Nonstationary Problems for Equations of Borel-Fuchs type

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Abstract

In the paper, the nonstationary problems for equations of Borel-Fuchs type are investigated. The asymptotic expansion are obtained for different orders of degeneration of operators in question. The approach to nonstationary problems based on the asymptotic theory on abstract algebras is worked out.

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Contents

Introduction

Let Y be a smooth manifold of dimension n (for simplicity, one can view this manifold as the Cartesian space $\mathbf{R}^n$). Consider a direct product

$$[0,1) \times Y. \tag{1}$$

The coordinates on this product will be denoted by (x, y) , where $x \in [0,1)$, and y are (local) coordinates on Y. We define the set of *operators of Borel-Fuchs type of degree* k as operators on the product (1) with special degeneration at $x = 0$. Namely, the general form of a Borel-Fuchs operator of order k is

$$\hat{H} = H\left(x, y, ix^{k+1}\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right),$$

where $H(x, y, p, q)$ are polynomials in (p, q) with smooth in (x, y) coefficients.

The aim of this paper is to investigate *hyperbolic* equations for operators of the above described type. Such equations are given on the product $\mathbf{R}_+ \times [0,1) \times Y$ and have the form

$$\hat{H}u = H\left(t, x, y, -i\frac{\partial}{\partial t}, ix^{k+1}\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right)u = 0,\ t \in \mathbf{R}_+,$$

the symbol (Hamiltonian) $H(t, x, y, E, p, q)$ of the operator $\hat{H}$ being a polynomial in (E, p, q) with smooth coefficients of some order m; we denote by $H_0(t, x, y, E, p, q)$

the homogeneous part of H of (higher) order m. The hyperbolicity of such operators mean that the roots $E = E(t, x, y, p, q)$ of the equation

$$H_0(t, x, y, E, p, q) = 0$$

are real and do not coincide with one another.

To show all the arising effects and to introduce all the necessary technique of investigation, it is sufficient to consider equations of the form

$$\frac{\partial^2 u}{\partial t^2} = -H\left(t, x, y, ix^{k+1}\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right) u, \tag{2}$$

with a second-order positive operator $H\left(t, x, y, ix^{k+1}\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}\right)$ in the right-hand side. The considerations in the general case are quite similar.

Clearly, equation (2) requires initial conditions. We choose these initial conditions in the form

$$u(t, x, y)|_{t=0} = u_0(x, y),$$
$$\left.\frac{\partial u(t, x, y)}{\partial t}\right|_{t=0} = u_1(x, y)$$

with $u_j(x, y)$ being *WKB-expansions of exponential degree* k at $x = 0$:

$$u_j(x, y) = \sum_m \exp\left\{i\frac{S_m^0(y)}{x^k}\right\}\sum_{l=0}^{\infty} x^l a_{mjl}^0(y),\ j = 0, 1 \tag{3}$$

for $k > 0$, and

$$u_j(x, y) = \sum_m x^{iS_m^0(y)}\sum_{l=0}^{\infty} a_{mjl}^0(y)\ln^l x,\ j = 0, 1 \tag{4}$$

for $k = 0$. Here, the outer sum on the right in (3) and (4) is supposed to be finite.

Due to the linearity of the problem, we can consider initial data with only one exponential in expansions (3) and (4). As a result, we arrive at the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -H\left(t, x, y, ix^{k+1}\dfrac{\partial}{\partial x}, -i\dfrac{\partial}{\partial y}\right) u, \\ u(t, x, y)|_{t=0} = u_0(x, y) = \exp\left\{i\dfrac{S^0(y)}{x^k}\right\}\sum_{l=0}^{\infty} x^l a_{0l}^0(y), \\ \left.-ix^k\dfrac{\partial u(t, x, y)}{\partial t}\right|_{t=0} = u_1(x, y) = \exp\left\{i\dfrac{S^0(y)}{x^k}\right\}\sum_{l=0}^{\infty} x^l a_{1l}^0(y) \end{cases} \tag{5}$$

(for simplicity, in the following two sections we consider the case $k > 0$, all modification needed for the case $k = 0$ are quite evident). In the last section, we consider the problem from the viewpoint of the general theory of differential equations in abstract algebras. This allows us to obtain different types of asymptotic expansions from the geometrical viewpoint.

1 Solution of the problem "in small"

In this section, we shall construct the solution to problem (5) "in small", that is, for sufficiently small values of t. We search for solutions to (5) in the form

$$u(t,x,y) = \exp\left\{i\frac{S(t,y)}{x^k}\right\}\sum_{l=0}^{\infty} x^l a_l(t,y).$$

1. To begin with, let us construct asymptotic solutions to the equation involved into problem (5); the initial data will be taken into account later.

Differentiating the latter expression with respect to t, r, and y, we obtain

$$ix^{k+1}\frac{\partial u}{\partial x}(t,x,y) = \exp\left\{i\frac{S(t,y)}{x^k}\right\}\left[kS(t,y) + ix^{k+1}\frac{\partial}{\partial x}\right]\sum_{l=0}^{\infty} x^l a_l(t,y),$$

$$-i\frac{\partial u}{\partial t}(t,x,y) = \exp\left\{i\frac{S(t,y)}{x^k}\right\}\left[x^{-k}\frac{\partial S(t,y)}{\partial t} - i\frac{\partial}{\partial t}\right]\sum_{l=0}^{\infty} x^l a_l(t,y),$$

and

$$-i\frac{\partial u}{\partial y}(t,x,y) = \exp\left\{i\frac{S(t,y)}{x^k}\right\}\left[x^{-k}\frac{\partial S(t,y)}{\partial y} - i\frac{\partial}{\partial y}\right]\sum_{l=0}^{\infty} x^l a_l(t,y).$$

Now, expanding the operator involved into problem (5) in powers of x, substituting the latter relations into the obtained equation, and cancelling out the exponential, we obtain the following equation for amplitude functions $a_l(t,y)$, $j = 0, 1, \ldots$.

$$\left[\left(x^{-k}\frac{\partial S}{\partial t} - i\frac{\partial}{\partial t}\right)^2 - \sum_{j=0}^{\infty} x^j H_j\left(t, y, kS + ix^{k+1}\frac{\partial}{\partial x}, x^{-k}\frac{\partial S}{\partial y} - i\frac{\partial}{\partial y}\right)\right]$$
$$\times\sum_{l=0}^{\infty} x^l a_l(t,y) = 0, \tag{6}$$

where the functions $H_j(t,y,p,q)$ defined by the relation

$$H(t,x,y,p,q)=\sum_{j=0}^{\infty}x^jH(t,y,p,q)$$

are polynomials of the second order in (p,q) with smooth in (t,y) coefficients.

The further computations is quite different for the cases $k=1$ and $k>1$. Let us consider first the more simple case $k=1$.

1.1 Degeneration of the first degree

In this case, denoting

$$H_j(t,y,p,q)=H_j^{(0)}(t,y)p^2+\left(\sum_{l=1}^{n}H_j^{(l)}(t,y)q_l\right)p+\sum_{l,m=1}^{n}H_j^{(lm)}(t,y)q_lq_m, \qquad (7)$$

we rewrite equation (6) in the form

$$\left[\left[x^{-2}\left(\frac{\partial S}{\partial t}\right)^2-2ix^{-1}\frac{\partial S}{\partial t}\frac{\partial}{\partial t}-ix^{-1}\frac{\partial^2 S}{\partial t^2}-\frac{\partial^2}{\partial t^2}\right]-\sum_{j=0}^{\infty}x^j\left\{H_j^{(0)}(t,y)\right.\right.$$
$$\times\left[S^2+2ix^2S\frac{\partial}{\partial x}-\left(x^2\frac{\partial}{\partial x}\right)^2\right]+\sum_{l=1}^{n}H_j^{(l)}(t,y)\left[x^{-1}S\frac{\partial S}{\partial y^l}-iS\frac{\partial}{\partial y^l}\right.$$
$$\left.-i\frac{\partial S}{\partial y^l}+ix\frac{\partial S}{\partial y^l}\frac{\partial}{\partial x}+x^2\frac{\partial^2}{\partial x\partial y^l}\right]+\sum_{l,m=1}^{n}H_j^{(lm)}(t,y)\left[x^{-2}\frac{\partial S}{\partial y^l}\frac{\partial S}{\partial y^m}\right.$$
$$\left.\left.\left.-2ix^{-1}\frac{\partial S}{\partial y^l}\frac{\partial}{\partial y^m}-ix^{-1}\frac{\partial^2 S}{\partial y^l\partial y^m}-\frac{\partial^2}{\partial y^l\partial y^m}\right]\right\}\right]\sum_{l=0}^{\infty}x^la_l(t,y)=0.$$

Equating terms with one and the same power of x, we arrive at the following recurrent system:

$$\left(\frac{\partial S}{\partial t}\right)^2-H_0\left(t,y,0,\frac{\partial S}{\partial y}\right)=0 \qquad (8)$$

(this is the *Hamilton-Jacobi equation*; this equation will be used for determining the function $S=S(t,y)$),

$$\left[2\frac{\partial S}{\partial t}\frac{\partial}{\partial t}-\frac{\partial H_0}{\partial q_l}\left(t,y,0,\frac{\partial S}{\partial y}\right)\frac{\partial}{\partial y^l}+\frac{\partial^2 S}{\partial t^2}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q_l\partial q_m}\left(t,y,0,\frac{\partial S}{\partial y}\right)\frac{\partial^2 S}{\partial y_l\partial y_m}\right.$$
$$\left.-iH_1\left(t,y,0,\frac{\partial S}{\partial y}\right)+iS\frac{\partial H_0}{\partial p}\left(t,y,0,\frac{\partial S}{\partial y}\right)\right]a_0(t,y)=0$$

(the *transport equation*[1] with respect to the unknown function $a_0(t,y)$), and the sequence of *higher transport equations* having the form

$$\left[2\frac{\partial S}{\partial t}\frac{\partial}{\partial t}-\frac{\partial H_0}{\partial q_l}\left(t,y,0,\frac{\partial S}{\partial y}\right)\frac{\partial}{\partial y^l}+\frac{\partial^2 S}{\partial t^2}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q_l\partial q_m}\left(t,y,0,\frac{\partial S}{\partial y}\right)\frac{\partial^2 S}{\partial y_l\partial y_m}\right.$$
$$\left.-iH_1\left(t,y,0,\frac{\partial S}{\partial y}\right)+iS\frac{\partial H_0}{\partial p}\left(t,y,0,\frac{\partial S}{\partial y}\right)\right]a_l(t,y)=\mathcal{F}\left[a_0,\ldots,a_{l-1}\right]$$

for $l=1,2,\ldots$, where $\mathcal{F}\left[a_0,\ldots,a_{l-1}\right]$ are linear combinations of the functions $a_0,\ldots,a_{l-1}$ and their derivatives with smooth coefficients; the form of the latter expressions is not essential for us in future.

We emphasize that the transport operator

$$\begin{aligned}\hat{P} &= 2\frac{\partial S}{\partial t}\frac{\partial}{\partial t}-\frac{\partial H_0}{\partial q_l}\left(t,y,0,\frac{\partial S}{\partial y}\right)\frac{\partial}{\partial y^l}+\frac{\partial^2 S}{\partial t^2}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q_l\partial q_m}\left(t,y,0,\frac{\partial S}{\partial y}\right)\\ &\quad\times\frac{\partial^2 S}{\partial y_l\partial y_m}-iH_1\left(t,y,0,\frac{\partial S}{\partial y}\right)+iS\frac{\partial H_0}{\partial p}\left(t,y,0,\frac{\partial S}{\partial y}\right)\end{aligned}$$

involved into the transport equations, has the additional term

$$iS\frac{\partial H_0}{\partial p}\left(t,y,0,\frac{\partial S}{\partial y}\right)$$

compared with that in the case of classical WKB-method (see, for example, [1], [2]). This term depends on the action function $S(t,y)$ itself, not on derivatives of this function. The latter remark shows that all the above equations must be considered on a *Legendre* manifold rather than on a Lagrangian one.

2. Let us now interpret the Hamilton-Jacobi equation and the transport equations as equations on some Legendre manifold. Taking into account the initial data of problem (5), we arrive at the following Cauchy problem for Hamilton-Jacobi equation (8):

$$\begin{aligned}&\left(\frac{\partial S}{\partial t}\right)^2-H_0\left(t,y,0,\frac{\partial S}{\partial y}\right)=0,\\ &S|_{t=0}=S^0(y).\end{aligned}\tag{9}$$

This problem can be interpreted in the following way. Consider the contact space

$$C(\mathbf{R}_t\times Y)=\mathbf{R}_s\times T_0^*(\mathbf{R}_t\times Y)\tag{10}$$

[1]Here and below we use the so-called summation convention according to which the summation is performed in each term of the expression over pairs of identical indices if one of them is the upper and another is lower.

with local coordinates (s, t, y, E, q), where, as earlier, y are local coordinates on Y, and q are corresponding impulses. The structure form defining the contact structure in (10) is locally given by

$$\alpha = ds - Edt - qdy;$$

one can easily see that this form does not depend on the choice of local coordinates in Y.

The function $S^0(y)$ determines a submanifold $\mathcal{L}^0$ in $C(\mathbf{R}_t \times Y)$ by the relations

$$\begin{cases} s = S(y), \\ t = 0, \\ q = \dfrac{\partial S(y)}{\partial y}, \\ E^2 - H_0(0, y, 0, q) = 0. \end{cases}$$

For this manifold to be regular, we require that $\partial S/\partial y \neq 0$, and, hence,

$$H_0\left(t, y, 0, \frac{\partial S(y)}{\partial y}\right) \neq 0$$

at any point of the manifold Y. It is easy to see that in this case the manifold $\mathcal{L}^0$ splits into a disjoint union of the two manifolds $\mathcal{L}^0_+$ and $\mathcal{L}^0_-$ corresponding to the two possible choices of the sign in the expression

$$E = \pm\sqrt{H_0(0, y, 0, q)}.$$

Later on, the Hamilton-Jacobi equation shows that the Legendre manifolds $\mathcal{L}_\pm$ determined by the functions $S_\pm(t, y)$ corresponding to the two possible signs in the determination of E, that is,

$$\begin{cases} s = S_\pm(y, t), \\ E = \dfrac{\partial S_\pm(y, t)}{\partial t}, \\ q = \dfrac{\partial S_\pm(y, t)}{\partial y}, \end{cases}$$

must lie on the zero levels of the Hamilton functions

$$\mathcal{H}_\pm(t, y, E, q) = E \pm \sqrt{H_0(t, y, 0, q)}, \tag{11}$$

respectively. Similar to the symplectic case, the latter condition is equivalent to the fact that the manifolds $\mathcal{L}_\pm$ are invariant with respect to the contact vector fields

$$\begin{aligned} X_{\mathcal{H}_\pm} &= \frac{\partial \mathcal{H}_\pm}{\partial E}\frac{\partial}{\partial t} - \frac{\partial \mathcal{H}_\pm}{\partial t}\frac{\partial}{\partial E} + \frac{\partial \mathcal{H}_\pm}{\partial q}\frac{\partial}{\partial y} - \frac{\partial \mathcal{H}_\pm}{\partial y}\frac{\partial}{\partial q} \\ &\quad + \left(E\frac{\partial \mathcal{H}_\pm}{\partial E} + q\frac{\partial \mathcal{H}_\pm}{\partial q}\right)\frac{\partial}{\partial s}, \end{aligned}$$

or, in view of the explicit expressions (11) for functions $\mathcal{H}_\pm$,

$$\begin{aligned} X_{\mathcal{H}_\pm} &= \frac{\partial}{\partial t} \mp \frac{1}{2\sqrt{H_0}} \frac{\partial H_0}{\partial t} \frac{\partial}{\partial E} \pm \frac{1}{2\sqrt{H_0}} \frac{\partial H_0}{\partial q} \frac{\partial}{\partial y} \mp \frac{1}{2\sqrt{H_0}} \frac{\partial H_0}{\partial y} \frac{\partial}{\partial q} \\ &+ \left(E \pm q \frac{1}{2\sqrt{H_0}} \frac{\partial H_0}{\partial q} \right) \frac{\partial}{\partial s}, \end{aligned}$$

where the function H_0 and its derivatives are computed at the same point as in (11).

Since these fields do not vanish and are transversal to the manifolds $\mathcal{L}^0_\pm$, we can define the manifolds $\mathcal{L}_\pm$ as the phase flows of $\mathcal{L}^0_\pm$ along $X_{\mathcal{H}_\pm}$, respectively. At points where these manifolds are isomorphically projected on the space with coordinates (t, y), they determine solutions $S_\pm(t, y)$ to problem (9). We remark that this will be true for all points such that $|t| < \varepsilon$ with sufficiently small ε. So, the Legendre manifolds $\mathcal{L}_\pm$ constructed above, give us the required geometric interpretation of solutions to problem (9) for the Hamilton-Jacobi equation and, what is more, this description is *global* in contrast to that given by the function $S(t, y)$ itself.

Let us now turn our mind to the interpretation of the transport equations. Clearly, to the above two actions $S_\pm(y, t)$ there correspond two sequences of the amplitude functions $a_l^\pm(t, y)$. Dividing the transport equations for these functions by $2\partial S_\pm/\partial t$ and taking into account the Hamilton-Jacobi equation, we arrive at the following equations with respect to $a_0^\pm(t, y)$:

$$\begin{aligned} &\left[\frac{\partial}{\partial t} \pm \frac{1}{2\sqrt{H_0}} \frac{\partial H_0}{\partial q} \frac{\partial}{\partial y} \mp \frac{1}{2\sqrt{H_0}} \frac{\partial^2 S}{\partial t^2} \pm \frac{1}{4\sqrt{H_0}} \frac{\partial^2 H_0}{\partial q_l \partial q_m} \frac{\partial^2 S}{\partial y_l \partial y_m} \right. \\ (3mm) \qquad &\left. \pm \frac{i}{2\sqrt{H_0}} H_1 \mp \frac{iS}{2\sqrt{H_0}} \frac{\partial H_0}{\partial p} \right] a_0^\pm(t, y) = 0. \qquad (12) \end{aligned}$$

One can easily see that the first two terms on the left in the latter equation are the coordinate expression (in the coordinates (t, y) of a nonsingular charts of $\mathcal{L}_\pm$) of the vector field $X_{\mathcal{H}_\pm}$ on the manifold $\mathcal{L}_\pm$. Later on, let us define the measure $\mu_\pm$ on $\mathcal{L}_\pm$ by the following two requirements:

i) The measure $\mu_\pm$ is invariant with respect to the vector field $X_{\mathcal{H}_\pm}$.

ii) The density $m_\pm$ of the measure $\mu_\pm$ in the coordinates[2] (t, y) equals to the density of the measure $dy \wedge dt$ at $t = 0$.

[2]As it was remarked before, a neighborhood of $t = 0$ lies as a whole in a nonsingular chart of the manifold $\mathcal{L}_\pm$.

It is evident that $\mu_\pm$ is uniquely determined as a measure on $\mathcal{L}_\pm$ by the requirements *i)* and *ii)*.

Now we perform a change of the unknown in equation (12) given by

$$a_0^\pm(t,y) = \varphi_0^\pm(t,y)\sqrt{m_\pm(t,y)},$$

where the branch of the square root is chosen equal to 1 at $t = 0$. The computations similar to that in the symplectic case (see, e. g. [2]) lead us to the equation with respect of the new unknown $\varphi_0^\pm(t,y)$ of the form

$$\left[X_{\mathcal{H}_\pm} \pm \left(\frac{1}{4\sqrt{H_0}}\frac{\partial^2 H_0}{\partial q_l \partial y^l} + \frac{i}{2\sqrt{H_0}}H_1 - \frac{is}{2\sqrt{H_0}}\frac{\partial H_0}{\partial p}\right)\Bigg|_{\mathcal{L}_\pm}\right]\varphi_0^\pm(t,y) = 0.$$

So, the transport operator is

$$\mathcal{P}_\pm = X_{\mathcal{H}_\pm} \pm \left(\frac{1}{4\sqrt{H_0}}\frac{\partial^2 H_0}{\partial q_l \partial y^l} + \frac{i}{2\sqrt{H_0}}H_1 - \frac{is}{2\sqrt{H_0}}\frac{\partial H_0}{\partial p}\right)\Bigg|_{\mathcal{L}_\pm},$$

and it is clear that this operator is defined on the *Legendre* (not Lagrangian) manifold.

The rest transport equations have the form

$$\mathcal{P}_\pm \varphi_l^\pm = \mathcal{F}^\pm\left[\varphi_0^\pm, \ldots, \varphi_{l-1}^\pm\right]$$

with the same transport operator $\mathcal{P}_\pm$ and some expressions $\mathcal{F}^\pm\left[\varphi_0^\pm, \ldots, \varphi_{l-1}^\pm\right]$ of the above described type.

3. Let us consider the initial conditions for the functions $\varphi_0^\pm$. The solution to problem (5) is searched in the form

$$\begin{aligned} u(t,x,y) &= \exp\left\{i\frac{S^+(t,y)}{x}\right\}\sqrt{m_+(t,y)}\sum_{l=0}^{\infty} x^l \varphi_l^+(t,y) \\ &\quad + \exp\left\{i\frac{S^-(t,y)}{x}\right\}\sqrt{m_-(t,y)}\sum_{l=0}^{\infty} x^l \varphi_l^-(t,y). \end{aligned}$$

Taking into account the initial conditions of problem (5), we arrive at the following relations

$$\varphi_0^+(0,y) + \varphi_0^-(0,y) = a_{00}^0(y),$$

$$\sqrt{H_0\left(0,y,0,\frac{\partial S^0}{\partial y}(y)\right)}\left[\varphi_0^+(0,y) - \varphi_0^-(0,y)\right] = a_{10}^0(y)$$

for functions $\varphi_0^{\pm}$ at $t=0$. Since, by assumption, $H_0\left(0,y,0,\frac{\partial S^0}{\partial y}(y)\right)\neq 0$, the latter system is solvable in smooth functions with respect to $\varphi_0^{\pm}\big|_{t=0}$, which supplies us with initial data for the first transport equation. The initial data for all the rest transport equations can be obtained in the quite similar manner. Now, solving all transport equations, one can obtain the solution to problem (5) up to any power of x.

1.2 Degeneration of higher degree

1. Since all the effects can be seen for the operators with degeneration of the second degree, we consider here the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = -H\left(t,x,y,ix^3\dfrac{\partial}{\partial x},-i\dfrac{\partial}{\partial y}\right)u, \\ u\left(t,x,y\right)\big|_{t=0} = u_0\left(x,y\right) = \exp\left\{i\dfrac{S^0\left(y\right)}{x^2}\right\}\sum\limits_{l=0}^{\infty} x^l a_{0l}^0\left(y\right), \\ -ix^2\dfrac{\partial u\left(t,x,y\right)}{\partial t}\bigg|_{t=0} = u_1\left(x,y\right) = \exp\left\{i\dfrac{S^0\left(y\right)}{x^2}\right\}\sum\limits_{l=0}^{\infty} x^l a_{1l}^0\left(y\right). \end{cases} \tag{13}$$

One can check that the asymptotic solutions to the latter problem of the form

$$u\left(t,x,y\right) = \exp\left\{i\frac{S\left(t,y\right)}{x^2}\right\}\sum_{l=0}^{\infty} x^l a_l\left(t,y\right)$$

does not exist, at least in the case when the dependence of the symbol

$$H\left(t,x,y,p,q\right)$$

in x is a generic one. To improve the situation it occurs to be sufficient (as we shall see below) to allow the regular dependence of functions S and a_l on the variable x.

However, there arises an ambiguity in expansions in powers of x of the type

$$\exp\left\{i\frac{S(t,r,y)}{x^2}\right\}\sum_{l=0}^{\infty} x^l a_l\left(t,x,y\right)$$

(for example, one can reexpand the coefficients $a_l(t,x,y)$ and regular terms in the phase function $r^{-2}S(t,x,y)$ again in the Taylor series thus obtaining another expansion of the same type). To fix this ambiguity, we remark that, in essence, we search the solution to problem (13) in the form

$$u\left(t,x,y\right) = U\left(t,x_1,x_2,y\right)$$

with $U(t, x_1, x_2, y)$ satisfying the problem

$$\begin{cases} \frac{\partial^2 U}{\partial t^2} = -H\left(t, x, y, ix_1^3\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x}\right), -i\frac{\partial}{\partial y}\right) U, \\ U(t, x_1, x_2, y)\big|_{t=0} = U_0(x_1, x_2, y) = \exp\left\{i\frac{S^0(y)}{x_1^2}\right\} \sum\limits_{l=0}^{\infty} x_1^l a_{0l}^0(y), \\ -ix^2 \frac{\partial U(t,x_1,x_2,y)}{\partial t}\Big|_{t=0} = U_1(x_1, x_2, y) = \exp\left\{i\frac{S^0(y)}{x_1^2}\right\} \sum\limits_{l=0}^{\infty} x_1^l a_{1l}^0(y), \end{cases}$$

searching solutions as expansions in the variable r_1. The solution to the latter problem must be searched in the form

$$U(t, x_1, x_2, y) = \exp\left\{i\frac{S(t, x, y)}{x_1^2}\right\} \sum_{l=0}^{\infty} x_1^l a_l(t, x, y),$$

We shall not carry out this substitution in the explicit manner, but one can have in mind the described representation to understand the separation of orders which will be used below.

2. So, let us search for solutions to problem (13) in the form

$$u(t, x, y) = \exp\left\{i\frac{S(t, x, y)}{x^2}\right\} a(t, x, y),$$

where

$$a(t, x, y) = \sum_{l=0}^{\infty} x^l a_l(t, x, y).$$

Clearly, one can generalize the initial data in problem (13) replacing the functions $u_0(x, y)$ and $u_1(x, y)$ by

$$\begin{aligned} u_0(x, y) &= \exp\left\{i\frac{S^0(x, y)}{x^2}\right\} \sum_{l=0}^{\infty} x^l a_{0l}^0(x, y), \\ u_1(x, y) &= \exp\left\{i\frac{S^0(x, y)}{x^2}\right\} \sum_{l=0}^{\infty} x^l a_{1l}^0(x, y), \end{aligned} \tag{14}$$

where the functions $S^0(x, y)$, $a_{0l}^0(x, y)$, and $a_{1l}^0(x, y)$ are smooth in x up to $x = 0$. Differentiating the above expression for $u(t, x, y)$, we obtain the following relations:

$$\begin{aligned} -i\frac{\partial u}{\partial t} &= \exp\left\{i\frac{S(t, x, y)}{x^2}\right\} \left[x^{-2}\frac{\partial S}{\partial t} - i\frac{\partial}{\partial t}\right] \sum_{l=0}^{\infty} x^l a_l(t, x, y), \\ ix^3\frac{\partial u}{\partial x} &= \exp\left\{i\frac{S(t, x, y)}{x^2}\right\} \left[2S - x\frac{\partial S}{\partial x} + ix^3\frac{\partial}{\partial x}\right] \sum_{l=0}^{\infty} x^l a_l(t, x, y), \\ -i\frac{\partial u}{\partial y} &= \exp\left\{i\frac{S(t, x, y)}{x^2}\right\} \left[x^{-2}\frac{\partial S}{\partial y} - i\frac{\partial}{\partial y}\right] \sum_{l=0}^{\infty} x^l a_l(t, x, y). \end{aligned}$$

Substituting the latter relations for derivatives of the solution to the equation from (13), we come to the equation

$$\left[x^{-2}\frac{\partial S}{\partial t} - i\frac{\partial}{\partial t}\right]^2 a = H\left(t, x, y, 2S - x\frac{\partial S}{\partial x} + ix^3\frac{\partial}{\partial x}, x^{-2}\frac{\partial S}{\partial y} - i\frac{\partial}{\partial y}\right) a \qquad (15)$$

for the amplitude function

$$a = a(t, x, y) = \sum_{l=0}^{\infty} x^l a_l(t, x, y).$$

Using the representation of the symbol $H\left(t, x, y, p, q\right)$

$$H\left(t, x, y, p, q\right) = H_0\left(t, x, y\right) p^2 + \left(\sum_{l=1}^{n} H_l\left(t, x, y\right) q_l\right) p + \sum_{l,m=1}^{n} H_{lm}\left(t, x, y\right) q_l q_m$$

similar to (7), we can rewrite equation (15) in the more detailed form

$$\left[x^{-4}\left(\frac{\partial S}{\partial t}\right)^2 - 2ix^{-2}\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - ix^{-2}\frac{\partial^2 S}{\partial t^2} - \frac{\partial^2}{\partial t^2}\right]\sum_{l=0}^{\infty} x^l a_l = \left\{H_0\left[4S^2 - 4xS\frac{\partial S}{\partial x}\right.\right.$$

$$\left. +4ir^3 S\frac{\partial}{\partial x} + x^2\left(\frac{\partial S}{\partial x}\right)^2 - 2ix^4\frac{\partial S}{\partial x}\frac{\partial}{\partial x} + ix^3\frac{\partial S}{\partial x} - ix^4\frac{\partial^2 S}{\partial x^2} - \left(x^3\frac{\partial}{\partial x}\right)^2\right]$$

$$+\sum_{l=1}^{n} H_l\left[2x^{-2}S\frac{\partial S}{\partial y^l} + x^{-1}\frac{\partial S}{\partial x}\frac{\partial S}{\partial y^l} - 2iS\frac{\partial}{\partial y^l} - 2i\frac{\partial S}{\partial y^l} + ix\frac{\partial S}{\partial y^l}\frac{\partial}{\partial x}\right.$$

$$\left. +ix\frac{\partial S}{\partial x}\frac{\partial}{\partial y^l} + ix\frac{\partial^2 S}{\partial x \partial y^l} + x^3\frac{\partial^2}{\partial x \partial y^l}\right] + \sum_{l,m=1}^{n} H_{lm}\left[x^{-4}\frac{\partial S}{\partial y^l}\frac{\partial S}{\partial y^m}\right.$$

$$\left.\left. -ix^{-2}\frac{\partial S}{\partial y^l}\frac{\partial}{\partial y^m} - ix^{-2}\frac{\partial S}{\partial y^m}\frac{\partial}{\partial y^l} - ix^{-2}\frac{\partial^2 S}{\partial y^l \partial y^m} - \frac{\partial^2}{\partial y^l \partial y^m}\right]\right\}\sum_{l=0}^{\infty} x^l a_l.$$

Equating terms with one and the same power of x in the both sides of the latter equality (in this process, we do not take into account the dependence of the functions H_0, H_l, H_{lm}, S, and a_l on this variable), we obtain the following equations:

- The comparison of coefficient of the highest negative power of x (namely, of x^{-4}) gives the equation

$$\left[\left(\frac{\partial S}{\partial t}\right)^2 - H\left(t, x, y, 0, \frac{\partial S}{\partial y}\right)\right] a_0\left(t, x, y\right) = 0.$$

Since we are intended to obtain nontrivial solutions to the equation in question, we can suppose that the function $a_0(t,x,y)$ does not vanish, so the *Hamilton-Jacobi equation*

$$\left(\frac{\partial S}{\partial t}\right)^2 - H\left(t,x,y,0,\frac{\partial S}{\partial y}\right) = 0 \tag{16}$$

must be fulfilled.

- Under the requirement that the function S is a solution to equation (16) (in what follows we suppose that it is so), the coefficient of x^{-3} vanishes automatically.

- Equating the coefficients of x^{-2} leads us to the equality

$$\left[2\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - \sum_{l=0}^{n}\frac{\partial H}{\partial q_l}\frac{\partial}{\partial y^l} + \frac{\partial^2 S}{\partial t^2} - \frac{1}{2}\sum_{l,m=1}^{n}\frac{\partial^2 H}{\partial q_l \partial q_m}\frac{\partial^2 S}{\partial y^l \partial y^m} \right.$$
$$\left. +2iS\frac{\partial H}{\partial p}\right] a_0(t,x,y) = 0 \tag{17}$$

(the function H and its derivatives are computed in the same point as in (16)).

- All the rest equations have the form

$$\hat{P}a_j(t,x,y) = \mathcal{F}[a_0,\ldots,a_{j-1}],\ j \geq 1,$$

where

$$\hat{P} = 2\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - \sum_{l=0}^{n}\frac{\partial H}{\partial q_l}\frac{\partial}{\partial y^l} + \frac{\partial^2 S}{\partial t^2} - \frac{1}{2}\sum_{l,m=1}^{n}\frac{\partial^2 H}{\partial q_l \partial q_m}\frac{\partial^2 S}{\partial y^l \partial y^m} - 2iS\frac{\partial H}{\partial p}$$

is the transport operator involved into equation (17), and $\mathcal{F}[a_0,\ldots,a_{j-1}]$ is a linear combination of functions $a_0,\ldots,a_{j-1}$ and their derivatives with smooth coefficients.

3. In this case, the interpretation of the Hamilton-Jacobi equation and the transport equation requires consideration of a *family* of Legendre manifolds, in contrast to the case considered in the previous subsection. Using the initial data (14) of problem (13), we arrive at the following Cauchy problem for Hamilton-Jacobi equation (16):

$$\begin{cases} \left(\frac{\partial S}{\partial t}\right)^2 - H\left(t,x,y,0,\frac{\partial S}{\partial y}\right) = 0, \\ S|_{t=0} = S^0(x,y). \end{cases}$$

The function $S^0(x,y)$ determines now a smooth family of manifolds $\mathcal{L}^0(x)$ in $C(\mathbf{R}_t \times Y)$ parameterized by x:

$$\begin{cases} s = S(x,y), \\ t = 0, \\ q = \dfrac{\partial S(x,y)}{\partial y}, \\ E^2 - H(0,x,y,0,q) = 0. \end{cases}$$

Since all our considerations are performed near the point $x = 0$, the requirement $\partial S/\partial y \neq 0$ must be fulfilled at this point. Similar to the previous case, the family $\mathcal{L}^0(x)$ splits into a disjoint union of the two families $\mathcal{L}^0_+(x)$ and $\mathcal{L}^0_-(x)$ corresponding to the choice of the sign in the expression

$$E = \pm\sqrt{H(0,x,y,0,q)}.$$

The two families $\mathcal{L}_\pm(x)$ of Legendre manifold determining the solution to Cauchy problem (1.2) can be obtained as the phase flows of $\mathcal{L}^0_+(x)$ and $\mathcal{L}^0_-(x)$ with respect to the contact fields $X_{\mathcal{H}_\pm}$ corresponding to the Hamilton functions

$$\mathcal{H}_\pm(t,y,E,q) = E \pm \sqrt{H(t,x,y,0,q)} \tag{18}$$

(it would be more precise to speak about the families of Hamilton functions, but we shall not emphasize this fact in what follows for brevity). These vector fields are given by the formula

$$\begin{aligned} X_{\mathcal{H}_\pm} &= \frac{\partial}{\partial t} \mp \frac{1}{2\sqrt{H}}\frac{\partial H}{\partial t}\frac{\partial}{\partial E} \pm \frac{1}{2\sqrt{H}}\frac{\partial H}{\partial q}\frac{\partial}{\partial y} \mp \frac{1}{2\sqrt{H}}\frac{\partial H}{\partial y}\frac{\partial}{\partial q} \\ &\quad + \left(E \pm q\frac{1}{2\sqrt{H}}\frac{\partial H}{\partial q}\right)\frac{\partial}{\partial s}. \end{aligned}$$

At points where these manifolds are isomorphically projected on the space with coordinates (t,y), they determine solutions $S_\pm(t,x,y)$ to problem (1.2). So, the Legendre manifolds $\mathcal{L}_\pm(x)$ give us the required geometric interpretation of solutions to this problem.

Let us now turn our mind to the interpretation of the transport equation. Clearly, to the above two actions $S_\pm(t,x,y)$ there correspond two sequences of the amplitude functions $a^\pm_l(t,x,y)$. Similar to the previous subsection, the first transport equation can be written down in the form

$$\begin{aligned} &\left[\frac{\partial}{\partial t} \pm \frac{1}{2\sqrt{H}}\frac{\partial H}{\partial q}\frac{\partial}{\partial y} \mp \frac{1}{2\sqrt{H}}\frac{\partial^2 S}{\partial t^2} \pm \frac{1}{4\sqrt{H}}\frac{\partial^2 H}{\partial q_l \partial q_m}\frac{\partial^2 S}{\partial y_l \partial y_m}\right. \\ &\left.\mp\frac{iS}{\sqrt{H_0}}\frac{\partial H}{\partial p}\right] a^\pm_0(t,x,y) = 0. \end{aligned} \tag{19}$$

The first two terms on the left in the latter equation are the coordinate expression of the vector field $X_{\mathcal{H}_\pm}$ on the manifold $\mathcal{L}_\pm(x)$.

Again, using the invariant measure $\mu_\pm(t,x,y)$ on $\mathcal{L}_\pm(x)$, which is constructed literally in the same way as in the previous subsection, we perform a change of the unknown in equation (19) given by

$$a_0^\pm(t,x,y) = \varphi_0^\pm(t,x,y)\sqrt{m_\pm(t,x,y)},$$

thus reducing the transport equation to the form

$$\left[X_{\mathcal{H}_\pm} \pm \left(\frac{1}{2}\frac{\partial^2 H}{\partial q_l \partial y^l} + \frac{is}{2\sqrt{H_0}}\frac{\partial H_0}{\partial p}\right)\Big|_{\mathcal{L}_\pm}\right]\varphi_0^\pm(t,x,y) = 0.$$

So, the transport operator in this case equals

$$\mathcal{P}_\pm = X_{\mathcal{H}_\pm} \pm \left(\frac{1}{2}\frac{\partial^2 H_0}{\partial q_l \partial y^l} - \frac{is}{2\sqrt{H_0}}\frac{\partial H_0}{\partial p}\right)\Big|_{\mathcal{L}_\pm}.$$

The rest transport equations have the form

$$\mathcal{P}_\pm \varphi_l^\pm = \mathcal{F}^\pm\left[\varphi_0^\pm, \ldots, \varphi_{l-1}^\pm\right]$$

with the same transport operator $\mathcal{P}_\pm$ and some expressions $\mathcal{F}^\pm\left[\varphi_0^\pm, \ldots, \varphi_{l-1}^\pm\right]$ of the above described type.

4. Let us consider the initial conditions for the functions $\varphi_0^\pm$. The solution to problem (13) (with the modified initial conditions (14)) is searched in the form

$$\begin{aligned} u(t,x,y) &= \exp\left\{i\frac{S^+(t,x,y)}{x^2}\right\}\sqrt{m_+(t,x,y)}\sum_{l=0}^\infty x^l \varphi_l^+(t,x,y) \\ &\quad + \exp\left\{i\frac{S^-(t,x,y)}{x^2}\right\}\sqrt{m_-(t,x,y)}\sum_{l=0}^\infty x^l \varphi_l^-(t,x,y), \end{aligned} \tag{20}$$

and, hence, the initial conditions for the functions $\varphi_l^+(t,x,y)$ and $\varphi_l^-(t,x,y)$ are given by

$$\begin{aligned} &\varphi_0^+(0,x,y) + \varphi_0^-(0,x,y) = a_{00}^0(x,y), \\ &\sqrt{H\left(0,x,y,0,\frac{\partial S^0}{\partial y}(y)\right)}\left[\varphi_0^+(0,x,y) - \varphi_0^-(0,x,y)\right] = a_{10}^0(x,y). \end{aligned} \tag{21}$$

Similar to the previous subsection, solving all transport equations, one can obtain the solution to problem (5) up to any power of x.

5. Now we are able to describe the main difference between the cases $k = 1$ and $k > 1$. Expanding the functions $S^{\pm}$ and $a_l^{\pm} = \sqrt{m_{\pm}}\varphi_l^{\pm}$ in powers of x, we obtain the asymptotic solution $u\,(t, x, y)$ in the form

$$\begin{aligned} u\,(t,x,y) &= \exp\left\{ i\left[\frac{S^+_{-2}\,(t,y)}{x^2} + \frac{S^+_{-1}\,(t,y)}{x} + S^+_0\,(t,y) + \ldots \right] \right\} \sum_{l=0}^{\infty} x^l b_l^+\,(t,y) \\ &\quad + \exp\left\{ i\left[\frac{S^-_{-2}\,(t,y)}{x^2} + \frac{S^-_{-1}\,(t,y)}{x} + S^-_0\,(t,y) + \ldots \right] \right\} \sum_{l=0}^{\infty} x^l b_l^-\,(t,y) \end{aligned}$$

with some smooth functions $b_l^{\pm}\,(t, y)$. Later on, expanding the regular parts of the exponentials in the latter expression again in powers of x, we finally arrive at the expansion

$$\begin{aligned} u\,(t,x,y) &= \exp\left\{ i\left[\frac{S^+_{-2}\,(t,y)}{x^2} + \frac{S^+_{-1}\,(t,y)}{x} \right] \right\} \sum_{l=0}^{\infty} x^l c_l^+\,(t,y) \\ &\quad + \exp\left\{ i\left[\frac{S^-_{-2}\,(t,y)}{x^2} + \frac{S^-_{-1}\,(t,y)}{x} \right] \right\} \sum_{l=0}^{\infty} x^l c_l^-\,(t,y) \end{aligned}$$

with smooth coefficients $c_l^{\pm}\,(t, y)$. The latter expression does not contain an ambiguity in determining the coefficients $c_l^{\pm}\,(t, y)$. However, in contrast to the expansion obtained in the previous subsection for the case $k = 1$, it contains *two* irregular terms $x^{-2}S^{\pm}_{-2}\,(t, y)$ and $x^{-1}S^{\pm}_{-1}\,(t, y)$ in both exponentials. It is clear that the number of the irregular terms in the exponentials will increase for increasing k, so that the expansion for an arbitrary values of k will have the form

$$\begin{aligned} u\,(t,x,y) &= \exp\left\{ i\left[\sum_{j=1}^{k} \frac{S^+_{-j}\,(t,y)}{r^j} \right] \right\} \sum_{l=0}^{\infty} x^l c_l^+\,(t,y) \\ &\quad + \exp\left\{ i\left[\sum_{j=1}^{k} \frac{S^-_{-j}\,(t,y)}{x^j} \right] \right\} \sum_{l=0}^{\infty} x^l c_l^-\,(t,y)\,. \end{aligned}$$

2 Solution of the problem “in large”

The aim of this section is to construct a solution to Cauchy problem “in large”. This means that we must construct a solution not only in a neighborhood of any nonsingular point of the corresponding Legendre manifold, but also in a neighborhood of all its singular points. The construction of asymptotic expansions will be carried out by Maslov's canonical operator method. Again, we shall consider first the simplest case of a singular point of degree 1.

2.1 Degeneration of the first degree

To construct asymptotic solutions to problem (5) near singular points of the corresponding Legendre manifold, we shall use the mixed coordinate-impulse representation. We recall ([2]) that in a neighborhood of any point of a Legendre manifold $\mathcal{L}$ there exists a chart of the special form (the so-called *canonical chart*). In such a chart the coordinates have the form

$$\begin{aligned} &\left(y^{i_1}, \ldots y^{i_l}, q_{j_1}, \ldots, q_{j_m}\right), \\ &\{i_1, \ldots, i_l\} \cup \{j_1, \ldots, j_m\} = \{1, \ldots, n\}, \\ &\{i_1, \ldots, i_l\} \cap \{j_1, \ldots, j_m\} = \emptyset. \end{aligned}$$

To shorten the notation, we shall denote by I the set of indices $I = \{i_1, \ldots, i_l\}$; then the set $\{j_1, \ldots, j_m\}$ is a complement of the set I in $\{1, \ldots, n\}$, and we denote it by $\overline{I}$. So, the coordinates in the canonical chart will be denoted by $\left(y^I, q_{\overline{I}}\right)$, and the chart itself will be denoted by U_I.

To write down a WKB-element in a singular chart (that is, in the chart where $\overline{I} \neq \emptyset$) we shall use the so-called x^{-1}-*Fourier transform* (see, e. g. [2]) in the variables $y^{\overline{I}}$. This transform is given by[3]

$$\tilde{f}\left(x, y^I, q_{\overline{I}}\right) = F^{1/x}_{y^{\overline{I}} \to q_{\overline{I}}}[f] = \left(-\frac{i}{2\pi x}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{-\frac{i}{x} y^{\overline{I}} q_{\overline{I}}\right\} f\left(x, y^I, y^{\overline{I}}\right) dy^{\overline{I}}, \quad (22)$$

and its inverse is

$$f\left(x, y^I, y^{\overline{I}}\right) = F^{1/x}_{q_{\overline{I}} \to y^{\overline{I}}}\left[\tilde{f}\right] = \left(\frac{i}{2\pi x}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{\frac{i}{x} y^{\overline{I}} q_{\overline{I}}\right\} \tilde{f}\left(x, y^I, q_{\overline{I}}\right) dq_{\overline{I}}. \quad (23)$$

The properties of the x^{-1}-Fourier transform are known rather well, and we need only to derive the commutation formula with the operator $ix^2\partial/\partial x$.

[3]Here and below we again use the natural summation rules. For example, the expression $y^{\overline{I}} q_{\overline{I}}$ must be read as

$$y^{\overline{I}} q_{\overline{I}} = \sum_{j \in \overline{I}} y^j q_j.$$

To do this, we can simply differentiate formula (23) by the operator $ix^2\partial/\partial x$. We obtain

$$ix^2\frac{\partial}{\partial x}f\left(x,y^I,y^{\overline{I}}\right) = \left(\frac{i}{2\pi x}\right)^{\frac{|\overline{I}|}{2}}\int \exp\left\{\frac{i}{x}y^{\overline{I}}q_{\overline{I}}\right\}\left[-\frac{i\left|\overline{I}\right|}{2}x + y^{\overline{I}}q_{\overline{I}} + ix^2\frac{\partial}{\partial x}\right]\tilde{f}\left(x,y^I,q_{\overline{I}}\right)\,dq_{\overline{I}}.$$

To get rid of the dependence on $y^{\overline{I}}$ in the function in the integrand on the right in the latter formula, we use the integration by parts:

$$\begin{aligned}
&\int \exp\left\{\frac{i}{x}y^{\overline{I}}q_{\overline{I}}\right\}y^{\overline{I}}q_{\overline{I}}\tilde{f}\left(x,y^I,q_{\overline{I}}\right)\,dq_{\overline{I}}\\
&= \int \exp\left\{\frac{i}{x}y^{\overline{I}}q_{\overline{I}}\right\}\left(ix\frac{\partial}{\partial q_{\overline{I}}}\right)q_{\overline{I}}\tilde{f}\left(x,y^I,q_{\overline{I}}\right)\,dq_{\overline{I}}\\
&= \int \exp\left\{\frac{i}{x}y^{\overline{I}}q_{\overline{I}}\right\}\left(ix\left|\overline{I}\right| + ixq_{\overline{I}}\frac{\partial}{\partial q_{\overline{I}}}\right)\tilde{f}\left(x,y^I,q_{\overline{I}}\right)\,dq_{\overline{I}}.
\end{aligned}$$

So, finally we obtain

$$\begin{aligned}
&ix^2\frac{\partial}{\partial x}f\left(x,y^I,y^{\overline{I}}\right) =\\
&= \left(\frac{i}{2\pi x}\right)^{\frac{|\overline{I}|}{2}}\int \exp\left\{\frac{i}{x}y^{\overline{I}}q_{\overline{I}}\right\}\left[\frac{i\left|\overline{I}\right|}{2}x + ixq_{\overline{I}}\frac{\partial}{\partial q_{\overline{I}}} + ix^2\frac{\partial}{\partial x}\right]\tilde{f}\left(x,y^I,q_{\overline{I}}\right)\,dq_{\overline{I}}.
\end{aligned}$$

Later on, the operator $-ix\partial/\partial y^{\overline{I}}$ is taken by transforms (22), (23) to the multiplication by $q_{\overline{I}}$, and the multiplication by $y^{\overline{I}}$ is taken to the operator $ix\partial/\partial q_{\overline{I}}$. So, applying these transforms to the equation involved into problem (5) (for $k=1$), we arrive at the equation

$$\frac{\partial^2\tilde{u}}{\partial t^2} = -H\left(t,x,y^I,ix\frac{\partial}{\partial q_{\overline{I}}},\frac{i\left|\overline{I}\right|}{2}x + ixq_{\overline{I}}\frac{\partial}{\partial q_{\overline{I}}} + ix^2\frac{\partial}{\partial x},-i\frac{\partial}{\partial y^I},q_{\overline{I}}\right)\tilde{u}, \tag{24}$$

where $\tilde{u} = \tilde{u}\left(t,x,y^I,q_{\overline{I}}\right)$ is the x^{-1}-Fourier transform of the function $u\left(t,x,y\right)$ with respect to the variables $y^{\overline{I}}$.

We remark that in general the operator on the right in the latter formula is a pseudodifferential one. We shall not present here the exact description of such operators, as well as of the spaces where they act; the reader can find all needed definitions and affirmations in the book cited above.

For equation (24), we shall again search a solution in the WKB-form

$$\tilde{u}\left(t,x,y^I,q_{\overline{I}}\right)=\exp\left\{\frac{i}{x}S_I\left(t,y^I,q_{\overline{I}}\right)\right\}\sum_{l=0}^{\infty}x^l a_{Il}\left(t,y^I,q_{\overline{I}}\right).$$

Substituting this expansion into equation (24) and equating terms with one and the same powers of x in both its sides, we arrive at the following relations:

- *Hamilton-Jacobi equation*

$$\left(\frac{\partial S_I}{\partial t}\right)^2-H_0\left(t,y^I,-\frac{\partial S_I}{\partial q_{\overline{I}}},0,\frac{\partial S_I}{\partial y^I},q_{\overline{I}}\right)=0; \tag{25}$$

- *First transport equation*

$$\left[2\frac{\partial S_I}{\partial t}-\frac{\partial H_0}{\partial q_I}\frac{\partial}{\partial y^I}+\frac{\partial H_0}{\partial y^{\overline{I}}}\frac{\partial}{\partial q_{\overline{I}}}+\left(\frac{\partial^2 S_I}{\partial t^2}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q_I\partial q_I}\frac{\partial^2 S_I}{\partial y^I\partial y^I}\right.\right.$$
$$\left.+\frac{\partial^2 H_0}{\partial q_I\partial y^{\overline{I}}}\frac{\partial^2 S_I}{\partial y^I\partial q_{\overline{I}}}-\frac{\partial^2 H_0}{\partial y^{\overline{I}}\partial y^{\overline{I}}}\frac{\partial^2 S_I}{\partial q_{\overline{I}}\partial q_{\overline{I}}}\right)-iH_1+i\left(-q_{\overline{I}}\frac{\partial S_I}{\partial q_{\overline{I}}}+S_I\right)$$
$$\left.\times\frac{\partial H_0}{\partial p}\right]a_{I0}\left(t,y^I,q_{\overline{I}}\right)=0;$$

- *Transport equations of higher order*

$$\hat{P}a_{Ij}\left(t,y^I,q_{\overline{I}}\right)=\mathcal{F}\left[a_{I0},\ldots,a_{Ij-1}\right].$$

Here $H_j\left(t,y,p,q\right)$ are, as above, the Taylor coefficients of the function H in the variable x, the functions H_j and their derivatives in the transport equations are computed at the same point as in the Hamilton-Jacobi equation, the operator $\hat{P}$ is given by the formula

$$\hat{P}=2\frac{\partial S_I}{\partial t}-\frac{\partial H_0}{\partial q_I}\frac{\partial}{\partial y^I}+\frac{\partial H_0}{\partial y^{\overline{I}}}\frac{\partial}{\partial q_{\overline{I}}}+\left(\frac{\partial^2 S_I}{\partial t^2}-\frac{1}{2}\frac{\partial^2 H_0}{\partial q_I\partial q_I}\frac{\partial^2 S_I}{\partial y^I\partial y^I}\right.$$
$$\left.+\frac{\partial^2 H_0}{\partial q_I\partial y^{\overline{I}}}\frac{\partial^2 S_I}{\partial y^I\partial q_{\overline{I}}}-\frac{\partial^2 H_0}{\partial y^{\overline{I}}\partial y^{\overline{I}}}\frac{\partial^2 S_I}{\partial q_{\overline{I}}\partial q_{\overline{I}}}\right)-iH_1+i\left(-q_{\overline{I}}\frac{\partial S_I}{\partial q_{\overline{I}}}+S_I\right)\frac{\partial H_0}{\partial p},$$

and $\mathcal{F}\left[a_{I0},\ldots,a_{Ij-1}\right]$ is a linear combination of functions $a_{I0},\ldots,a_{Ij-1}$ and their derivatives with smooth coefficients.

Let us consider first the Hamilton-Jacobi equation. It is known that if U_I is a canonical chart on the Lagrangian manifold $\mathcal{L}_+$ (or $\mathcal{L}_-$), then the equations of this manifold can be written down in the form

$$q_I = \frac{\partial S_I^+}{\partial y^I},$$
$$y^{\overline{I}} = -\frac{\partial S_I^+}{\partial q_{\overline{I}}},$$

where the function S_I^+ is a coordinate expression of the function

$$s - q_{\overline{I}} y^{\overline{I}}\Big|_{\mathcal{L}^+} \tag{26}$$

in the chart U_I (for the manifold $\mathcal{L}_-$ the function S_I^+ must be replaced by the function S_I^- defined in the similar manner). So, Hamilton-Jacobi equation (25) will be satisfied on the manifold $\mathcal{L}_+$ if we choose S_I to be equal to function (26) (the same relates to the manifold $\mathcal{L}_-$ if we replace S_I^+ by S_I^-; we shall omit such remarks below).

Further, performing the change of the unknown

$$a_{Ij}^+ = \sqrt{m_I^+}\varphi_{Ij}^+$$

in the transport equations (here m_I^+ is a density of the measure μ^+ with respect to the coordinates $(y^I, q_{\overline{I}})$) and taking into account the relation

$$-q_{\overline{I}}\frac{\partial S_I^+}{\partial q_{\overline{I}}} + S_I^+ = q_{\overline{I}} y^{\overline{I}}\Big|_{\mathcal{L}^+} + s - q_{\overline{I}} y^{\overline{I}}\Big|_{\mathcal{L}^+} = s|_{\mathcal{L}^+},$$

we reduce the transport equations to the form

$$\begin{aligned} \mathcal{P}_\pm \varphi_{I0}^\pm &= 0, \\ \mathcal{P}_\pm \varphi_{Il}^\pm &= \mathcal{F}^\pm\left[\varphi_{I0}^\pm, \dots, \varphi_{Il-1}^\pm\right],\ j \geq 1 \end{aligned}$$

on the manifolds $\mathcal{L}_\pm$, where the transport operator $\mathcal{P}_\pm$ is given by the formula

$$\mathcal{P}_\pm = X_{\mathcal{H}_\pm} \pm \left(\frac{1}{2}\frac{\partial^2 H_0}{\partial q_l \partial y^l} + \frac{i}{2\sqrt{H_0}} H_1 - \frac{is}{2\sqrt{H_0}}\frac{\partial H_0}{\partial p}\right)\Bigg|_{\mathcal{L}_\pm},$$

as above (see Subsection 2.1).

The only thing rest now is to establish a correspondence between amplitude functions $\varphi_{I0}^\pm$ in different charts of the manifolds $\mathcal{L}_\pm$ (here we restrict ourselves by constructing the main term of the asymptotic expansion; the procedure of constructing higher-order terms is standard and the reader can find it in the book [2]). This can be done with the help of the usual stationary phase formula, and the result is that if all functions $\varphi^{\pm}_{I0}$ are coordinate expressions of one and the same function $\varphi^{\pm}_{0}$ on the Lagrangian manifold $\mathcal{L}_{\pm}$, then the local expressions for the Maslov's canonical operator

$$F^{1/x}_{q_{\overline{I}}\to y^{\overline{I}}}\left[\exp\left\{\frac{i}{x}S^{\pm}_{I}\left(t,y^{I},q_{\overline{I}}\right)\right\}\sqrt{m^{\pm}_{I}}\varphi^{\pm}_{I0}\left(t,y^{I},q_{\overline{I}}\right)\right]$$

coincide on projections of the intersections $U_I\cap U_J$ of different charts of the considered Legendre manifolds with some concrete choice of the signs of the square roots $\sqrt{m^{\pm}_{I}}$; this choice is governed by the *Maslov's index* (see [1], [2]). Hence, the usual procedure using partition of unity leads us to the definition of Maslov's canonical operator $\mathcal{K}_{(\mathcal{L}^{\pm},\mu^{\pm})}$, and the solution to problem (5) for $k=1$ is given by

$$\mathcal{K}_{(\mathcal{L}_{+},\mu^{+})}\left[\varphi^{+}_{I0}\right]+\mathcal{K}_{(\mathcal{L}_{-},\mu^{-})}\left[\varphi^{-}_{I0}\right],$$

where all the objects involved into the latter expression were defined above.

2.2 Degeneration of higher degree

The consideration of the case when the degree of degeneration of the considered operator is more than one is in essence an easy compilation of the results and the considerations of Subsections 1.2 and 2.1. Clearly, one has to construct the canonical operator on a family of Legendre manifolds parameterized by x rather than on a single Legendre manifold. The only question worth considering is the application of the stationary phase method in the case when the phase function of the rapidly oscillating integral in question depends (regularly) on the small parameter of the expansion. So, in this subsection we present the corresponding version of the stationary phase method and formulate the main results.

To be short, we shall consider here a one-dimensional rapidly oscillating integral of the form

$$I\left(x,y\right)=\left(\frac{i}{2\pi x^{k}}\right)^{1/2}\int\exp\left\{\frac{i}{x^{k}}\Phi\left(x,y,p\right)\right\}a\left(x,y,p\right)\,dp, \tag{27}$$

where $p\in\mathbf{R}^{1}$, and functions $\Phi\left(x,y,p\right)$ and $a\left(x,y,p\right)$ depend on x regularly up to $x=0$ (being clearly infinitely smooth functions in all their variables). We suppose that the function $a(x,y,p)$ has compact support and there exists exactly one stationary point $p=p_0\left(y\right)$

$$\frac{\partial\Phi}{\partial p}\left(0,y,p_0\left(y\right)\right)=0$$

of the phase function $\Phi(0,y,p)$ on the support supp$a$ (we carry out all our considerations in a sufficiently small neighborhood of $x=0$), Moreover, we suppose that this point is non-degenerate:

$$\frac{\partial^2\Phi}{\partial p^2}(0,y,p_0(y)) \neq 0.$$

From the above assumptions, it follows that the equation

$$\frac{\partial\Phi}{\partial p}(x,y,p)=0$$

has a smooth up to $x=0$ solution $p=p(x,y)$ for sufficiently small x. Later on, using the Morse lemma, one can find the variable change $p=p(x,y,z)$ such that:

- the function $p(x,y,z)$ is smooth in all its variables up to $x=0$;
- the point $p=p(x,y)$ corresponds to $z=0$;
- the function $\Phi(x,y,p)$ is taken to the function

  $$\Phi(x,y,p(x,y,z)) = \Phi_0(x,y)+z^2,$$

  where

  $$\Phi_0(x,y)=\Phi(x,y,p(x,y)).$$

Performing the described variable change in integral (27), we reduce it to the form

$$\begin{aligned} I(x,y) &= \left(\frac{i}{2\pi x^k}\right)^{1/2} \exp\left\{\frac{i}{x^k}\Phi_0(x,y)\right\} \int \exp\left\{\frac{i}{x^k}z^2\right\} \left|\frac{\partial p(x,y,z)}{\partial z}\right| \\ &\times\ a(x,y,p(x,y,z))\,dz. \end{aligned}$$

Now, expanding the integrand of the latter integral in powers of z and computing the obtained integrals, we arrive at the relation

$$I(x,y)=\exp\left\{\frac{i}{x^k}\Phi_0(x,y)\right\} \frac{a(x,y,p(x,y))}{\sqrt{-\frac{\partial^2\Phi}{\partial p^2}(x,y,p(x,y))}} + O(x),$$

where the branch of the square root is chosen in accordance to the relation

$$-\frac{3\pi}{2} < \arg\left[-\frac{\partial^2\Phi}{\partial p^2}(x,y,p(x,y))\right] < \frac{\pi}{2}.$$

We remark also that the definitions of the x^{-k}-Fourier transforms are

$$\begin{aligned} F^{1/x^k}_{y^{\overline{I}}\to q_{\overline{I}}}[f] &= \left(-\frac{i}{2\pi x^k}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{-\frac{i}{x^k} y^{\overline{I}} q_{\overline{I}}\right\} f\left(x, y^I, y^{\overline{I}}\right) dy^{\overline{I}}, \\ F^{1/x^k}_{q_{\overline{I}}\to y^{\overline{I}}}\left[\tilde{f}\right] &= \left(\frac{i}{2\pi x^k}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{\frac{i}{x^k} y^{\overline{I}} q_{\overline{I}}\right\} \tilde{f}\left(x, y^I, q_{\overline{I}}\right) dq_{\overline{I}}. \end{aligned}$$

After these remarks all the construction of the canonical operator $\mathcal{K}^{(k)}_{(\mathcal{L}_\pm,\mu^\pm)}$ in the situation of degeneration of degree k goes quite similar to the constructions of the previous subsection. We shall only formulate the main result.

Let $(\mathcal{L}_\pm, \mu^\pm)$ be the Legendre manifold with measure constructed in the previous subsection. Let $\varphi_0^\pm$ be functions on $\mathcal{L}_\pm$ satisfying the transport equations

$$\mathcal{P}_\pm \varphi_0^\pm = 0$$

and the initial conditions (21), where the transport operator $\mathcal{P}_\pm$ is given by

$$\mathcal{P}_\pm = X_{\mathcal{H}_\pm} \pm \left(\frac{1}{2}\frac{\partial^2 H}{\partial q_l \partial y^l} - \frac{iks}{2\sqrt{H}}\frac{\partial H}{\partial p}\right)\Bigg|_{\mathcal{L}_\pm}.$$

Then the function

$$\mathcal{K}^{(k)}_{(\mathcal{L}_+,\mu^+)}\left[\varphi^+_{I0}\right] + \mathcal{K}^{(k)}_{(\mathcal{L}_-,\mu^-)}\left[\varphi^-_{I0}\right]$$

satisfies problem (5) up to terms of the first order in x. Moreover, the local expressions for the canonical operators involved into the latter relation can be rewritten in the form

$$F^{1/x^k}_{q_{\overline{I}}\to y^{\overline{I}}}\left[\exp\left\{i\sum_{j=1}^{k}\frac{S^\pm_{I,-j}\left(t, y^I, q_{\overline{I}}\right)}{x^j}\right\}\sqrt{m^\pm_I}\psi^\pm_{I0}\left(t, y^I, q_{\overline{I}}\right)\right]$$

(see the discussion in the end of Subsection 1.2).

3 Investigation of nonstationary problems in abstract algebras

3.1 General theory

Let us now try to investigate the problem from the viewpoint of abstract algebras (see [3]). To do this, we suppose that $\mathcal{U}$ is an abstract topological algebra with unity,

$(\cdot,\cdot)$ is a partial scalar product[4] on $\mathcal{U}$, and

$$G : C \to \mathcal{U} \tag{28}$$

is an analytic group, that is, an analytic mapping such that

$$G(p_1 + p_2) = G(p_1)\, G(p_2)\,.$$

We suppose that all the conditions posed on these objects in the above cited paper are fulfilled. Then the group (28) determines the following objects:

- An analytic transform

$$\begin{aligned} \mathcal{F}_G &: \mathcal{A}(\mathbf{C}) \to \mathcal{U}, \\ \mathcal{F}_G[U] &= \int_{-\infty}^{\infty} G(ip)\, U(p)\, dp \end{aligned}$$

  defined on the space $\mathcal{A}(\mathbf{C})$ of endlessly continuable analytic hyperfunctions; the image of this transform will be denoted by $\mathcal{U}^r$; the elements from $\mathcal{U}^r$ will be called resurgent elements of the algebra $\mathcal{U}$.

- A Hilbert space $\mathcal{H}$ being a completion of $\mathcal{U}_0$ with respect to the pre-Hilbert structure defined by the scalar product $(\cdot,\cdot)$.

- A differentiation $\hat{p}$ of the algebra $\mathcal{U}^r$ defined by the formula

$$\hat{p}\mathcal{R}_G = \mathcal{R}_G(ip)\,,$$

  where by ip we denote also the operator of multiplication by ip.

- A generating element

$$A = \frac{dG}{dp}(0)$$

  of the group G; it is supposed that this element is invertible in $\mathcal{U}$ and determines a positive self-adjoint operator in $\mathcal{H}$.

Let now $\mathbf{R}^n$ denote the Cartesian space of dimension n with coordinates

$$\left(y^1, \ldots, y^n\right)$$

[4]The latter means that there exist a dense subspace $\mathcal{U}_0$ of $\mathcal{U}$ and a semilinear form (u, v) defined at least each time when one of the factors u or v belongs to $\mathcal{U}_0$.

Consider some invertible element $B \in \mathcal{U}$ determining a self-adjoint operator on $\mathcal{H}$, and the tuple

$$(a_1, \ldots, a_k)$$

of elements from U such that each a_j, $j = 1, \ldots k$ determines a bounded self-adjoint operator in $\mathcal{H}$ and so are $\hat{p} a_j$, $j = 1, \ldots k$ (in such a situation the expression $\varphi(a_1, \ldots, a_k)$ is well-defined for any C^∞-function $\varphi(x)$).

Denote

$$D_y = -iB^{-1}\frac{\partial}{\partial y}, \ D_t = -iB^{-1}\frac{\partial}{\partial t};$$

these operators act on the set $C^\infty(\mathbf{R}^n, \mathcal{U})$ of C^∞-functions defined on the space $\mathbf{R}^n$ (or on some its open subset) with values in the algebra $\mathcal{U}$. We shall consider here the problem

$$\begin{cases} D_t^2 = H(a_1, \ldots, a_k, y, \hat{p}, D_y)\, u, \\ u|_{t=0} = u_0(y), \\ D_t u|_{t=0} = v_0(y), \end{cases} \tag{29}$$

where the operator $H(a_1, \ldots, a_k, y, \hat{p}, D_y)$ is given by the formula

$$H(a_1, \ldots, a_k, y, \hat{p}, D_y) = \sum_{j=0}^{2} \sum_{|\alpha| \le 2-j} a_\alpha(a_1, \ldots, a_k, y)\, D_y^\alpha \hat{p}^j$$

with $a_{j\alpha}(x, y) \in C^\infty(\mathbf{R}^{n+k})$ (we denote $x = (x_1, \ldots, x_k)$, $a = (a_1, \ldots a_k)$), and the initial data $u_0(y)$ and $v_0(y)$ are expansions of the generalized WKB-form

$$\begin{aligned} u_0(y) &= \exp(iBS_0(a, y)) \sum_{j=0}^{\infty} B^{-j} u_{0j}(a, y), \\ v_0(y) &= \exp(iBS_0(a, y)) \sum_{j=0}^{\infty} B^{-j} v_{0j}(a, y) \end{aligned}$$

with values in the algebra $\mathcal{U}$.

Let us search for a solution to problem (29) in the form

$$u(t, y) = \exp(iBS(t, a, y)) \sum_{j=0}^{\infty} B^{-j} u_j(t, a, y).$$

We have

$$D_y u(t, y) = \exp(iBS(t, a, y)) \left[\frac{\partial S}{\partial y}(t, a, y) + D_y\right] \sum_{j=0}^{\infty} B^{-j} u_j,$$

$$D_t u(t,y) = \exp(iBS(t,a,y))\left[\frac{\partial S}{\partial t}(t,a,y) + D_t\right]\sum_{j=0}^{\infty} B^{-j} u_j,$$

and

$$\begin{aligned} \hat{p}u(t,y) &= \exp(iBS(t,a,y))\,[i\hat{p}(B)\,S(t,a,y) \\ &\quad + \sum_{j=1}^{k} \hat{p}(a_j)\frac{\partial S}{\partial x^j}(t,a,y)\Bigg]\sum_{j=0}^{\infty} B^{-j} u_j, \end{aligned}$$

so that the equation involved into problem (29) can be rewritten in the form (we omit the arguments (t,a,y) of the functions S and u_j)

$$\begin{aligned} \left(\frac{\partial S}{\partial t} + D_t\right)^2 \sum_{j=0}^{\infty} B^{-j} u_j &= H\left(a, y, i\hat{p}(B)\,S + \sum_{j=1}^{k}\hat{p}(a_j)\frac{\partial S}{\partial x^j} + \hat{p}, \frac{\partial S}{\partial y} + D_y\right) \\ &\quad \times \sum_{j=0}^{\infty} B^{-j} u_j. \end{aligned} \tag{30}$$

Below, we shall consider the two following special cases of the latter equation:

Case 1. The element $\hat{p}(iB)$ is a scalar element of the algebra $\mathcal{U}$ (we can assume, without loss of generality, that $\hat{p}(iB) = 1$), and

$$\hat{p}(a_j) = B^{-1}\varphi_j(a_1,\ldots,a_k),\ j = 1,\ldots k$$

with bounded real-valued C^∞-functions $\varphi_j(x)$

Case 2. The relations

$$\hat{p}(iB) = B^{-l},\ \hat{p}(a_j) = B^{-l-1}\varphi_j(a_1,\ldots,a_k),\ j = 1,\ldots k$$

hold for some positive integer l.

Let us begin the consideration of Case 1. Passing to the spectral representation associated with the (self-adjoint) operator A, one can easily see that the relation $\hat{p}(B) = 1$ yields

$$B = A + c$$

with some scalar element c. It is not hard to see that this scalar element can be removed by the expansion in the Taylor series, so that we can assume

$$B = A.$$

Let us rewrite relation (30) taking into account the obtained information:

$$\left(\frac{\partial S}{\partial t} - iA^{-1}\frac{\partial}{\partial t}\right)^2 \sum_{j=0}^{\infty} A^{-j}u_j = H\left(a, y, S + A^{-1}\sum_{j=1}^{k}\varphi_j(a)\frac{\partial S}{\partial x^j} + \hat{p}, \frac{\partial S}{\partial y} - i\frac{\partial}{\partial y}\right) \times \sum_{j=0}^{\infty} A^{-j}u_j.$$

We remark that, being applied to expressions of the form $\sum_{j=0}^{\infty} A^{-j}u_j(t,a,y)$, the operator $\hat{p}$ has the order -1 with respect to the filtration defined by negative powers of A. As above, we expand the latter relation in powers of A^{-1}, thus obtaining a recurrent system of equations with respect to the unknown functions S, u_0, u_1, ... The main relation reads

$$\left(\frac{\partial S}{\partial t}\right)^2 = H\left(a, y, S, \frac{\partial S}{\partial y}\right)$$

with the initial conditions

$$S|_{t=0} = S_0(a,y).$$

Passing to the spectral representation corresponding to the set of commuting operators a_1, ..., a_k, one can replace these operators by the corresponding variables x^1, ..., x^k, so obtaining the following Cauchy problem for the Hamilton-Jacobi equation

$$\begin{cases} \left(\frac{\partial S}{\partial t}\right)^2 = H\left(x, y, S, \frac{\partial S}{\partial y}\right), \\ S|_{t=0} = S_0(x,y). \end{cases} \tag{31}$$

depending on the variables x as on parameters. This is exactly the Hamilton-Jacobi equation obtained above for the particular cases but in this case it contains the function S explicitly, and, by this reason, the corresponding Hamilton system must be considered on the contact space $J^1\left(\mathbf{R}^{n+1}_{t,y}\right)$ of the first jets on the Cartesian space $\mathbf{R}^{n+1}_{t,y}$. The latter has the coordinates

$$(s, t, y, E, q)$$

and the structure form

$$ds - qdy - Edt.$$

To write down the equations of the corresponding Legendre manifolds, we remark that the equation involved into problem (31), splits into the two equations

$$\frac{\partial S_\pm}{\partial t} = \pm\sqrt{H\left(x, y, S, \frac{\partial S}{\partial y}\right)} \tag{32}$$

(we suppose that the function S is such that the expression under the square root sign does not vanish for all values of y under consideration). The initial conditions for both functions $S_{\pm}$ coincide and are given by the relation

$$S_{\pm}|_{t=0} = S_0(x, y). \tag{33}$$

So, problem (31) determines two Legendre manifolds $\mathcal{L}_{\pm}$ corresponding to the two different choices of the sign in (32). Each of them is defined as a phase flow along the corresponding Hamilton system of the initial manifold

$$\mathcal{L}^0_{\pm} = \begin{cases} t = 0, \\ q_i = \dfrac{\partial S_0(x, y)}{\partial y^i}, \\ s = S_0(x, y) \\ E = \pm\sqrt{H(x, y, s, q)}. \end{cases}$$

The above mentioned Hamilton systems are given by the contact vector fields

$$\begin{aligned} X_{\mathcal{H}_{\pm}} &= \frac{\partial}{\partial t} + \frac{\partial \mathcal{H}_{\pm}}{\partial q_i}\frac{\partial}{\partial y^i} - E\frac{\partial \mathcal{H}_{\pm}}{\partial s}\frac{\partial}{\partial E} - \left(\frac{\partial \mathcal{H}_{\pm}}{\partial y^i} + q_i\frac{\partial \mathcal{H}_{\pm}}{\partial s}\right)\frac{\partial}{\partial q_i} \\ &\quad - \left(\mathcal{H}_{\pm} - E - q_i\frac{\partial \mathcal{H}_{\pm}}{\partial q_i}\right)\frac{\partial}{\partial s} \end{aligned}$$

corresponding to the following two Hamilton functions

$$\mathcal{H}_{\pm}(x, t, y, s, E, q) = E \mp \sqrt{H(x, y, s, q)}$$

(summation rule is used in the latter expression). So, in this case the Legendre manifolds corresponding to Hamilton-Jacobi equations (32) with initial conditions (33) are defined with the help of the following Cauchy problem for the contact Hamilton system:

$$\begin{cases} \dot{y}^i = \dfrac{\partial \mathcal{H}_{\pm}}{\partial q_i}, \\ \dot{E} = -\dfrac{\partial \mathcal{H}_{\pm}}{\partial s}, \\ \dot{q}_i = -\dfrac{\partial \mathcal{H}_{\pm}}{\partial y^i} - q_i\dfrac{\partial \mathcal{H}_{\pm}}{\partial s}, \\ \dot{s} = -\mathcal{H}_{\pm} + E + q_i\dfrac{\partial \mathcal{H}_{\pm}}{\partial q_i}, \end{cases} ; \quad \begin{cases} y^i(0) = y^i_0, \\ E(0) = \pm\sqrt{H\left(x, y_0, S_0(x, y_0), \dfrac{\partial S_0(x, y_0)}{\partial y_0}\right)}, \\ q_i(0) = \dfrac{\partial S_0(x, y_0)}{\partial y_0}, \\ s(0) = S_0(x, y_0). \end{cases} .$$

We shall not derive the corresponding transport equations as well as the representation of the solution in singular charts of the corresponding Legendre manifolds since this investigations are quite similar to that carried out above. We only mention that for constructing the mixed coordinate-impulse representation one should use the A-Fourier transform in a part of variables y:

$$\tilde{f}\left(x, y^{I}, q_{\overline{I}}\right) = F^{A}_{y^{\overline{I}} \to q_{\overline{I}}}[f] = \left(-\frac{iA}{2\pi}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{-iAy^{\overline{I}} q_{\overline{I}}\right\} f\left(x, y^{I}, y^{\overline{I}}\right) dy^{\overline{I}},$$

$$f\left(x, y^{I}, y^{\overline{I}}\right) = F^{A}_{q_{\overline{I}} \to y^{\overline{I}}}\left[\tilde{f}\right] = \left(\frac{iA}{2\pi}\right)^{\frac{|\overline{I}|}{2}} \int \exp\left\{iAy^{\overline{I}} q_{\overline{I}}\right\} \tilde{f}\left(x, y^{I}, q_{\overline{I}}\right) dq_{\overline{I}}.$$

We shall illustrate these considerations on the example in the end of this section.

Now, let us turn our mind to the consideration of Case 2. In this case equation (30) can be rewritten in the form

$$\left(\frac{\partial S}{\partial t} + D_t\right)^2 \sum_{j=0}^{\infty} B^{-j} u_j = H\left(a, y, B^{-l} S + \sum_{j=1}^{k} B^{-l-1} \varphi_j(a) \frac{\partial S}{\partial x^j} + \hat{p}, \frac{\partial S}{\partial y} + D_y\right) \times \sum_{j=0}^{\infty} B^{-j} u_j.$$

One can easily see that the operator $\hat{p}$ being applied to functions of the form $\sum_{j=0}^{\infty} B^{-j} u_j(t, a, y)$ has in this case order at least $-l-1$ with respect to the filtration given by powers of B (this can be proved with the help of the spectral representation associated with the operator A). Hence, in this case the Cauchy problem for the Hamilton-Jacobi equation has the form

$$\begin{cases} \left(\frac{\partial S}{\partial t}\right)^2 = H\left(x, y, 0, \frac{\partial S}{\partial y}\right), \\ S|_{t=0} = S_0(x, y), \end{cases} \tag{34}$$

which differs from that of the previous case by the absence of the action function S itself in this equation. However, it is clear that this function will appear in the transport equations, and we again have to work with Legendre manifolds rather than with Lagrangian ones.

As usual, the equation in problem (34) is split into the two equations

$$\frac{\partial S}{\partial t} = \pm\sqrt{H\left(x, y, 0, \frac{\partial S}{\partial y}\right)}$$

(clearly, we suppose that the function on the right in the latter relation is smooth for values of y in question). So, for this case the corresponding Legendre manifolds $\mathcal{L}_\pm$ can be obtained with the help of solution of the following initial data problem for the Hamilton system

$$\begin{cases} \dot{y}^i = \dfrac{\partial \mathcal{H}_\pm}{\partial q_i}, \\ \dot{E} = 0, \\ \dot{q}_i = -\dfrac{\partial \mathcal{H}_\pm}{\partial y^i}, \\ \dot{s} = -\mathcal{H}_\pm + E + q_i \dfrac{\partial \mathcal{H}_\pm}{\partial q_i}, \end{cases} ; \quad \begin{cases} y^i(0) = y_0^i, \\ E(0) = \pm\sqrt{H\left(x, y_0, 0, \dfrac{\partial S_0(x, y_0)}{\partial y_0}\right)}, \\ q_i(0) = \dfrac{\partial S_0(x, y_0)}{\partial y_0}, \\ s(0) = S_0(x, y_0). \end{cases},$$

where

$$\mathcal{H}_\pm(x, t, y, E, q) = E \mp \sqrt{H(x, y, 0, q)}.$$

The reader can see that this problem has exactly the same form as in the examples above, so that these examples serve as a good guide to the construction of the rest of the theory, and we shall not stand here on this point.

3.2 Example

We conclude this section with an example of the situation described in the Case 1 (as we have written above, the preceding sections give us good examples of the situation of Case 2).

Consider the following Cauchy problem

$$\begin{cases} x^2 \dfrac{\partial^2 u}{\partial t^2} = \left(x^2 \dfrac{\partial}{\partial x}\right)^2 u + x^2 \dfrac{\partial^2 u}{\partial y^2}, \\ u\big|_{t=0} = \exp\left\{\frac{i}{x} S_0(y)\right\} a_0(y), \\ -ix \dfrac{\partial u}{\partial t}\Big|_{t=0} = \exp\left\{\frac{i}{x} S_0(y)\right\} b_0(y) \end{cases} \tag{35}$$

with some smooth functions $S_0(y)$, $a_0(y)$, and $b_0(y)$, where x and y are one-dimensional variables. Searching for solution to this problem in the form

$$u = \exp\left\{\frac{i}{x} S(t, y)\right\} a(t, y),$$

we arrive at the following equation

$$\left(\frac{\partial S}{\partial t} - ix \frac{\partial}{\partial t}\right)^2 a(t, y) = \left[\left(S - ix^2 \frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y} - ix \frac{\partial}{\partial y}\right)^2\right] a(t, y)$$

or

$$\left[\left(\frac{\partial S}{\partial t}\right)^2 - 2ix\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - ix\frac{\partial^2 S}{\partial t^2} + \left(-ix\frac{\partial}{\partial t}\right)^2\right] a = \left\{\left[S^2 - 2ix^2 S\frac{\partial}{\partial x}\right.\right.$$

$$\left.\left. -ix^2\frac{\partial S}{\partial x} + \left(-ix^2\frac{\partial}{\partial x}\right)^2\right] + \left[\left(\frac{\partial S}{\partial y}\right)^2 - 2ix\frac{\partial S}{\partial y}\frac{\partial}{\partial y} - ix\frac{\partial^2 S}{\partial y^2} + \left(-ix\frac{\partial}{\partial y}\right)^2\right]\right\} a.$$

Expanding the latter equation in powers of x and equating the coefficients of x^0 and x^1 in both its parts, we obtain the following equations:

$$\left(\frac{\partial S}{\partial t}\right)^2 = \left(\frac{\partial S}{\partial y}\right)^2 + S^2,$$

$$\left[2\frac{\partial S}{\partial t}\frac{\partial}{\partial t} - 2\frac{\partial S}{\partial y}\frac{\partial}{\partial y} + \frac{\partial^2 S}{\partial t^2} - \frac{\partial^2 S}{\partial y^2}\right] a\,(t,y) = 0. \tag{36}$$

Let us consider in detail the first of these two equations — the Hamilton-Jacobi equation. As earlier, it splits into the following two equations

$$\frac{\partial S^\pm}{\partial t} = \pm\sqrt{\left(\frac{\partial S^\pm}{\partial y}\right)^2 + \left(S^\pm\right)^2} \tag{37}$$

with the initial data

$$S^\pm\big|_{t=0} = S_0\,(y)\,.$$

Clearly, we must require that the expression under the square root sign on the right in (37) does not vanish[5]. Let us solve the Hamilton system

$$\begin{cases} \dot y = \mp q\left[q^2 + s^2\right]^{-1/2}, \\ \dot q = \pm q s\left[q^2 + s^2\right]^{-1/2}, \\ \dot s = \pm s^2\left[q^2 + s^2\right]^{-1/2}. \end{cases}$$

corresponding to the Hamilton-Jacobi equation in question. Denoting by (y_0, q_0, s_0) the initial data for this system, we obtain the solution to the latter problem in the form

$$\begin{cases} y = y_0 \mp q_0 t\left[q_0^2 + s_0^2\right]^{-1/2}, \\ q = q_0 \exp\left\{\pm s_0 t\left[q_0^2 + s_0^2\right]^{-1/2}\right\}, \\ s = s_0 \exp\left\{\pm s_0 t\left[q_0^2 + s_0^2\right]^{-1/2}\right\}. \end{cases}$$

[5]We emphasize that this condition can be fulfilled even in the case when y are local coordinates on the compact manifold.

To obtain the equation for the Legendre manifolds $\mathcal{L}_\pm$, one has to substitute

$$q_0 = \frac{\partial S_0(y_0)}{\partial y_0}, \; s_0 = S_0(y_0).$$

So, the equations of the Legendre manifolds $\mathcal{L}_\pm$ are

$$\begin{aligned}
y^\pm(t, y_0) &= y_0 \mp t\frac{\partial S_0(y_0)}{\partial y_0}\left[\left(\frac{\partial S_0(y_0)}{\partial y_0}\right)^2 + [S_0(y_0)]^2\right]^{-1/2}, \\
q^\pm(t, y_0) &= \frac{\partial S_0(y_0)}{\partial y_0}\exp\left\{\pm t S_0(y_0)\left[\left(\frac{\partial S_0(y_0)}{\partial y_0}\right)^2 + [S_0(y_0)]^2\right]^{-1/2}\right\}, \\
s^\pm(t, y_0) &= S_0(y_0)\exp\left\{\pm t S_0(y_0)\left[\left(\frac{\partial S_0(y_0)}{\partial y_0}\right)^2 + [S_0(y_0)]^2\right]^{-1/2}\right\}.
\end{aligned}$$

Now, the solutions to the Hamilton-Jacobi equation in nonsingular charts of Legendre manifolds $\mathcal{L}_\pm$ can be obtained with the help of the formula

$$S^\pm(t, y) = s^\pm(t, y_0)\big|_{y_0 = y_0^\pm(t,y)},$$

where the function $y_0^\pm(t, y)$ is defined as a solution to the equation

$$y = y^\pm(t, y_0)$$

with respect to y_0 (the existence of this solution is the necessary and sufficient condition for the point to be in a nonsingular chart of the manifold $\mathcal{L}_\pm$). Similar, the action $\tilde{S}^\pm(t, p)$ in the singular chart of the manifold $\mathcal{L}_\pm$ is given by

$$\tilde{S}^\pm(t, p) = \left[s^\pm(t, y_0) - y^\pm(t, y_0)\, q^\pm(t, y_0)\right]\big|_{y_0 = y_0^\pm(t,q)},$$

where $y_0^\pm(t, q)$ is a solution to

$$q = q^\pm(t, y_0)$$

with respect to y_0.

Let us now turn our mind to the solution to the transport equation (the second equation in (36)). Similar to the case considered in the examples above, we can transform this equation with the help of the invariant measure $\mu^\pm$ which is defined literally in the same way as above. The result is

$$\frac{d\varphi^\pm}{dt} = 0,$$

where d/dt is the derivative along the trajectories of the Hamilton system, the function $\varphi^{\pm}$ is a function on $\mathcal{L}_{\pm}$ such that $a^{\pm}(t,y)$ and $\tilde{a}^{\pm}(t,q)$ are defined by the formulas

$$a^{\pm}(t,y) = \sqrt{m^{\pm}(t,y)}\varphi^{\pm}(t,y), \; \tilde{a}^{\pm}(t,q) = \sqrt{\tilde{m}^{\pm}(t,q)}\varphi^{\pm}(t,q),$$

and $m^{\pm}(t,y)$, $\tilde{m}^{\pm}(t,q)$ are densities of the measure $\mu^{\pm}$ with respect to the coordinates (t,y) and (t,q), respectively. We suppose that the signs of the square roots in the above expressions are chosen in accordance to the Maslov's index.

Now, defining the initial data for the functions $\varphi^{\pm}$ exactly as it was done in the above considered examples, we obtain the following expressions for these functions:

$$\varphi^{\pm}(t,y) = \frac{1}{2}\left\{a_0(y_0) \pm b_0(y_0)\left[\left(\frac{\partial S_0(y_0)}{\partial y_0}\right)^2 + (S_0(y_0))^2\right]\right\}\Bigg|_{y_0=y_0^{\pm}(t,y)}$$

in a nonsingular chart, and

$$\varphi^{\pm}(t,q) = \frac{1}{2}\left\{a_0(y_0) \pm b_0(y_0)\left[\left(\frac{\partial S_0(y_0)}{\partial y_0}\right)^2 + (S_0(y_0))^2\right]\right\}\Bigg|_{y_0=y_0^{\pm}(t,q)}$$

in a singular one. Therefore, the solution to problem (35) is described by the formula

$$\begin{aligned} u(t,x,y) &= \exp\left\{\frac{i}{x}S^{+}(t,y)\right\}\sqrt{m^{+}(t,y)}\varphi^{+}(t,y) \\ &\quad + \exp\left\{\frac{i}{x}S^{-}(t,y)\right\}\sqrt{m^{-}(t,y)}\varphi^{-}(t,y) \end{aligned}$$

in a nonsingular chart, and

$$\begin{aligned} u(t,x,y) &= F^{1/x}_{q\to y}\left[\exp\left\{\frac{i}{x}\tilde{S}^{+}(t,q)\right\}\sqrt{\tilde{m}^{+}(t,q)}\varphi^{+}(t,q)\right. \\ &\quad \left. + \exp\left\{\frac{i}{x}\tilde{S}^{-}(t,q)\right\}\sqrt{\tilde{m}^{-}(t,q)}\varphi^{-}(t,q)\right] \end{aligned}$$

in a singular one, where all functions involved into the latter formulas as well as the inverse $1/x$-Fourier transform $F^{1/x}_{q\to y}$ were defined above.

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