{R}$, $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ can be found, e.g., in [7, 1.2.1]. □

The characterization of the Mellin transforms of functions in $\mathcal{Z}_\nu(\mathbb{R}_+)$, together with the description of H_ν as $I\,\mathrm{op}_M^1(a_\nu)$, reproves the result that H_ν is a topological isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto itself: for $u \in \mathcal{Z}_\nu(\mathbb{R}_+)$, $a_\nu(z)\tilde{u}(z)$ is meromorphic on $\mathbb{C}$ with at most simple poles at $\nu + 2j + 2$, $j \in \mathbb{N}$. Then, in the Mellin image, the application of I brings these poles to the places $-\nu - 2j$, $j \in \mathbb{N}$, as required. Moreover, H_ν is self-inverse on $\mathcal{Z}_\nu(\mathbb{R}_+)$, since $\mathcal{S}_{Q_\nu}^{-1/2}(\mathbb{R}_+) \subset \mathcal{H}^{\infty,-1/2}(\mathbb{R}_+)$, and H_ν is self-inverse on $\mathcal{H}^{\infty,-1/2}(\mathbb{R}_+)$. The above consideration also reveals that $\mathcal{Z}_\nu(\mathbb{R}_+)$ is the smallest space containing $C_0^\infty(\mathbb{R}_+)$ having the property that H_ν is an topological isomorphism on it.

2. *Hankel transformation of arbitrary order.* The relation (26), with $j = l$, was used in [13] to define the Hankel transformation H_ν on $\mathcal{Z}_\nu(\mathbb{R}_+)$ for arbitrary $\nu \in \mathbb{R}$. This was particularly based on the fact that $N_{\nu+1}$ induces an isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto $\mathcal{Z}_{\nu+1}(\mathbb{R}_+)$ with inverse $N_{\nu+1}^{(1)}$.

The observation that $H_\nu = I\,\mathrm{op}_M^1(a_\nu)$ holds provides another possibility to extend to orders $\nu < 0$. The definition of the operator $I\,\mathrm{op}_M^{1-\delta}(a_\nu)$ makes sense on $\mathcal{H}^{-\infty,-1/2+\delta}(\mathbb{R}_+)$ for arbitrary $\nu \in \mathbb{R}$, if only $\delta > -1-\nu$. Theorem 1 continues to hold, when again denoting the operator obtained by H_ν. For instance, H_ν is an isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ for $\nu > -1$. The corresponding variants of Theorem 2 and Theorem 3 are valid. Thus we also get modified Hankel transformations $H_\nu^{(j)}$ for $j \in \mathbb{N}$, $j \geq 1$.

This situation is pictured in Figure 1, where the axis are labelled with the order of the Hankel transformation ν and the parameter δ in (12). Thus the ν-axis corresponds to the conormal order $-1/2$ for which the isometric mapping property holds. The picture is symmetric with respect to the ν-axis in the sense that reflection yields the *inverse operator*. The grey region indicates where the Hankel transformation was initially defined, while the hatched region exemplary specifies the region in which we have $H_\nu^{(1)}$. The lines $\pm\delta = 1+\nu+2l$, $l \in \mathbb{N}$, cut out the intervals $I_j(\nu)$ from the vertical line drawn at width ν. In particular, for $-1-2l < \nu < 1-2l$, $l \in \mathbb{N}$, $l \geq 1$, we only have the intervals $I_j(\nu)$ with $|j| \geq l$, where $I_j(\nu) = (-1-\nu-2l, 1+\nu+2l)$ for $|j| = l$. Therefore, for $-1-2l < \nu < 1-2l$, $l \in \mathbb{N}$, $l \geq 1$, and $\delta \in I_l(\nu)$, $H_\nu^{(l)}$ is a bijection from $\mathcal{H}^{s,-1/2+\delta}(\mathbb{R}_+)$ onto $\mathcal{H}^{s-\delta,-1/2-\delta}(\mathbb{R}_+)$, with inverse $H_\nu^{(l)}$, especially, $H_\nu^{(l)}$ is a self-inverse isometry on $\mathcal{H}^{s,-1/2}(\mathbb{R}_+)$ in case $\delta = 0$.

Especially interesting are the points of intersection of the lines $\pm\delta = 1+\nu+2l$, $l \in \mathbb{N}$. The lines $\delta = -1-\nu-2l$ correspond to the poles of $a_\nu(z)$ at $z = \nu + 2l + 2$, whereas the dashed lines $\delta = 1+\nu+2l$ correspond to the zeros of $a_\nu(z)$ at $z = -\nu - 2l$. For $\nu = -1-2j$, $j \in \mathbb{N}$, we see that, at $z = \nu + 2l + 2$ for $l \in \mathbb{N}$, $l \leq j$, the poles of the numerator of $a_\nu(z)$ are cancelled by the poles of the denominator of $a_\nu(z)$. Equally the zeros of $a_\nu(z)$ at $z = -\nu - 2l$ for $l \in \mathbb{N}$, $l \leq j$ disappear. Therefore, $H_\nu = H_\nu^{(l)}$ for $\nu = -1-2j$, $l \in \mathbb{N}$, $l \leq j$. Afterwards this in part justifies the notation H_ν used in [13] to denote the extension of the Hankel transformation to $\mathcal{Z}_\nu(\mathbb{R}_+)$ for negative values of ν.

3. *Structure of the dual to $\mathcal{Z}_\nu(\mathbb{R}_+)$.* It is stated in [13, Problem 5.2-3] that every element in $\mathcal{Z}_\nu(\mathbb{R}_+)'$ restricted to (a,∞) for $a > 0$ is a distribution of temperated growth. Lemma 6 leads to a more accurate description of the structure of $\mathcal{Z}_\nu(\mathbb{R}_+)'$. First we have the dual to (29), which yields that every $\Phi \in \mathcal{Z}_\nu(\mathbb{R}_+)'$ may be represented in the form

$$\langle \Phi, \phi \rangle = \langle \Phi_1, \phi_1 \rangle, \quad \phi \in \mathcal{Z}_\nu(\mathbb{R}_+)$$

for a uniquely determined $\Phi_1 \in \mathcal{S}_0'(\overline{\mathbb{R}}_+)$, where $\phi(r) = r^\nu \phi_1(r^2)$.

Lemma 8. *Let $\nu \in \mathbb{R}$. Then every $\Phi \in \mathcal{Z}_\nu(\mathbb{R}_+)'$ may be represented in the form*

$$\langle \Phi, \phi \rangle = \frac{1}{2} \langle \Phi_0, E(r^{-\nu}\phi) \rangle, \quad \phi \in \mathcal{Z}_\nu(\mathbb{R}_+), \tag{30}$$

where $\Phi_0 \in \mathcal{S}'(\mathbb{R})$ is an even distribution and E denotes the even extension of a function defined on $\mathbb{R}_+$ to a function on $\mathbb{R}$. Moreover, this representation is unique.

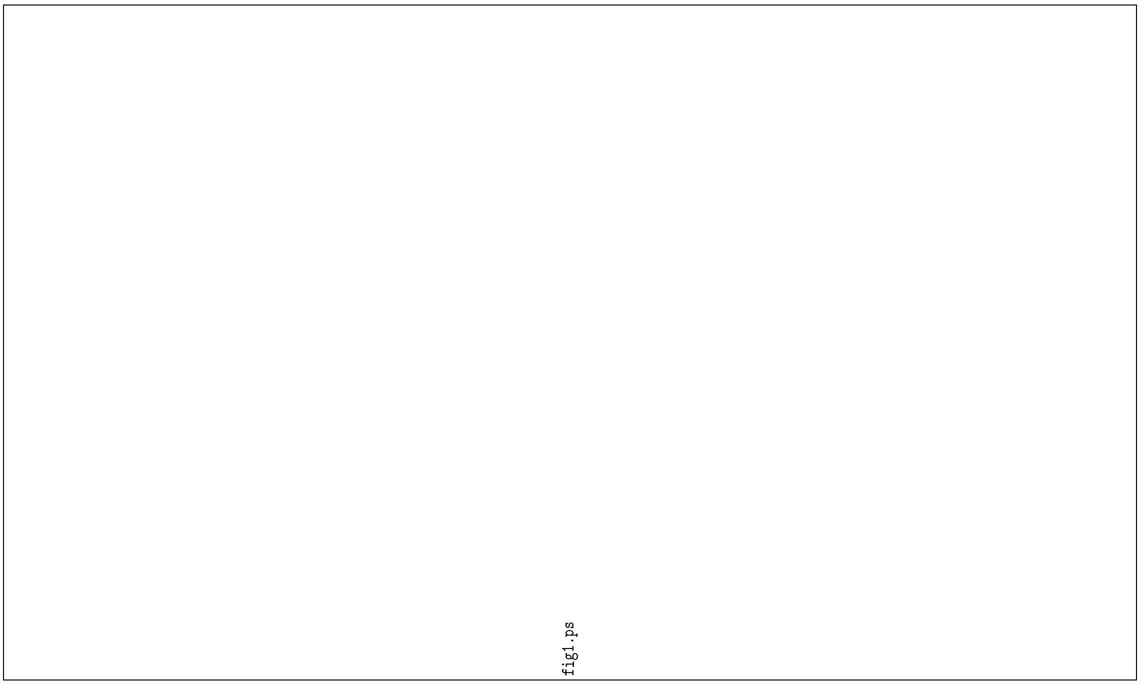

FIGURE 1. Range of parameters for the Hankel transformation

Proof. Since multiplication by $r^{-\nu}$ realizes an isomorphism from $\mathcal{Z}_\nu(\mathbb{R}_+)$ onto $\mathcal{Z}_0(\mathbb{R}_+)$, it suffices to prove the lemma for $\nu = 0$. Let $\mathcal{S}_{\text{ev}}(\mathbb{R})$, $\mathcal{S}_{\text{ev}}(\mathbb{R}_+)$ be resp. the space of all even Schwartz functions on $\mathbb{R}$ and the space of all restrictions of such functions to $\mathbb{R}_+$. The spaces $\mathcal{S}'_{\text{ev}}(\mathbb{R})$, $\mathcal{S}'_{\text{ev}}(\mathbb{R}_+)$ as well as $\mathcal{S}_{\text{odd}}(\mathbb{R})$, $\mathcal{S}'_{\text{odd}}(\mathbb{R})$ for odd distributions are similarly defined. Since $E\colon \mathcal{S}_{\text{ev}}(\mathbb{R}_+) \to \mathcal{S}_{\text{ev}}(\mathbb{R})$ is an isomorphism, with the inverse being the restriction operator, and obviously $\mathcal{Z}_0(\mathbb{R}_+) = \mathcal{S}_{\text{ev}}(\mathbb{R}_+)$, the dual to $\mathcal{Z}_0(\mathbb{R})$ can be identified with the dual to $\mathcal{S}_{\text{ev}}(\mathbb{R})$ which is $\mathcal{S}'_{\text{ev}}(\mathbb{R})$. The latter can be seen by noting that $\mathcal{S}(\mathbb{R}) = \mathcal{S}_{\text{ev}}(\mathbb{R}) \oplus \mathcal{S}_{\text{odd}}(\mathbb{R})$, $\mathcal{S}'(\mathbb{R}) = \mathcal{S}'_{\text{ev}}(\mathbb{R}) \oplus \mathcal{S}'_{\text{odd}}(\mathbb{R})$, and the polar to $\mathcal{S}_{\text{ev}}(\mathbb{R})$ with respect to the dual pairing $\langle \mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R})\rangle$ is $\mathcal{S}'_{\text{odd}}(\mathbb{R})$. The identification of $\mathcal{Z}_0(\mathbb{R}_+)'$ and $\mathcal{S}'_{\text{ev}}(\mathbb{R})$ is written down, for $\nu = 0$ and up to the factor $1/2$, in (30). $\square$

Noting further that when denoting the space of distributions in $\mathcal{S}'_{\text{ev}}(\mathbb{R})$ that are supported in $r = 0$ by $\mathcal{S}'_{\partial\overline{\mathbb{R}}_+,\text{ev}}(\overline{\mathbb{R}}_+)$, i.e., $\mathcal{S}'_{\partial\overline{\mathbb{R}}_+,\text{ev}}(\overline{\mathbb{R}}_+) \subset \mathcal{S}'_0(\overline{\mathbb{R}}_+)$ and

$$\mathcal{S}'_{\partial\overline{\mathbb{R}}_+,\text{ev}}(\overline{\mathbb{R}}_+) = \Big\{ \sum_{j=0}^{N} \alpha_{2j}\, \delta^{(2j)};\ \alpha_{2j} \in \mathbb{C}, j \in \mathbb{N}, j \le N, N \in \mathbb{N} \Big\},$$

where $\delta^{(2j)}$ are the even order derivatives of the Dirac measure at $r = 0$, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{S}'_{\partial\overline{\mathbb{R}}_+,\text{ev}}(\overline{\mathbb{R}}_+) \longrightarrow \mathcal{Z}_0(\mathbb{R}_+)' \xrightarrow{\ \iota'\ } \mathcal{S}'(\mathbb{R}_+) \longrightarrow 0,$$

via the dual to the canonical embedding $\iota\colon \mathcal{S}_0(\overline{\mathbb{R}}_+) \to \mathcal{S}_{\text{ev}}(\mathbb{R}_+)$. The normalization in (30) is chosen in such a way that $\iota'\Phi$ for $\Phi \in \mathcal{Z}_0(\mathbb{R}_+)'$ is won by restricting Φ_0 to $\mathbb{R}_+$. Noting that, for $\mathcal{Z}_1(\mathbb{R}_+) = \mathcal{S}_{\text{odd}}(\mathbb{R}_+)$, an analogous representation to (30) based on odd distributions can be supplied.

4. Application to Waves in Infinite Wedges

In this final section we consider an application to wave equations on infinite wedges. More precisely, we discuss the model equation in 1-1 dimensions

$$\partial_t^2 u - r^{-2}\big(r\partial_r\big)^2 u + r^{-2}\nu^2 u = f \text{ on } \mathbb{R}\times\mathbb{R}_+, \tag{31}$$

assuming that the right-hand side f fulfils $f = 0$ for $t < 0$. Thereby, throughout this section, we shall suppose that $\nu > 0$. Geometrically, the degeneracy at $r = 0$ in Eq. (31) corresponds to a conical point when t is fixed, then an edge emanating from this conical point arises in time. It is not difficult to verify that (31) possesses a unique solution $u \in \bigcap_{j=0}^{s} C^j(\mathbb{R};\mathcal{K}^{s-j+1,1/2}(\mathbb{R}_+)) \cap C^{s+1}(\mathbb{R};\mathcal{K}^{0,-1/2}(\mathbb{R}_+))$ satisfying $u = 0$ for $t < 0$ provided that $f \in \bigcap_{j=0}^{s+1} C^j(\mathbb{R};\mathcal{K}^{s-j,-3/2}(\mathbb{R}_+))$ for a certain $s \in \mathbb{N}$, $s \geq 1$. For details, see [1], [12]. But the general expectation from the elliptic theory, cf. [7], [9], is that (31) exhibits a solution $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)$, for $\eta > 0$, if $f \in \mathcal{W}_\eta^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)$, and s is any given real number. This section is devoted to the proof of that fact and some of its generalizations. Especially, it is stated in Remark 7 that $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)$ belongs to the space $H_\eta^{s+1}(\mathbb{R}\times\mathbb{R}_+)$ away from the edge $\{(t,r) \in \mathbb{R}\times\overline{\mathbb{R}}_+; r = 0\}$.

The elliptic differential operator $L_\nu = -r^{-2}\big(r\partial_r\big)^2 + r^{-2}\nu^2$ that appears on the left-hand side of (31) is diagonalized by the Hankel transformation, i.e., we have $H_\nu L_\nu = \xi^2 H_\nu$, cf. (10). This makes it possible to perform a global approach for solving Eq. (31), i.e., in some sense to deal with the operator $\partial_t^2 - r^{-2}\big(r\partial_r\big)^2 + r^{-2}\nu^2$ as being an operator having constant coefficients. Its Fourier-Hankel symbol is $-\tau^2 + \xi^2$, and its zeros $\xi = |\tau|$ for $(\tau,\xi) \neq 0$ causes a shift to the complex line $\operatorname{Im}\tau = -\eta$, $\eta > 0$. Notice the asymptotic behaviour of the Bessel function $J_\nu(r)$,

$$J_\nu(r) \sim \begin{cases} \Gamma(\nu+1)^{-1}\big(r/2\big)^\nu & \text{as } r \to +0, \\ \big(\pi r/2\big)^{-1/2}\cos\big(r - \nu\pi/2 - \pi/4\big) & \text{as } r \to \infty. \end{cases} \tag{32}$$

Therefore, localizing essentially splits the Hankel transformation in a part near $r = 0$ that behaves like the Mellin transformation, what is adequate in dealing with a conical point, and in a part near $r = \infty$ that behaves like the Fourier transformation.

In this paper, we treat the case where f is assumed to satisfy $f = 0$ for $t < 0$. Then the solution u should also satisfy $u = 0$ for $t < 0$. The Cauchy problem in which initial data are prescribed at $t = 0$ and the solution is sought for $t > 0$ shall be dealt with in a forthcoming paper. We remark that for the Cauchy problem additional compatibility conditions that should be fulfilled near $r = 0$ are required in order to obtain solutions of higher Sobolev regularity.

Theorem 4. *Let s, γ, $\eta \in \mathbb{R}$, $s \geq 0$, $\gamma \in [-3/2, s]$, $\eta > 0$. Then, for any $f \in \mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$ with $f = 0$ for $t < 0$, there exists a unique solution $u \in \mathcal{W}_{\eta,Q_\nu}^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)_{[-3/2-\gamma,0]}$ to (31) satisfying $u = 0$ for $t < 0$.*

Remark 4. (a) For $\gamma = -3/2$, the right-hand side $f \in \mathcal{W}_\eta^{s,-3/2}(\mathbb{R}\times\mathbb{R}_+)$ gives a solution $u \in \mathcal{W}_\eta^{s+1,1/2}(\mathbb{R}\times\mathbb{R}_+)$ if $s \geq 0$. Using a duality argument with respect to an extension of the $\mathcal{W}_\eta^{0,-1/2}(\mathbb{R}\times\mathbb{R}_+)$-duality, we get the same conclusion for all $s \in \mathbb{R}$. Thus part of Theorem 4 extends to assert the existence of a unique solution $u \in \mathcal{W}_\eta^{-\infty,1/2}(\mathbb{R}\times\mathbb{R}_+)$ to (31) if $f \in \mathcal{W}_\eta^{-\infty,-3/2}(\mathbb{R}\times\mathbb{R}_+)$, and the support assumptions are fulfilled.
(b) Then the other statements concern regularity of these solutions in case $s \geq 0$. Notice the particular instance $\gamma = s$, in which a right-hand side in $e^{\eta t}H_0^s(\mathbb{R}\times\overline{\mathbb{R}}_+)$ gives a solution in $e^{\eta t}H_0^{s+1}(\mathbb{R}\times\overline{\mathbb{R}}_+)$, plus terms having constant discrete asymptotics of type Q_ν.

Remark 5. It is seen that $f \in C^\infty(\mathbb{R};\mathcal{Z}_\nu(\mathbb{R}_+))$ with $f = 0$ for $t < 0$ leads to a unique solution $u \in C^\infty(\mathbb{R};\mathcal{Z}_\nu(\mathbb{R}_+))$ to (31) satisfying $u = 0$ for $t < 0$. This solution is given by the formula

$$u(t) = \int_0^t H_\nu \xi^{-1} \sin(\xi(t-s)) H_\nu f(s)\, ds.$$

Notice that $C^\infty(\mathbb{R}; \mathcal{Z}_\nu(\mathbb{R}_+)) = [\omega]\, \mathcal{W}^{\infty,1/2}_{Q_\nu,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+) + C^\infty(\mathbb{R}; \mathcal{S}_0(\overline{\mathbb{R}}_+))$.

As we will see in a moment in the proof of Theorem 4, the basic equation to discuss is

$$\boxed{\kappa(\tau)^{-1}\hat{u}(\tau) = \langle\tau\rangle^{-2}\, H_\nu \,\frac{1}{-(\tau/\langle\tau\rangle)^2+\xi^2}\, H_\nu \kappa(\tau)^{-1}\hat{f}(\tau)} \tag{33}$$

which holds for $\operatorname{Im}\tau = -\eta$ and relates the given right-hand side f to the solution u. But before we need two lemmas. The kind of argumentation that applies in the proof of the first one, which we call the $[\omega]$–$[1-\omega]$ *cut-off technique*, will be used on several occasions in the sequel. Later on a cut-off function $\omega \in C^\infty(\overline{\mathbb{R}}_+)$, i.e., it fulfils $\omega = 1$ near $r = 0$ and $\omega = 0$ for large r, shall be fixed once and for all.

Lemma 9. *Let s, $\gamma \in \mathbb{R}$, $s \geq 0$. Then*

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \begin{cases} \big(\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + \mathcal{H}^{s,s}(\mathbb{R}_+)\big) \cap \mathcal{H}^{0,0}(\mathbb{R}_+) & \text{if } 0 \leq \gamma \leq s, \\ \mathcal{H}^{s,\gamma}(\mathbb{R}_+) + \big(\mathcal{H}^{s,s}(\mathbb{R}_+) \cap \mathcal{H}^{0,0}(\mathbb{R}_+)\big) & \text{if } \gamma \leq 0 \leq s. \end{cases}$$

Proof. It suffices to note that $[1-\omega]\, H^s(\mathbb{R}_+) = [1-\omega]\,\big(\mathcal{H}^{s,s}(\mathbb{R}_+) \cap \mathcal{H}^{0,0}(\mathbb{R}_+)\big)$ for $s \geq 0$ and then to discuss the resulting cases. □

Lemma 10. *Let s, γ, $\eta \in \mathbb{R}$, and $s \geq 0$, $\gamma \in [-3/2, s]$, $\eta > 0$. Then*

$$H_\nu \,\frac{1}{-(\tau/\langle\tau\rangle)^2+\xi^2}\, H_\nu \colon\, \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}$$

is a family of bounded linear operators with parameter $\tau \in \mathbb{C}$, $\operatorname{Im}\tau = -\eta$, where

$$\Big\| H_\nu \,\frac{1}{-(\tau/\langle\tau\rangle)^2+\xi^2}\, H_\nu \Big\|_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)\to\mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}} \leq C_\eta \langle\tau\rangle \tag{34}$$

for some constant $C_\eta > 0$.

Proof. The operator under consideration is the composition of three linear operators. We look separately how these operators act.

By Theorem 1 and Lemma 9, we have

$$\begin{aligned} H_\nu \colon \mathcal{K}^{s,\gamma}(\mathbb{R}_+) &\to \begin{cases} \big(\mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) + \mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+)\big) \cap \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) \\ \mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) + \big(\mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+) \cap \mathcal{H}^{-1/2,-1}(\mathbb{R}_+)\big) \end{cases} \\ &= [\omega]\, \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) + [1-\omega]\,\big(\mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) + \mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+)\big), \end{aligned}$$

where the statement of the second line is true independently of the sign of γ. Furthermore, for the multiplication operator we have

$$\begin{aligned} \frac{1}{-(\tau/\langle\tau\rangle)^2+\xi^2} \colon [\omega]\, \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) + [1-\omega]\,\big(\mathcal{H}^{s-\gamma-1/2,-\gamma-1}(\mathbb{R}_+) + \mathcal{H}^{-1/2,-s-1}(\mathbb{R}_+)\big) \to \\ \to [\omega]\, \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) + [1-\omega]\,\big(\mathcal{H}^{s-\gamma-1/2,-\gamma-3}(\mathbb{R}_+) + \mathcal{H}^{-1/2,-s-3}(\mathbb{R}_+)\big) \\ = \mathcal{H}^{s-\gamma-1/2,-\gamma-3}(\mathbb{R}_+) \cap \mathcal{H}^{-1/2,-1}(\mathbb{R}_+), \end{aligned}$$

where the norm of the operator

$$\frac{1}{-(\tau/\langle\tau\rangle)^2+\xi^2} \colon [\omega]\, \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) \to [\omega]\, \mathcal{H}^{-1/2,-1}(\mathbb{R}_+)$$

is estimated by $C_\eta\langle\tau\rangle$. This follows by interpolation and duality, since the norm of the multiplication operators $\big(-(\tau/\langle\tau\rangle)^2+\xi^2\big)^{-1} \colon [\omega]\, \mathcal{H}^{r,-1/2}(\mathbb{R}_+) \to [\omega]\, \mathcal{H}^{r,-1/2}(\mathbb{R}_+)$ for $r =$

$0, 1$ are estimated by $C_\eta\langle\tau\rangle$. Finally, from Theorem 2 and Theorem 3, cf. also Remark 3, we infer that

$$H_\nu\colon \mathcal{H}^{s-\gamma-1/2,-\gamma-3}(\mathbb{R}_+) \cap \mathcal{H}^{-1/2,-1}(\mathbb{R}_+) \to$$
$$\to \big(\mathcal{H}^{s+2,\gamma+2}(\mathbb{R}_+) + \mathcal{E}^{1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}\big) \cap \mathcal{H}^{0,0}(\mathbb{R}_+) = \mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}.$$

The lemma is proved. □

Proof of Theorem 4. We start with the energetic inequality. Let $u \in \mathcal{W}^{s+2,\gamma+2}_\eta(\mathbb{R}\times\mathbb{R}_+)$. Define f to be the right-hand side of (31). Then $f \in \mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$. Applying the Fourier transformation in t and the Hankel transformation in r to (31), we arrive at the equation

$$-\tau^2 H_\nu \hat{u}(\tau) + \xi^2 H_\nu \hat{u}(\tau) = H_\nu \hat{f}(\tau)$$

valid for $\operatorname{Im}\tau = -\eta$. Hence we obtain Eq. (33), i.e.,

$$\kappa(\tau)^{-1}\hat{u}(\tau) = \langle\tau\rangle^{-2}\, H_\nu \,\frac{1}{-(\tau/\langle\tau\rangle)^2 + \xi^2}\, H_\nu \kappa(\tau)^{-1}\hat{f}(\tau)$$

for $\kappa_\lambda H_\nu = \lambda^{-1} H_\nu \kappa_\lambda^{-1}$ for $\lambda > 0$. This gives

$$\int_{-\infty-i\eta}^{\infty-i\eta} \langle\tau\rangle^{2s+2} \|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{\mathcal{K}^{s+2,1/2}_{Q_\nu}(\mathbb{R}_+)_{[-\gamma-3/2,0]}}\, d\tau \le$$
$$\le C_\eta^2 \int_{-\infty-i\eta}^{\infty-i\eta} \langle\tau\rangle^{2s} \|\kappa(\tau)^{-1}\hat{f}(\tau)\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)}\, d\tau,$$

where C_η is the constant in (34), i.e.,

$$\|u\|_{\mathcal{W}^{s+1,1/2}_{\eta,Q_\nu}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}} \le C_\eta\, \|f\|_{\mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)}.$$

Now using Remark 5 and the fact that $C_0^\infty(\mathbb{R}\times\mathbb{R}_+)$ is dense in $\mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$, the proof is concluded by standard functional-analytic arguments. □

To formulate the result in Theorem 5 we introduce some further notation. For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$, $\gamma \in \mathbb{R}$, let $\tilde{P}$ be the smallest asymptotic type larger or equal to P, $\tilde{P} \succcurlyeq P$, that satisfies the "shadow condition" $T^{-2}\tilde{P} \preccurlyeq \tilde{P}$, and let $Q_\nu \vee P$ be the maximum of Q_ν, P with respect to the partial ordering $\preccurlyeq$. For $P \in C^\infty(\mathbb{R}; \underline{\mathrm{As}}^\gamma)^\bullet$ these constructions apply pointwise.

Theorem 5. *Let $s, \gamma, \eta \in \mathbb{R}$, $s \ge 0$, $\gamma \in [-3/2, s]$, $\eta > 0$. Let $P \in C^\infty(\mathbb{R}; \underline{\mathrm{As}}^{-3/2})^\bullet$ be some asymptotic type for branching discrete asymptotics. Then, for any $f \in \mathcal{W}^{s,-3/2}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ with $f = 0$ for $t < 0$, there exists a unique solution $u \in \mathcal{W}^{s+1,1/2}_{\eta,Q_\nu\vee T^{-2}\tilde{P}}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ to (31) satisfying $u = 0$ for $t < 0$.*

Sketch of proof. For $\mathcal{W}^{s,-3/2}_{P,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]} = \mathcal{W}^{s,\gamma}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+) \oplus \mathcal{F}^{s,-3/2}_P(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$, and Eq. (31) with $f \in \mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$, $f = 0$ for $t < 0$, has already been solved, it suffices to suppose that $f \in \mathcal{F}^{s,-3/2}_P(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]} \cap \mathcal{W}^{s,-3/2}_\eta(\mathbb{R}\times\mathbb{R}_+)$. Because we need the result on asymptotics only locally, we are further allowed to assume that f has compact support in t. Here we deal with the case that f has the form

$$f(t,r) = F^{-1}_{\tau\to t}\Big\{\alpha(t)\omega(r\langle\tau\rangle)\langle\tau\rangle^{1/2}\big\langle\zeta, (r\langle\tau\rangle)^{-z}\big\rangle\hat{v}(\tau)\Big\} \tag{35}$$

with $\zeta \in \mathcal{A}'(K)^\bullet$ for some compact K, $K \subset \{z \in \mathbb{C}; 1/2-\gamma \le \operatorname{Re} z < 2\}$, $\alpha \in C_0^\infty(\mathbb{R})$, $\alpha = 0$ for $t < 0$, and $v \in H^s(\mathbb{R}_+)$.

Employing (33) and the series expansion

$$\frac{1}{-\tau^2+\xi^2} = \sum_{k=0}^{l-1} \frac{P_k(\tau_1,\xi)}{(-\tau_1^2+\xi^2)^{k+1}}\,(\tau-\tau_1)^k + \\ + \int_0^1 (1-\sigma)^{l-1} \frac{P_l(\tau_1+\sigma(\tau-\tau_1),\xi)}{(-(\tau_1+\sigma(\tau-\tau_1))^2+\xi^2)^{l+1}}\,d\sigma\,(\tau-\tau_1)^l,$$

where $P_k(\tau,\xi)$ for $k=0,\ldots,l$ is a homogeneous polynomial of degree k in (τ,ξ) in the monomials of which ξ appears only in even powers, by Taylor's formula, we obtain

$$\begin{aligned}\hat{u}(\tau,r) =&(2\pi)^{-1}\int_{-\infty-i\eta}^{\infty-i\eta} \hat{\alpha}(\tau-\tau_1)\,H_\nu \frac{1}{-\tau^2+\xi^2} H_\nu\,\kappa(\tau_1)\langle\zeta,\omega(r)r^{-z}\rangle \hat{v}(\tau_1)\,d\tau_1 \qquad (36)\\ =&\sum_{k=0}^{l-1}(2\pi)^{-1}\int_{-\infty-i\eta}^{\infty-i\eta}\hat{\alpha}_k(\tau-\tau_1)\kappa(\tau_1)\,H_\nu \frac{P_k(\tau_1/\langle\tau_1\rangle,\xi)}{(-(\tau_1/\langle\tau_1\rangle)^2+\xi^2)^{k+1}}\,H_\nu \\ &\langle\zeta,\omega(r)r^{-z}\rangle\hat{v}_k(\tau_1)\,d\tau_1 + (2\pi)^{-1}\int_{-\infty-i\eta}^{\infty-i\eta}\int_0^1(1-\sigma)^{l-1}\hat{\alpha}_l(\tau-\tau_1)\kappa(\tau_1)\\ &H_\nu \frac{P_l(\tau_1/\langle\tau_1\rangle+\sigma(\tau/\langle\tau_1\rangle-\tau_1/\langle\tau_1\rangle),\xi)}{(-(\tau_1/\langle\tau_1\rangle+\sigma(\tau/\langle\tau_1\rangle-\tau_1/\langle\tau_1\rangle))^2+\xi^2)^{l+1}} H_\nu\,\langle\zeta,\omega(r)r^{-z}\rangle\hat{v}_l(\tau_1)\,d\sigma d\tau_1,\end{aligned}$$

where $\alpha_k = (-i\partial/\partial t)^k\alpha \in C_0^\infty(\mathbb{R})$, $\alpha_k = 0$ for $t<0$, $v_k = (1-\Delta)^{-(k+2)/2}v \in H^{s+k+2}(\mathbb{R})$ such that $\hat{\alpha}_k(\tau) = \tau^k\hat{\alpha}(\tau)$ and $\hat{v}_k(\tau) = \langle\tau\rangle^{-k-2}\hat{v}(\tau)$.

After applying inverse Fourier transformation, the first l summands in the last equality contribute expressions,

$$F^{-1}_{\tau\to t}\Big\{\alpha_k(t)\kappa(\tau)\,H_\nu \frac{P_k(\tau/\langle\tau\rangle,\xi)}{\big(-(\tau/\langle\tau\rangle)^2+\xi^2\big)^{k+1}}\,H_\nu\langle\zeta,\omega(r)r^{-z}\rangle\hat{v}_k(\tau)\Big\},$$

that are basically of the same form as f in (35). For an arbitrary finite weight interval Δ, these expressions are seen to belong resp. to $\mathcal{F}^{s+k+2,1/2}_{Q_\nu\vee T^{-k-2}\tilde{P}}(\mathbb{R}\times\mathbb{R}_+)_\Delta$ for k even and to $\mathcal{F}^{s+k+2,1/2}_{Q_\nu\vee T^{-k-3}\tilde{P}}(\mathbb{R}\times\mathbb{R}_+)_\Delta$ for k odd, where $P=\mathrm{sg}^\bullet(\zeta)$. This is true mainly because

$$H_\nu \frac{P_k(\tau,\xi)}{(-\tau^2+\xi^2)^{k+1}}\,H_\nu\langle\zeta,\omega(r)r^{-z}\rangle \in \begin{cases}\mathcal{S}^{1/2}_{Q_\nu\vee T^{-k-2}\tilde{P}}(\mathbb{R}_+) & \text{if } k \text{ is even},\\ \mathcal{S}^{1/2}_{Q_\nu\vee T^{-k-3}\tilde{P}}(\mathbb{R}_+) & \text{if } k \text{ is odd},\end{cases}$$

for $\tau\in\mathbb{C}$, $\operatorname{Im}\tau<0$, and that can be shown starting with $\langle\zeta,\omega(r)r^{-z}\rangle \in \mathcal{S}^{-3/2}_P(\mathbb{R}_+)$, which holds by definition, and then applying the reasoning leading to the proof of Lemma 10. Moreover, it can be shown that the remainder

$$\begin{aligned}F^{-1}_{\tau\to t}\Big\{&(2\pi)^{-1}\int_{-\infty-i\eta}^{\infty-i\eta}\int_0^1(1-\sigma)^{l-1}\hat{\alpha}_l(\tau-\tau_1)\kappa(\tau_1)\\ &H_\nu \frac{P_l(\tau_1/\langle\tau_1\rangle+\sigma(\tau/\langle\tau_1\rangle-\tau_1/\langle\tau_1\rangle),\xi)}{(-(\tau_1/\langle\tau_1\rangle+\sigma(\tau/\langle\tau_1\rangle-\tau_1/\langle\tau_1\rangle))^2+\xi^2)^{l+1}} H_\nu\,\langle\zeta,\omega(r)r^{-z}\rangle\hat{v}_l(\tau_1)\,d\sigma d\tau_1\Big\}\end{aligned}$$

belongs to $\mathcal{W}^{s+l+1,1/2}_{Q_\nu}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ provided that $l\geq\gamma+3/2$. Thus we have found that $u\in\mathcal{W}^{s+1,1/2}_{Q_\nu\vee T^{-2}\tilde{P}}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]}$ as required.

This completes the proof in the case that f has the form (35). The general case follows by tensor product arguments. □

Remark 6. Theorem 4 is a special case of Theorem 5 for $P=\mathcal{O}$, since $Q_\nu\vee T^{-2}\tilde{\mathcal{O}} = Q_\nu$ and $\mathcal{W}^{s,-3/2}_{\eta,\mathcal{O}}(\mathbb{R}\times\mathbb{R}_+)_{[-\gamma-3/2,0]} = \mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)$.

Appendix A.

In this appendix, we collect facts concerning cone Sobolev spaces, Mellin operators acting on them, discrete asymptotics, wedge Sobolev spaces, and so on. For further details and the general ideas behind, see [7], [9]. In the notation of asymptotics we closely follow [7], what is well-adapted in our situation, but slightly different from [9], where half-open weight intervals were considered. Note that we are very brief in describing the concept of branching discrete asymptotics, see [8], since we only mention the generalization of Theorem 4 to Theorem 5 without going to much into the details.

A.1. **Cone Sobolev spaces and Mellin operators.** For given $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, the Hilbert space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is defined to consist of all functions u on $\mathbb{R}_+$ satisfying $r^{j-\gamma} d^j u/dr^j \in L^2(\mathbb{R}_+)$ for all $j \in \mathbb{N}$, $j \leq s$. For arbitrary s, $\gamma \in \mathbb{R}$, the Hilbert space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is then defined by duality and interpolation. In particular, $r^\delta \mathcal{H}^{s,\gamma}(\mathbb{R}_+) = \mathcal{H}^{s,\gamma+\delta}(\mathbb{R}_+)$.

It turns out that the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ can be introduced using the Mellin transformation,

$$Mu(z) = \tilde{u}(z) = \int_0^\infty r^{z-1} u(r)\, dr \tag{37}$$

for $u \in C_0^\infty(\mathbb{R}_+)$. It can be shown that the Mellin transformation extends by continuity to an isomorphism $M\colon r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{1/2-\gamma})$, where $\Gamma_{1/2-\gamma}$ denotes the *weight line* $\{z \in \mathbb{C};\ \operatorname{Re} z = 1/2 - \gamma\}$ and $L^2(\Gamma_{1/2-\gamma})$ is explained via identification of $\Gamma_{1/2-\gamma}$ with $\mathbb{R}$. More generally, an equivalent norm on $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is given by

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} = \left\{ \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} (1+|z|^2)^s\, |Mu(z)|^2\, dz \right\}^{1/2}. \tag{38}$$

Henceforth, the expression $(1+|z|^2)^{1/2}$ shall be abbreviated as $\langle z \rangle$. Speaking of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ as a Hilbert space, we always have the norm (38) in mind. For further reference, we also introduce

$$\mathcal{H}^{\infty,\gamma}(\mathbb{R}_+) = \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+), \;\; \mathcal{H}^{-\infty,\gamma}(\mathbb{R}_+) = \bigcup_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+).$$

Given a function $h(z)$ on $\Gamma_{1/2-\gamma}$ satisfying the estimate $|h(z)| \leq C \langle z \rangle^\mu$ for certain constants C, $\mu \in \mathbb{R}$, a so-called *Mellin symbol*, we associate with h the Mellin operator $\operatorname{op}_M^{1/2-\gamma}(h)$ by the convention

$$\operatorname{op}_M^{1/2-\gamma}(h)u(r) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty \left(\frac{r}{r'}\right)^{-z} h(z) u(r')\, dr' dz$$

for $u \in C_0^\infty(\mathbb{R}_+)$, i.e., formally we have $\operatorname{op}_M^{1/2-\gamma}(h) = M^{-1} h(z) M$, with the Mellin transformation and its inverse being related to the weight line $\Gamma_{1/2-\gamma}$. Obviously, $\operatorname{op}_M^{1/2-\gamma}(h)$ extends by continuity to an operator

$$\operatorname{op}_M^{1/2-\gamma}(h)\colon \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+). \tag{39}$$

If $h \in S^\mu(\Gamma_{1/2-\gamma})$ is a pseudo-differential symbol on $\Gamma_{1/2-\gamma}$, then $\operatorname{op}_M^{1/2-\gamma}(h)$ becomes a pseudo-differential operator of order μ on $\mathbb{R}_+$.

We still need a variant of the cone Sobolev spaces, that is defined by

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = [\omega]\, \mathcal{H}^{s,\gamma}(\mathbb{R}_+) + [1-\omega]\, H^s(\mathbb{R}_+),$$

where $s \in \mathbb{R} \cup \{\pm\infty\}$, $\gamma \in \mathbb{R}$. ω is a cut-off function, i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ with $\omega(r) = 1$ close to $r = 0$, fixed once and for all. The notation $[\omega]\, \mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ stands for the closure of $\omega(r)\, \mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ in $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, similarly for $[1-\omega]\, H^s(\mathbb{R})$. The parameter γ is interpreted as *conormal order* of distributions near $r = 0$. Especially, $\mathcal{K}^{0,0}(\mathbb{R}_+) = L^2(\mathbb{R}_+)$. The multiplicative group $\mathbb{R}_+$ acts on $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ as a strongly continuous group $\{\kappa_\lambda\}_{\lambda>0}$ by

$$\kappa_\lambda u(r) = r^{1/2}\, u(\lambda r). \tag{40}$$

$\mathbb{R}_+$ acts as a group of isometries on $L^2(\mathbb{R}_+)$.

Now we introduce discrete asymptotics for distributions in $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. These asymptotics are of the form

$$u(r) \sim \sum_{j=1}^{N} \sum_{k=0}^{m_j} c_{jk}\, r^{-p_j} (\log r)^k \text{ as } r \to 0 \tag{41}$$

for $N \in \mathbb{N} \cup \{\infty\}$, $m_j \in \mathbb{N}$, and certain $c_{jk} \in \mathbb{C}$. Notice that, for $p \in \mathbb{C}$, $k \in \mathbb{N}$, we have $\omega(r)\, r^{-p}(\log r)^k \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ if and only if $\operatorname{Re} p < 1/2 - \gamma$. An asymptotic type P is then a sequence $\{(p_j, m_j)\}_{j=1}^N \subset \mathbb{C} \times \mathbb{N}$ having the property that $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ in case $N = \infty$. In addition, some conormal order γ will be fixed, then it is required that $\operatorname{Re} p_j < 1/2 - \gamma$ holds for all j, $1 \le j \le N$. The set of such asymptotic types is denoted by $\underline{\mathrm{As}}^{\gamma,\bullet}$, the empty asymptotic type, for $N = 0$, by $\mathcal{O}$. For there are also continuous asymptotics, $^\bullet$ always indicates discrete asymptotics. For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$, $P = \{(p_j, m_j)\}_{j=1}^N$, and $\varrho \in \mathbb{R}$, $T^\varrho P = \{(p_j + \varrho, m_j)\}_{j=1}^N$ denotes the shift of P by ϱ.

By a weight interval for asymptotics as $r \to 0$ we mean a closed interval Δ of the form $[\vartheta, 0]$, $\vartheta \le 0$, or $(-\infty, 0]$. For two asymptotic types $P = \{(p_j, m_j)\}_{j=1}^N$, $P' = \{(p'_k, m'_k)\}_{k=1}^{N'}$, we write $P \preccurlyeq P'$ if for every j, $1 \le j \le N$, there is a k, $1 \le k \le N'$, such that $p_j = p'_k$ and $m_j \le m'_k$. Asymptotic types P, P' satisfying $P \preccurlyeq P'$ and $P' \preccurlyeq P$ are regarded as being equal. Every asymptotic type $P = \{(p_j, m_j)\}_{j=1}^N$ has a *minimal representation* characterized by the property that $p_j \ne p_k$ for $j \ne k$. Given $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ and a finite weight interval Δ, we write $\pi_{\mathbb{C}}(P) = \bigcup_{j=1}^N \{p_j\}$ and $m^\gamma(P, \Delta) = \sum_{j:\, \operatorname{Re} p_j - 1/2 + \gamma \in \Delta} (m_j + 1)$, where $P = \{(p_j, m_j)\}_{j=1}^N$ is a minimal representation of P. $P_\Delta^\gamma = \{(p_j, m_j)\}_{j:\, \operatorname{Re} p_j - 1/2 + \gamma \in \Delta}$ is the restriction of P to Δ.

For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ and a finite weight interval Δ, $\mathcal{E}_P^\gamma(\mathbb{R}_+)_\Delta$ denotes the linear span of all functions $\omega(r) r^{-p_j}(\log r)^k$, with (p_j, m_j) being a term of the sequence P, $\operatorname{Re} p_j \ge 1/2 - \gamma - \vartheta$, and $k \in \mathbb{N}$, $k \le m_j$. Notice that $\mathcal{E}_P^\gamma(\mathbb{R}_+)_\Delta \subset \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, while $\mathcal{E}_P^\gamma(\mathbb{R}_+)_\Delta \cap \mathcal{K}^{s,\gamma-\vartheta}(\mathbb{R}_+) = \emptyset$. We put

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)_\Delta = \mathcal{K}^{s,\gamma-\vartheta}(\mathbb{R}_+) \oplus \mathcal{E}_P^\gamma(\mathbb{R}_+)_\Delta.$$

In particular, $\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)_{[0,0]} = \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$. Moreover, we put

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)_{(-\infty,0]} = \bigcap_{k=0}^{\infty} \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)_{[-k,0]}.$$

Example 2. For $s \ge 0$, we have $H^s(\mathbb{R}_+) = \mathcal{K}_{P_0}^{s,0}(\mathbb{R}_+)_{[-s,0]}$, where $P_0 = \{(-j, 0); j \in \mathbb{N}\}$ is Taylor asymptotics, and $H_0^s(\overline{\mathbb{R}}_+) = \mathcal{K}_{\mathcal{O}}^{s,0}(\mathbb{R}_+)_{[-s,0]}$, where $H_0^s(\overline{\mathbb{R}}_+)$ is the space of all functions in $H^s(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$. This is in essence shown in [2, Appendix A].

We also put

$$\mathcal{S}_P^\gamma(\mathbb{R}_+) = [\omega]\, \mathcal{K}_P^{\infty,\gamma}(\mathbb{R}_+) + [1-\omega]\, \mathcal{S}(\mathbb{R}_+).$$

Here $\mathcal{S}(\mathbb{R}_+)$ is the space of restrictions of functions in $\mathcal{S}(\mathbb{R})$ to $\mathbb{R}_+$. Note the particular cases $\mathcal{S}_{P_0}^{1/2}(\mathbb{R}_+) = \mathcal{S}(\mathbb{R}_+)$ and $\mathcal{S}_{\mathcal{O}}^{1/2}(\mathbb{R}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+)$, where $\mathcal{S}_0(\overline{\mathbb{R}}_+)$ is the space of all functions in $\mathcal{S}(\mathbb{R})$ supported in $\overline{\mathbb{R}}_+$.

Notice that, for $N < \infty$, the asymptotics appearing in (41) can be rewritten as

$$u \sim \langle \zeta, r^{-z} \rangle \text{ as } r \to 0, \tag{42}$$

where $\zeta \in \mathcal{A}'(\mathbb{C})$ is the analytical functional given by

$$\langle \zeta, h \rangle = \sum_{j=1}^{N} \sum_{k=0}^{m_j} (-1)^k c_{jk} \frac{d^k h}{dz^k}(p_j), \quad h \in \mathcal{A}(\mathbb{C}),$$

and the pairing between ζ and r^{-z} is in the variable z. The special form of the analytic functional ζ is referred to as being *of discrete type* and *of finite order*. In particular, ζ

is carried by $\bigcup_{j=1}^N\{p_j\}$, i.e., $\zeta \in \mathcal{A}'(\bigcup_{j=1}^N\{p_j\})$. We write $\mathrm{sg}^\bullet(\zeta)$ for the asymptotic type $\{(p_j, m_j)\}_{j=1}^N$.

A.2. **Wedge Sobolev spaces.** For $s, \gamma, \eta \in \mathbb{R}$, we introduce the wedge Sobolev space $\mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$ as the space of all $u \in \mathcal{S}'(\mathbb{R};\mathcal{K}^{s,\gamma}(\mathbb{R}_+))$ such that $\hat{u} \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{K}^{s,\gamma}(\mathbb{R}_+))$ and the norm

$$\|u\|_{\mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)} = \left\{\frac{1}{2\pi}\int_{-\infty-i\tau}^{\infty-i\tau}\langle\tau\rangle^{2s}\,\|\kappa(\tau)^{-1}\hat{u}(\tau)\|^2_{\mathcal{K}^{s,\gamma}(\mathbb{R}_+)}\,d\tau\right\}^{1/2} \tag{43}$$

is finite. Here $\hat{u}$ denotes the Fourier transform of u, $Fu(\tau) = \hat{u}(\tau) = \int_{-\infty}^{\infty} e^{-it\tau}u(t)\,dt$, where $\operatorname{Im}\tau = -\eta$. Further, $\kappa(\tau) = \kappa_{\langle\tau\rangle}$, where the group action κ_λ is defined in (40). The space $\mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$ is a Hilbert space. Obviously, $\mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+) = e^{\eta t}\mathcal{W}^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$, where $\mathcal{W}^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+) = \mathcal{W}_0^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$. $\mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$ consists of all distributions $u(t,r)$ of the form

$$u(t,r) = F^{-1}_{\tau\to t}\{\langle\tau\rangle^{1/2}\hat{v}(\tau, r\langle\tau\rangle)\}, \tag{44}$$

where $v \in H^s_\eta(\mathbb{R};\mathcal{K}^{s,\gamma}(\mathbb{R}_+))$, $\hat{v}(\tau,r) = F_{t\to\tau}\{v(t,r)\}$, and Fourier transformation and its inverse transformation are related to the line $\{\tau\in\mathbb{C}; \operatorname{Im}\tau = -\eta\}$.

Remark 7. It is not hard to see that, for a distribution u on $\mathbb{R}\times\mathbb{R}_+$ with $\operatorname{supp} u \subseteq \mathbb{R}\times[a,\infty)$ for some $a > 0$, $u \in \mathcal{W}_\eta^{s,\gamma}(\mathbb{R}\times\mathbb{R}_+)$ if and only if $u \in H^s_\eta(\mathbb{R}\times\mathbb{R}_+)$, with equivalence of norms and constants only depending on a, cf. [10, Proposition 2.1.19].

Example 3. Using in (43), for $s \geq 0$, the spaces resp. $H^s(\mathbb{R}_+)$ and $H^s_0(\overline{\mathbb{R}}_+)$ instead of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$, and the same group action κ_λ, yields resp. $H^s_\eta(\mathbb{R}\times\mathbb{R}_+)$ and $e^{\eta t}H^s_0(\mathbb{R}\times\overline{\mathbb{R}}_+)$.

For $P \in \underline{\mathrm{As}}^{\gamma,\bullet}$ and Δ a finite weight interval, we obtain distributions u on $\mathbb{R}\times\mathbb{R}_+$ obeying constant discrete asymptotics of type P over the interval Δ by inserting $v \in H^s_\eta(\mathbb{R};\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)_\Delta)$ in (44) instead of $v \in H^s_\eta(\mathbb{R};\mathcal{K}^{s,\gamma}(\mathbb{R}_+))$. This gives us the space $\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_\Delta$. For $\Delta = (-\infty, 0]$, we set $\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+) = \bigcap_{k=0}^\infty \mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_{[-k,0]}$. We want to generalize this construction to the case when the asymptotic type is allowed to vary with the edge parameter $t\in\mathbb{R}$. The main difficulty is that the singular exponents p_j in (41), that are associated with the concrete differential operator under consideration, are in general solutions to nonlinear eigenvalue problems and thus possibly possess a very complicated branching behaviour in t. The ansatz to the solution of this problem comes from the introduction of analytic functionals, cf. (42).

We describe here the local picture. Let $U \subseteq \mathbb{R}$ be open, $\gamma\in\mathbb{R}$ be a fixed conormal order. Further let Δ be a finite weight interval. Denote $S^\gamma_\Delta = \{z\in\mathbb{C};\ \operatorname{Re} z - 1/2 + \gamma \in \Delta\}$. Then $C^\infty(U;\mathcal{A}'(S^\gamma_\Delta))^\bullet$ is the space of all t-dependent analytic functionals $\zeta = \zeta(t)$, $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta))$, such that, for each $t\in U$, $\zeta(t)$ is discrete and of finite order and, for every open set U', $U' \Subset U$, there exists a compact set $K\subset S^\gamma_\Delta$, $K\cap\Gamma_{1/2-\gamma} = \emptyset$, such that $\zeta\in C^\infty(U';\mathcal{A}'(K))$ and

$$\sup_{t\in U'} m^\gamma(\mathrm{sg}^\bullet(\zeta(t)),\Delta) < \infty. \tag{45}$$

$P = \{P(t)\}_{t\in U}$ is said to be an asymptotic type for branching discrete asymptotics if, for every finite weight interval Δ, there is a $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta))^\bullet$ such that

$$\mathrm{sg}^\bullet(\zeta(t)) = P(t)^\gamma_\Delta$$

for all $t\in U$. Given such an asymptotic type, we write $P\in C^\infty(U;\underline{\mathrm{As}}^\gamma)^\bullet$. If $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta))^\bullet$ and $\mathrm{sg}^\bullet(\zeta(t)) \subseteq P(t)^\gamma_\Delta$ for all $t\in U$, then we write $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta))^\bullet_P$.

There are vector-valued versions of the above constructions possible. For example, we have the space $C^\infty(U;\mathcal{A}'(S^\gamma_\Delta;H^s(\mathbb{R})))^\bullet_P$ for $P\in C^\infty(U;\underline{\mathrm{As}}^\gamma)^\bullet$, where $\mathcal{A}'(S^\gamma_\Delta;H^s(\mathbb{R})) = \mathcal{A}'(S^\gamma_\Delta)\hat{\otimes}_\pi H^s(\mathbb{R})$, and $\hat{\otimes}_\pi$ is the completed projective tensor product, due to the nuclearity of $\mathcal{A}'(S^\gamma_\Delta)$. Notice that $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta;H^s(\mathbb{R})))^\bullet$ means that $\zeta = \zeta(t)$ is a t-dependent,

$H^s(\mathbb{R})$-*valued* analytic functional, of discrete type and of finite order, carried in S^γ_Δ, that fulfils the conditions stated before and in (45).

The space $\mathcal{F}^{s,\gamma}_P(U\times\mathbb{R}_+)_\Delta$ of asymptotic terms, for $P\in C^\infty(U;\underline{\mathrm{As}}^\gamma)^\bullet$, $\Delta=[\vartheta,0]$, $\vartheta\le 0$, is the space of all distributions $u(t,r)$ having the form

$$u(t,r)=F^{-1}_{\tau\to t}\Big(\omega(r\langle\tau\rangle)\langle\tau\rangle^{1/2}\big\langle\hat\zeta(t,\tau),(r\langle\tau\rangle)^{-z}\big\rangle\Big) \tag{46}$$

for a certain $\zeta\in C^\infty(U;\mathcal{A}'(S^\gamma_\Delta;H^s(\mathbb{R})))^\bullet_P$. Here $\hat\zeta(t,\tau)=F_{t'\to\tau}\{\zeta(t,t')\}$, where $t'\in\mathbb{R}$ is the variable for which $\zeta=\zeta(t)$ belongs to $H^s(\mathbb{R})$. Notice that u defined in (46) is in $\mathcal{W}^{s,\gamma}_{\mathrm{loc}}(U\times\mathbb{R}_+)$, but not in $\mathcal{W}^{s,\gamma-\vartheta}_{\mathrm{loc}}(U\times\mathbb{R}_+)$, where *loc*-versions of the wedge Sobolev spaces have the usual meaning. We then put

$$\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}}(U\times\mathbb{R}_+)_\Delta=\mathcal{W}^{s,\gamma-\vartheta}_{\mathrm{loc}}(U\times\mathbb{R}_+)\oplus\mathcal{F}^{s,\gamma}_P(U\times\mathbb{R}_+)_\Delta.$$

For $\Delta=(-\infty,0]$, we set $\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}}(U\times\mathbb{R}_+)=\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}}(U\times\mathbb{R}_+)_{(-\infty,0]}=\bigcap_{k=0}^\infty\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}}(U\times\mathbb{R}_+)_{[-k,0]}$. Finally, for an arbitrary weight interval Δ, we put

$$\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_\Delta=\mathcal{W}^{s,\gamma}_\eta(\mathbb{R}\times\mathbb{R}_+)\cap\mathcal{W}^{s,\gamma}_{P,\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)_\Delta, \tag{47}$$

again with the notation $\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)=\mathcal{W}^{s,\gamma}_{\eta,P}(\mathbb{R}\times\mathbb{R}_+)_{(-\infty,0]}$.

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