Lefschetz Theory on Manifolds with Edges

Introduction

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Abstract

The aim of this book is to develop the Lefschetz fixed point theory for elliptic complexes of pseudodifferential operators on manifolds with edges.

The general Lefschetz theory contains the index theory as a special case, while the case to be studied is much more easier than the index problem.

The main topics are:

- The calculus of pseudodifferential operators on manifolds with edges, especially symbol structures (inner as well as edge symbols).
- The concept of ellipticity, parametrix constructions, elliptic regularity in Sobolev spaces.
- Hodge theory for elliptic complexes of pseudodifferential operators on manifolds with edges.
- Development of the algebraic constructions for these complexes, such as homotopy, tensor products, duality.
- A generalization of the fixed point formula of Atiyah and Bott for the case of simple fixed points.
- Development of the fixed point formula also in the case of non-simple fixed points, provided that the complex consists of differential operators only.
- Investigation of geometric complexes (such as, for instance, the de Rham complex and the Dolbeault complex).

Results in this direction are desirable because of both purely mathematical reasons and applications in natural sciences.

Ziel des Buches ist es, die Lefschetz-Theorie der Fixpunkte für elliptische Komplexe von Pseudodifferentialoperatoren auf Mannigfaltigkeiten mit Kanten zu gewinnen. Die allgemeine Lefschetz-Theorie enthält die Index-Theorie als Spezialfall, aber der Fall, den wir analysieren werden, ist viel leichter als das Index-Problem. Ergebnisse in dieser Richtung sind wünschenswert, einerseits aus innermathematischen Gründen, aber auch im Hinblick auf Anwendungen in den Naturwissenschaften.

Contents

Introduction

If M is a closed manifold and $f: M \to M$ is a continuous mapping, then the Lefschetz number of f is defined by $L(f) = \sum_i (-1)^i \mathrm{tr}\,(Hf)_i$, where $(Hf)_i$ denotes the induced endomorphism in the cohomology with real coefficients $H^i(M, \boldsymbol{R})$ and tr the trace. In 1926 Lefschetz published his famous fixed point formula (cf. [Lef26]) expressing this global characteristic of f in case all fixed points of f are isolated as the sum of local indices at fixed points (these indices are mapping degrees of $1-f$ considered as a mapping between small spheres centered at the fixed points). His argument is based on the intersection theory applied to the cycles Δ and Γ_f representing the diagonal and the graph of f in $M \times M$, respectively. Their homological intersection number corresponds via the Künneth formula and the Poincaré duality to the Lefschetz number, and a straightforward calculation yields the equality of the geometric intersection number and the sum of the local indices.

A few years later, considering simplicial mappings of finite simplicial complexes, Hopf proved an alternating sum formula which by simplicial approximation lead to an alternative proof of the Lefschetz formula (cf. [Hop29]).

In their paper [AB67], Atiyah and Bott established an analogue of the Lefschetz fixed point formula for geometric endomorphisms of elliptic complexes. The original proof of the formula in [AB67] can be considered as a generalisation of Hopf's argument. Its central point is again an alternating trace formula for endomorphisms of elliptic complexes given by pseudodifferential operators.

There is a well-established relationship between homological traces of mappings and their fixed point sets. This relationship comes from two results: first, the relation between the fixed points of a function and its trace as a composition operator on a space of functions; and second, the relation between traces on spaces in a complex and associated traces on the homology of the complex. The Atiyah-Bott fixed point formula goes through these two steps in its proof; the middle step, involving the meaning of the trace of a composition operator, requires some extensions of the notion of a trace. To this end, Atiyah and Bott made essential use of the structure and the properties of classical pseudodifferential operators introduced by Kohn and Nirenberg [KN65] and Hörmander [Hör66].

In the '80s the interest in the Atiyah-Singer index theorem and closely related to it Atiyah-Bott formula increased enormously. This is first of all explained by the connection discovered between the index theorem and sypersymmetric quantum theories (cf. Alvarez-Gaumé [AG83] and Witten [Wit82]). There appeared new proofs of the Atiyah-Singer index theorem (cf. Atiyah, Bott and Patodi [ABP73], Berline and Vergne [BV85], Bismut [Bis84a], Getzler [Get83]) as well as of the Atiyah-Bott formula (cf. Atiyah and Segal [AS68], Bismut [Bis84b, Bis85]).

In their celebrated paper [APS75], Atiyah, Patodi and Singer proved an index theorem for Dirac operators on compact manifolds with boundary, under the assumption that the metric is a product near the boundary and using a global boundary condition arising from the induced Dirac operator on the boundary. In their formula the index-defect, i.e. the difference between the analytic index and the interior term (which is the integral over the manifold of the form representing the appropriate absolute characteristic class), is expressed in terms of the 'eta' invariant of the boundary operator and the dimension of its null space. Their definition of the 'eta' invariant extends directly to all "admissible" Dirac operators and was later shown to extend to all self-adjoint elliptic pseudodifferential operators on compact manifolds without boundary. In the Dirac setting there are various further extensions to non-compact manifolds (by Brüning and Seeley [BS88], by Müller [Mül87], by Stern [Ste89] and by Melrose [Mel93]), to singular manifolds (by Cheeger [Che87]), to boundary value problems (by Branson and Gilkey [BG92], by Douglas and Wojciechowski [DW91] and by Müller [Mül94], by Grubb [Gru92], by Grubb and Seeley [GS95]), to families (by Bismut and Cheeger [BC89, BC90], by Melrose [Mel95]) and also to define "higher" 'eta' invariants (by Lott [Lot92], by Getzler [Get93] and by Wu [Wu93]).

Melrose [Mel93] reinterpreted and systematized the proof of the original Atiyah-Patodi-Singer theorem using the calculus of b-pseudodifferential operators. One advantage of this method is the natural way the 'eta' invariant appears in the course of the proof.

The index theorem for deformation quantisation by Fedosov [Fed95] sheds new light to the Atiyah-Singer index theorem. It associates to every element of the K-functor with compact support on a symplectic manifold some topological invariant which in the simplest case is a polynomial in inverse powers of the Planck constant. This invariant is similar to the topological index of an elliptic operator. There is no analogue of the analytic index in deformation quantisation unless the deformation quantum algebra springs up from a "genuine" operator quantisation. In this case the index theorem asserts the coincidence of the analytic and topological indices. Thus, the quantisation conditions arise: the Planck constant may take only the values at which the topological index is an integer.

In [FS96, FST96] Fedosov's techniques (cf. [Fed74, Fed78]) is used to obtain analytic index formulas for elliptic operators on a cone and on a wedge.

Elliptic complexes of pseudodifferential operators on manifolds arise in various problems of geometry and analysis rather than single elliptic operators. For example, the well-known de Rham and Dolbeault complexes appear as resolutions of the natural sheaves of constant and holomorphic functions, respectively (cf. Wells [Wel73]).

Elliptic complexes of differential operators are a necessary tool in the study of overdetermined elliptic systems of partial differential equations (Spencer's resolution, cf. [Spe69]). Elliptic complexes of boundary value problems appear as solvability conditions of overdetermined elliptic boundary value problems (cf. Samborskii [Sam81]). They were studied in the papers of Dynin [Dyn72], Pillat and Schulze [PS80], Dudnikov and Samborskii [DS91].

The classical fixed point theorem of Lefschetz [Lef26] is easily formulated in terms of the de Rham complex. Indeed, let M be a closed compact smooth manifold of dimension n and let $\Lambda^i = \boldsymbol{C} \otimes_R \Lambda^i T^* M$ be the bundle of complex-valued exterior forms of degree i over M. The operator of exterior derivation d, when restricted to differential forms of degree i, provides a mapping $d_i : \mathcal{E}(\Lambda^i) \to \mathcal{E}(\Lambda^{i+1})$ satisfying $d_{i+1} d_i = 0$. The de Rham complex

$$\mathcal{E}(\Lambda^{\cdot}) : \ 0 \longrightarrow \mathcal{E}(\Lambda^0) \xrightarrow{d_0} \mathcal{E}(\Lambda^1) \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} \mathcal{E}(\Lambda^n) \longrightarrow 0$$

is known to be elliptic, its cohomology $H^i(\mathcal{E}(\Lambda^{\cdot})) = \frac{\ker d_i}{\operatorname{im} d_{i-1}}$ is finite-dimensional and isomorphic to the i-dimensional cohomology of the manifold X with coefficients in $\boldsymbol{C}$ (cf. de Rham [dR55]). Each smooth mapping $f : M \to M$ has a natural lift to the complex $\mathcal{E}(\Lambda^{\cdot})$ given by the "pull-back" operator $f^\sharp$ on differential forms. In other words, $f^\sharp$ restricts to a family of mappings $f_i^\sharp : \mathcal{E}(\Lambda^i) \to \mathcal{E}(\Lambda^i)$ commuting with the differential of $\mathcal{E}(\Lambda^{\cdot})$, i.e., $d_i f_i^\sharp = f_{i+1}^\sharp d_i$. Hence it follows that $f^\sharp$ induces an endomorphism $(Hf^\sharp)_i$ of the de Rham cohomology $H^i(\mathcal{E}(\Lambda^{\cdot}))$, for each $i = 0, 1, \dots, n$. As described above, the Lefschetz number of f is the alternating sum of the traces of $(Hf^\sharp)_i$, $i = 0, 1, \dots, n$. In particular, if $f = \mathrm{Id}$ is the identity mapping of M, then $L(f)$ coincides with the Euler characteristic of the manifold M. The Lefschetz theorem deals with a situation which is, in a sense, at the opposite extreme from the case of the identity mapping. It asserts that if f is a smooth mapping of M of "general position," i.e., each fixed point of

f is simple, then

$$L(f) = \sum_{f(p)=p} \text{sign} \det(1 - df(p));$$

thus, $L(f)$ is equal to the number of fixed points of f taken along with their multiplicities.

As mentioned, Atiyah and Bott [AB67] extended the Lefschetz fixed point formula to arbitrary elliptic complexes over a closed compact smooth manifold M. To state their result, let

$$\mathcal{E}(V^{\cdot}): \ 0 \longrightarrow \mathcal{E}(V^0) \xrightarrow{d_0} \mathcal{E}(V^1) \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} \mathcal{E}(V^N) \longrightarrow 0$$

be such a complex, where V^i are complex vector bundles over M and d_i classical pseudodifferential operators of type $V^i \to V^{i+1}$ satisfying $d_{i+1}d_i = 0$. The ellipticity of $\mathcal{E}(V^{\cdot})$ means that the corresponding sequence of principal symbols

$$0 \longrightarrow \pi^* V^0 \xrightarrow{\sigma(d_0)} \pi^* V^1 \xrightarrow{\sigma(d_1)} \dots \xrightarrow{\sigma(d_{N-1})} \pi^* V^N \longrightarrow 0$$

is exact in the complement of the zero section of T^*M. Here, $\pi^* V^i \to T^*M$ stands for the pull-back of the bundle V^i under the canonical mapping $\pi : T^*M \to M$. Just as in the case of the de Rham complex, the cohomology $H^i(\mathcal{E}(V^{\cdot})) = \frac{\ker d_i}{\text{im } d_{i-1}}$ of an elliptic complex is finite-dimensional at each step i. Suppose E is an endomorphism of the complex $\mathcal{E}(V^{\cdot})$, i.e., a sequence $E_i : \mathcal{E}(V^i) \to \mathcal{E}(V^i)$ of linear mappings such that $d_i E_i = E_{i+1} d_i$. Then E preserves the spaces of cocycles and coboundaries of $\mathcal{E}(V^{\cdot})$, hence after passing to quotient spaces it induces an endomorphism $(HE)_i$ of the cohomology $H^i(\mathcal{E}(\Lambda^{\cdot}))$, for every $i = 0, 1, \dots, n$. As these are finite-dimensional, the traces $\text{tr}\,(HE)_i$ are well-defined which yields the Lefschetz number of E by

$$L(E) = \sum_{i=0}^{N} (-1)^i \text{tr}\,(HE)_i.$$

Once again, if $E = \text{Id}$ is the identity endomorphism of $\mathcal{E}(V^{\cdot})$, then

$$\begin{aligned} L(\text{Id}) &= \sum_{i=0}^{N} (-1)^i \dim H^i(\mathcal{E}(V^{\cdot})) \\ &= \chi(\mathcal{E}(V^{\cdot})) \end{aligned}$$

is just the Euler characteristic of the complex $\mathcal{E}(V^{\cdot})$. In particular, if $N = 1$, this becomes the index of the elliptic operator d_0. The question of how to compute $L(E)$ is therefore a generalisation of the index problem for elliptic operators. Atiyah and Bott [AB67] evaluated the Lefschetz number $L(E)$ in the case when E is a geometric endomorphism of $\mathcal{E}(V^{\cdot})$. The latter is constructed via a smooth mapping f of the underlying manifold M and a family of smooth bundle homomorphisms $h_{V^i} : f^* V^i \to V^i$. For abbreviation, let us use the same letters h_{V^i} to designate the corresponding mappings $\mathcal{E}(f^* V^i) \to \mathcal{E}(V^i)$ of sections. An endomorphism E is said to be geometric if all E_i are of the form $E_i = h_{V^i} \circ f^*$. Then, the Atiyah-Bott formula reads

$$L(E) = \sum_{f(p)=p} \frac{\sum_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{V^i}(p)}{|\det(1 - df(p))|} \qquad (0.0.1)$$

provided f is of general position. Note that the bundle homomorphism $h_{V^i} : f^* V^i \to V^i$ is just a family of linear mappings $h_{V^i}(p) : V^i_{f(p)} \to V^i_p$. Hence, at a fixed point p of f we have $V^i_{f(p)} = V^i_p$, and so $h_{V^i}(p)$ is an endomorphism of the vector space V^i_p. Thus, $\mathrm{tr}\, h_{V^i}(p)$ is defined.

In the case of the de Rham complex we have $h_{V^i}(p) = \Lambda^i df(p)'$, the ith exterior power of the transpose to $df(p)$.

Thus, the Atiyah-Bott formula expresses the Lefschetz number of a geometric endomorphism of an elliptic complex on a closed compact manifold via infinitesimal invariants of f and $h_{V^{\cdot}}$ at the fixed points of the mapping f. It is worth pointing out that formula (0.0.1) does not explicitly involve the pseudodifferential operators d_i. Thus it is much simpler than the Atiyah-Singer index formula. Of course the d_i are implicitly involved by the condition $d_i E_i = E_{i+1} d_i$.

In [AB67] two more ways are sketched to prove the fixed point formula for elliptic complexes. The first approach relies on the results of Seeley [See67] on complex powers of pseudodifferential operators and the ζ-function. A closely related approach was chosen by Kotake in [Kot69]; it consists of the study of the fundamental solution for the heat equation defined by the Laplacians of the complex after having transformed the Lefschetz number by means of the Hodge theory.

Taking up the second suggestion of Atiyah and Bott is what the paper of Nestke [Nes81] aims at. It runs along the lines of the original proof of Lefschetz for the classical formula.

Toledo went one step further in proving a fixed point formula only assuming isolated fixed points (cf. [Tol73]). He constructs a differential form on the complement of the set of fixed points, of which the differential can be extended to the whole manifold and gives the Lefschetz number by integration. The formula then follows from Stokes' theorem by taking the limit for a contracting family of neighbourhoods.

A fixed point formula for higher-dimensional sets of fixed points was found by Gilkey in [Gil79] by means of heat equation methods.

A particular case of (0.0.1) is the Lefschetz fixed point formula for the Dolbeault complex which is referred to as the holomorphic Lefschetz formula. For direct constructions along more classical lines we refer the reader to Patodi [Pat73], Toledo and Tong [TT75], Inoue [Ino82], et al. Donnelly and Fefferman [DF86] found an analogue of the holomorphic Lefschetz formula for strictly pseudoconvex domains in $\boldsymbol{C}^n$ provided with the Bergman metric. This corresponds to the case of a non-compact manifold (cf. also Donnelly and Fefferman [DF83b, DF83a], Donnelly and Li [DL84] and Donnelly [Don88]).

In the paper of Efremov [Efr88] the Atiyah-Bott fixed point formula is extended to universal coverings of a closed manifold.

In the L^2-cohomology setting there are various further extensions of the Atiyah-Bott formula to non-compact manifolds (by Brüning [Bru90], by Shubin [Shu92] and by Shubin and Seifarth [SS90]).

A new idea suggested by Fedosov [Fed93] is to consider endomorphisms of elliptic complexes which are induced by classical Hamiltonian flows $t : T^*M \to T^*M$ rather than by a mapping f of the underlying manifold. He showed an asymptotic expansion of the Lefschetz number as $\hbar \to 0$, in terms of fixed points of t. Such endomorphisms can be realized on sections of vector bundles as Fourier integral operators obtained by quantising these symplectic canonical transformations. Sternin and Shatalov [SS94b] generalized this result to arbitrary symplectic canonical transformations of T^*M whose sets of fixed points are allowed to be compact smooth submanifolds of T^*M of various dimensions.

Yet another aspect of the general Lefschetz theory consists of generalising the classical Riemann-Roch theorem to solutions of elliptic equations with point singularities (cf. Gromov and Shubin [GS92, GS93a, GS93b] and Shubin [Shu93]).

In 1971, following his construction of an algebra of pseudodifferential boundary value problems for operators satisfying the transmission property, Boutet de Monvel [BdM71] gave an analogue of the Atiyah-Singer index theorem for boundary value problems. For elliptic complexes on a compact smooth manifold with boundary whose differentials are operators in the Boutet de Monvel algebra, Brenner and Shubin [BS81] proved an analogue of the Atiyah-Bott formula. We also mention the infinitesimal version of the classical Lefschetz formula for manifolds with boundary by Arnold [Arn79].

In this book elliptic complexes on manifolds with fibred boundaries are studied, whose differentials are operators of a pseudodifferential algebra introduced by the first author (cf. Rempel and Schulze [RS82b] and Schulze

[Sch91, Sch94, Sch97b]). As but a few instances of manifolds with fibred boundaries, we mention smooth manifolds with boundaries, closed manifolds with conical points and those with edges. For such complexes we prove an analogue of the Lefschetz fixed point theorem.

In Chapter 1 we give a brief exposition of manifolds with fibred boundary. By such a manifold we mean a smooth compact manifold M whose boundary is a fibred space over a smooth closed compact manifold S. Let $F : \partial M \to S$ stand for the fibration mapping. We assume that F is smooth and that all the fibres $F^{-1}(y)$, $y \in S$, are diffeomorphic to a smooth closed compact manifold X. The Riemannian metric on a manifold with fibred boundary degenerates along the boundary fibres. Such manifolds arise, as will be shown, in various ways. Constructions leading to manifolds with fibred boundary include the desingularisation of singular varieties via a blow-up procedure and the compactification of non-compact spaces. In the 'category' of manifolds with fibred boundary, local diffeomorphisms of M and smooth bundles over M are required to respect the boundary fibration.

Chapter 2 is devoted to the study of Sobolev spaces on manifolds with fibred boundary. By the theorem on a collar neighbourhood, the manifold M close to the boundary can be identified with the product $[0, 1) \times \partial M$. We say that a function or a distribution on M is supported close to the boundary of M if it vanishes away from a compact subset of the collar neighbourhood of ∂M. Fix a smooth function ω on M which is supported close to the boundary and equal to 1 in a smaller neighbourhood of ∂M. Then, each distribution u in the interior of M can be written as $u = u_1 + u_2$, where $u_1 = \omega u$ is supported close to the boundary and the support of $u_2 = (1 - \omega)u$ does not meet the boundary. For $s, \gamma \in \boldsymbol{R}$, we introduce a weighted Sobolev space $H^{s,\gamma}(M)$ of smoothness s and weight γ on M. To this end, let $N_+\partial M$ stand for the bundle of inner normals to the boundary of M and let $F_*N_+\partial M$ be the push-forward of this bundle under the fibration F. Thus, $F_*N_+\partial M$ is a bundle over S whose fibre over a point $y \in S$ is the semicylinder $N_{+y}\partial M \times F^{-1}(y)$. Denote by $H^{s,\gamma}(F_*N_+\partial M)$ the Hilbert bundle over S whose fibre over a point $y \in S$ is $H^{s,\gamma}((F_*N_+\partial M)_y)$, a weighted Sobolev space on the semicylinder over $F^{-1}(y)$. The definition of the space $H^{s,\gamma}((F_*N_+\partial M)_y)$ relies on the nature of the fibration mapping $F : \partial M \to S$. Then, a distributions u belongs to $H^{s,\gamma}(M)$ if $u_1 \in H^s(H^{s,\gamma}(F_*N_+\partial M))$ and $u_2 \in H^s(M)$, where $H^s(H^{s,\gamma}(F_*N_+\partial M))$ stands for a "twisted" Sobolev space of sections of $H^{s,\gamma}(F_*N_+\partial M)$ over S and $H^s(M)$ for the usual Sobolev space on M. A central result of Chapter 2 is that the space $H^{s,\gamma}(M)$ is locally invariant under smooth mappings of M (cf. Schrohe [Sch97a]). This definition is naturally extended to sections of smooth vector bundles over M.

In Chapter 3 we discuss pseudodifferential operators on a manifold M

with fibred boundary. We start with typical differential operators close to the boundary of M, i.e., in the collar neighbourhood $[0,1) \times \partial M$. This neighbourhood inherits a fibration from the boundary, whose base is S and whose fibre over a point $y \in S$ is $[0,1) \times F^{-1}(y)$. Then, typical differential operators are generated by vector fields along the normal direction to ∂M and those along the base S, which are smooth up to the boundary, and by vector fields along the fibers $F^{-1}(y)$ which are blown up on the boundary. These are spanned by $b\,\partial/\partial x_j$ in local coordinates of X, where b is a positive function in the collar neighbourhood of ∂M growing at the boundary. In order to get asymptotic results, we ignore the dependence of b on x and y thus requiring b to be a smooth positive function on the interval $(0,1)$ with $b(0+) = \infty$. We specify the singularity of b at $t = 0$ by assuming that $1/b$ is smooth up to $t = 0$. The explicit form of b originates with the "singular" nature of the fibration mapping $F : \partial M \to S$. The typical differential operators give rise to a symbol class $\mathrm{Symb}^m(T^*M)$ consisting of those classical symbols of order $m \in \boldsymbol{R}$ in the interior of M which have prescribed degeneracy on the boundary of M. More precisely, these are of the form

$$a(t,x,y,\tau,\xi,\eta) = e^{-m\int b}\, \tilde{a}\left(t,x,y,\frac{1}{b}\tau,\xi,e^{\int b}\eta\right) \tag{0.0.2}$$

in local coordinates close to the boundary of M, where $\tilde{a}$ is a classical symbol of order m smooth up to $t = 0$ and $\int b$ stands for a primitive of b. It is a simple matter to see that symbol classes $\mathrm{Symb}^m(T^*M)$ form an algebra with respect to the Leibnitz product modulo symbols of order $-\infty$ in the interior of M. Our task is then to find a proper quantisation of the symbol algebra as an operator algebra over Sobolev spaces $H^{s,\gamma}(M)$. To any symbol $a \in \mathrm{Symb}^m(T^*M)$ we can assign a classical pseudodifferential operator $\mathrm{op}(a)$ of order m in the interior of M. In fact, $\mathrm{op}(a)$ is defined up to smoothing operators in the interior of M. Set $\varphi_b = \omega$, $\varphi_i = 1 - \omega$ and pick C^∞ functions ψ_b, ψ_i on M such that ψ_b is supported close to the boundary of M, ψ_i is supported away from the boundary of M and ψ_ν "covers" φ_ν. This yields $\mathrm{op}(a) = \varphi_b \mathrm{op}(a)\psi_b + \varphi_i \mathrm{op}(a)\psi_i$ up to a smoothing operator in the interior of M. The operator $\varphi_i \mathrm{op}(a)\psi_i$ is well-defined on the standard Sobolev spaces $H^s(M)$ and thus extends to the weighted spaces $H^{s,\gamma}(M)$. The operator $\varphi_b \mathrm{op}(a)\psi_b$ has in general no extension to $H^{s,\gamma}(M)$. To cope with those difficulties, the idea suggested by the first author in [Sch89, Sch90] and elaborated recently by Gil, Schulze and Seiler in [GSS96] is to reformulate, modulo smoothing operators in the interior of M, the operator $\mathrm{op}(a)$ in the collar neighbourhood of the boundary as a pseudodifferential operator along S with a symbol taking its values in an algebra of pseudodifferential operators in the fibres of M over

S. In fact, we obtain a bundle homomorphism

$$a_S : \pi^* H^{s,\gamma}(F_* N_+ \partial M) \to \pi^* H^{s-m,\gamma-m}(F_* N_+ \partial M)$$

over T^*S, where $\pi^* H^{s,\gamma}(F_* N_+ \partial M)$ denotes the pull-back of the Hilbert bundle $H^{s,\gamma}(F_* N_+ \partial M)$ under the canonical projection $\pi : T^*S \to S$. This function satisfies symbol estimates which include a group action in the fibres $H^{s,\gamma}((F_* N_+ \partial M)_y)$ given by

$$\kappa_\lambda u(t,x) = const(\lambda)\ u(\delta^{-1}(\log \lambda + \delta(t)), x), \quad \lambda \in \boldsymbol{R}_+, \tag{0.0.3}$$

where δ is a diffeomorphism of $\boldsymbol{R}_+$ onto $\boldsymbol{R}$ which agrees with $\int b$ for small $t > 0$. [1] The property of "being homogeneous" for operator-valued symbols always refers to a given group action in the fibres. When referred to the group action (0.0.3), the symbol a_S is not classical while it admits a principal symbol. The latter can be written (at least formally) as a limit

$$\sigma_S(\mathrm{op}(a_S))(y,\eta) = \lim_{\lambda \to \infty} \lambda^{-m} \kappa_{\lambda^{-1}} a_S(y, \lambda\eta) \kappa_\lambda, \quad (y,\eta) \in T^*S \setminus \{0\}, \tag{0.0.4}$$

and thus may be thought of as a homogeneous component of $a_S(y,\eta)$ of the highest degree m. In fact, the "twisted" homogeneity of $\sigma_S(\mathrm{op}(a_S))$ away from the zero section of T^*S is easily seen from (0.0.4). The advantage of using the representation $\varphi_b \mathrm{op}(a) \psi_b = \varphi_b \mathrm{op}(a_S) \psi_b$ modulo smoothing operators in the interior of M lies in the fact that the operator in the right side behaves properly on the spaces $H^s(H^{s,\gamma}(F_* N_+ \partial M))$ and hence extends to a continuous mapping $H^{s,\gamma}(M) \to H^{s-m,\gamma-m}(M)$, for all $s, \gamma \in \boldsymbol{R}$. It follows that the operator $A = \varphi_b \mathrm{op}(a_S) \psi_b + \varphi_i \mathrm{op}(a) \psi_i$ provides the desired quantisation of a symbol $a \in \mathrm{Symb}^m(M)$. Such operators form an operator algebra over weighted Sobolev spaces on M with two symbolic levels. The *inner symbol* $\sigma_M(A)$ of A is well-defined because the operators $\mathrm{op}(a_S)$ and $\mathrm{op}(a_S)$ are compatible as described above. In fact, $\sigma_M(A)$ is given by the principal symbol of the operator $\mathrm{op}(a)$ in the interior of M, of which the "compressed" version is defined up to $t = 0$. Yet another symbol $\sigma_S(A)$ of A is given by the principal symbol of $\mathrm{op}(a_S)$ (cf. (0.0.4)) and will be referred to as the *singular symbol* of A along S. The invertibility of the singular symbol (*Lopatinskii condition*) is a necessary condition for the Fredholm property of the operator in question. From this we deduce that the parametrix construction, when carried out on the symbolic level, invests the class of admissible operator-valued symbols along S. While the

[1] To achieve the property $\|\kappa_\lambda u\|_{H^{s,\gamma}((F_* N_+ \partial M)_y)} = \|u\|_{H^{s,\gamma}((F_* N_+ \partial M)_y)}$ for all $\lambda > 0$, put $const(\lambda) = \lambda^{-\gamma}$.

symbols a_S originating with symbols in $\mathrm{Symb}^m(M)$ are families of pseudodifferential operators on the semicylinders $(F_*N_+\partial M)_y$ with holomorphic symbols relative to the Fourier transform lifted to the semiaxis $\boldsymbol{R}_+$ by the diffeomorphism δ, their inverses have poles in the complex plane. It is therefore adequate to have all the inverse symbols from the very beginning in the class. This results in forbidding the weights $\gamma \in \boldsymbol{R}$ such that the line $\Gamma_\gamma = \{z \in \boldsymbol{C} : \Im z = \gamma\}$ meets a pole. Moreover, for fixed weight data, we add so-called *Green symbols* with asymptotics, $g(y, \eta)$. These are classical operator-valued symbols of order m defined via their mapping properties between Sobolev spaces in the fibres. The corresponding operators $\mathrm{op}(g) : H^{s,\gamma}(M) \to H^{\infty,\gamma-m}(M)$ are smoothing in the interior of M. Having thus completed the algebra of operator-valued symbols along S, we arrive at an operator algebra $\Psi\mathrm{Diff}^m(M; w)$, where w stands for weight data. In this way we obtain what is known as pseudodifferential operators on a manifold with fibred boundary. We also define pseudodifferential operators between weighted Sobolev spaces of sections of vector bundles V and $\tilde{V}$ over M and write $\Psi\mathrm{Diff}^m(V, \tilde{V}; w)$ for the corresponding operator algebra.

In Chapter 4 we give a brief exposition of elliptic boundary value problems on a manifold M with fibred boundary. In the 'category' of manifolds with fibred boundary, the boundary data are required to be constant along the fibers $F^{-1}(y)$, $y \in S$. Therefore, they are actually interpreted as data on the underlying manifold S which leads to the concept of an "edge problem" (cf. Schulze [Sch91, 3.3.4]). To be more precise, let $A = \mathrm{op}(a)$ be a typical differential operator of order m on M. We say that A is *elliptic* if the inner symbol $\sigma_M(A)$ is invertible away from the zero section of the (compressed) cotangent bundle T^*M. For the boundary points $(0, x, y)$, this means that

$$\sigma_M(A)\left(0, x, y, be^{\int b}\tau, e^{\int b}\xi, \eta\right) \neq 0 \quad \text{if} \quad (\tau, \xi, \eta) \neq 0 \qquad (0.0.5)$$

(cf. (0.0.2)). Under this last condition, for each point $y \in S$ there is a discrete set $D(y)$ in the complex plane such that

$$\sigma_S(A)(y, \eta) \colon H^{s,\gamma}((F_*N_+\partial M)_y) \to H^{s-m,\gamma-m}((F_*N_+\partial M)_y)$$

is a Fredholm operator for all $s \in \boldsymbol{R}$ and for all $\gamma \in \boldsymbol{R}$ with $\Gamma_\gamma \cap D(y) = \emptyset$, unless $\eta = 0$. It is now a topological condition on the original operator A that there exist smooth vector bundles W and $\tilde{W}$ over S and homogeneous operator-valued symbols p, t and b on T^*S such that

$$\begin{pmatrix} \sigma_S(A) & p \\ t & b \end{pmatrix} : \begin{matrix} \pi^* H^{s,\gamma}(F_*N_+\partial M) \\ \oplus \\ \pi^* W \end{matrix} \to \begin{matrix} \pi^* H^{s-m,\gamma-m}(F_*N_+\partial M) \\ \oplus \\ \pi^* \tilde{W} \end{matrix} \qquad (0.0.6)$$

is an isomorphism outside the zero section of T^*S, for each $s \in \boldsymbol{R}$. This isomorphism can be used as the singular symbol of some boundary value problem associated with A. By such a problem we mean an operator

$$\mathcal{A} = \varphi_b \begin{pmatrix} A_S & P \\ T & B \end{pmatrix} \psi_b + \varphi_i \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \psi_i \qquad (0.0.7)$$

where $A \in \Psi\mathrm{Diff}(M; w)$ and A_S, P, T, B are pseudodifferential operators along S whose symbols are bundle homomorphisms as in (0.0.6). Thus, P has the meaning of a *potential* or *corestriction operator* with respect to S, T of a *trace operator*, and B is nothing else than a pseudodifferential operator between sections of vector bundles W and $\tilde{W}$ along S. It is worth pointing out that these last three operators are "classical." We require the operators A_S and A to be compatible close to the boundary, hence $\mathcal{A}$ has a well-defined principal inner symbol given by $\sigma_M(\mathcal{A}) = \sigma_M(A)$. Moreover, $\mathcal{A}$ has a principal singular symbol $\sigma_S(\mathcal{A})$ defined by (0.0.6). In this way we obtain what looks like the operators in Boutet de Monvel's theory of boundary value problems and encompasses this theory when F is a point fibration. Let $\mathrm{Alg}^m(M; W, \tilde{W}; w)$ denote the resulting operator algebra, w being weight data. Each operator $\mathcal{A} \in \mathrm{Alg}^m(M; W, \tilde{W}; w)$ defines a continuous linear mapping

$$\mathcal{A}\colon \begin{array}{c} H^{s,\gamma}(M) \\ \oplus \\ H^s(W) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(M) \\ \oplus \\ H^{s-m}(\tilde{W}) \end{array}, \qquad (0.0.8)$$

the weight exponent $\gamma \in \boldsymbol{R}$ being from the weight data w. An operator $\mathcal{A}$ is said to be *elliptic* if both symbol mappings $\sigma_M(\mathcal{A})$ and $\sigma_S(\mathcal{A})$ are isomorphisms away from the zero sections of T^*M and T^*S, respectively. The main result of Chapter 4 is that the ellipticity of $\mathcal{A}$ is equivalent to the Fredholm property of the corresponding boundary value problem (cf. (0.0.8)), for all $s \in \boldsymbol{R}$. Namely, under ellipticity there is an operator $\mathcal{P} \in \mathrm{Alg}^{-m}(M; \tilde{W}, W; w)$ such that both $\mathcal{P}\mathcal{A} - 1$ and $\mathcal{A}\mathcal{P} - 1$ are compact operators in the corresponding Hilbert spaces. Such an operator is called the *regularizer* or *parametrix* of the boundary problem $\mathcal{A}$. The compactness is obtained as a consequence of the fact that $\mathcal{P}\mathcal{A} - 1$ and $\mathcal{A}\mathcal{P} - 1$ are smoothing operators improving also the weight exponents γ and $\gamma - m$, respectively. Moreover, the nature of the parametrix $\mathcal{P}$ makes it obvious that $\mathcal{A}$ behaves properly in weighted Sobolev spaces with asymptotics $H^{s,\gamma}_{\mathrm{as}}(M) \oplus H^s(W)$, 'as' being an asymptotic type subordinated to w. The concept of function spaces with continuous asymptotics was first introduced by Schulze [Sch88b]. For a recent account of the theory the reader can consult Dorschfeldt [Dor95] and Schulze [Sch97b]. We also consider boundary value problems for operators in $\Psi\mathrm{Diff}^m(V,\tilde{V};w)$ and write $\mathrm{Alg}^m(V,\tilde{V};W,\tilde{W};w)$ for the corresponding operator algebra. As mentioned, a bundle $V \in \mathrm{Vect}(M)$ is required to respect the boundary fibration, whence $F^*(V|_S) = V|_{\partial M}$. In particular, the fibres of V are constant along the fibres of F, i.e., $V|_{F^{-1}(y)} = F^{-1}(y) \times V_y$ is a trivial bundle, for each $y \in S$. Thus, the singular symbol of an operator $\mathcal{A} \in \mathrm{Alg}^m(V,\tilde{V};W,\tilde{W};w)$ is a well-defined bundle homomorphism

$$\sigma_S(\mathcal{A}):\ \pi^* \begin{array}{c} H^{s,\gamma}(F_*N_+\partial M)\otimes V|_S \\ \oplus \\ W \end{array} \to \pi^* \begin{array}{c} H^{s-m,\gamma-m}(F_*N_+\partial M)\otimes \tilde{V}|_S \\ \oplus \\ \tilde{W} \end{array}$$

over T^*S. Yet another important point to note here is the order-reducing isomorphisms within the algebra. More precisely, given any $m \in \boldsymbol{R}$ and smooth vector bundles V and W over M and S, respectively, there is an elliptic operator $\mathcal{R}^m_{V,W} \in \mathrm{Alg}^m(V,V;W,W;w)$ such that

$$\mathcal{R}^m_{V,W}: \begin{array}{c} H^{s,\gamma}(V) \\ \oplus \\ H^s(W) \end{array} \to \begin{array}{c} H^{s-m,\gamma-m}(V) \\ \oplus \\ H^{s-m}(W) \end{array} \tag{0.0.9}$$

is an isomorphism for all $s \in \boldsymbol{R}$. One can even choose $\mathcal{R}^m$ without potential and trace conditions.

Chapter 5 deals with elliptic complexes of boundary value problems on a manifold M with fibred boundary. This idea goes back at least as far as Schulze [Sch88a] where elliptic complexes on manifolds with conical singularities are studied. We consider complexes of the form

$$L^{\cdot}:\ 0 \longrightarrow \begin{array}{c} H^{s,\gamma}(V^0) \\ \oplus \\ H^s(W^0) \end{array} \stackrel{d_0}{\longrightarrow} \begin{array}{c} H^{s_1,\gamma_1}(V^1) \\ \oplus \\ H^{s_1}(W^1) \end{array} \stackrel{d_1}{\longrightarrow} \dots \stackrel{d_{N-1}}{\longrightarrow} \begin{array}{c} H^{s_N,\gamma_N}(V^N) \\ \oplus \\ H^{s_N}(W^N) \end{array} \longrightarrow 0, \tag{0.0.10}$$

where

$$\begin{array}{rcl} s_i & = & s - m_0 - \dots - m_{i-1}, \\ \gamma_i & = & \gamma - m_0 - \dots - m_{i-1} \end{array}$$

and $d_i \in \mathrm{Alg}^{m_i}(V^i,V^{i+1};W^i,W^{i+1};w_i)$. The weight data w_i are assumed to be compatible in order that all the compositions $d_{i+1}\circ d_i$ be defined. In fact, the value $\gamma \in \boldsymbol{R}$ is specified by the weight data while $s \in \boldsymbol{R}$ can be chosen arbitrary. As described above, we associate two symbol sequences with complex (0.0.10). The first of the two is the sequence of inner symbols compressed close to the boundary, namely

$$0 \longrightarrow \pi^*V^0 \stackrel{\sigma_M(d_0)}{\longrightarrow} \pi^*V^1 \stackrel{\sigma_M(d_1)}{\longrightarrow} \dots \stackrel{\sigma_M(d_{N-1})}{\longrightarrow} \pi^*V^N \longrightarrow 0, \tag{0.0.11}$$

$\pi : T^*M \to M$ being the canonical projection. It follows from the multiplicative property of the symbol mapping that (0.0.11) is a complex. The exactness of this complex away from the zero section of T^*M is a straightforward generalisation of ellipticity in the interior of the manifold M up to its boundary. In general, this is not sufficient for (0.0.10) to be a Fredholm complex. To cope with those difficulties, we introduce yet another symbol sequence which controls the lack of the Fredholm property on the boundary of M. This is

$$0 \longrightarrow \pi^* \begin{array}{c} H^0 \otimes V^0|_S \\ \oplus \\ W^0 \end{array} \xrightarrow{\sigma_S(d_0)} \pi^* \begin{array}{c} H^1 \otimes V^1|_S \\ \oplus \\ W^1 \end{array} \xrightarrow{\sigma_S(d_1)} \dots \xrightarrow{\sigma_S(d_{N-1})} \pi^* \begin{array}{c} H^N \otimes V^N|_S \\ \oplus \\ W^N \end{array} \longrightarrow 0, \tag{0.0.12}$$

where $H^i = H^{s_i,\gamma_i}(F_* N_+ \partial M)$ and $\pi : T^*S \to S$ is the canonical projection. Once again (0.0.12) is a complex. The principal significance of this complex is that its exactness outside the zero section of T^*S is a generalisation of Lopatinskii's condition for classical boundary value problems. We call a complex (0.0.10) *elliptic* if both (0.0.11) and (0.0.12) are exact sequences away from the zero sections of the corresponding cotangent bundles. To prove that any elliptic complex is a Fredholm complex, i.e., it has finite-dimensional cohomology $H^i(L^\cdot)$, the usual way is to reduce the matter to a single elliptic operator. For this purpose, one uses the Laplacians $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$ of the complex. However, for an operator $\mathcal{A} \in \mathrm{Alg}^m(V, \tilde{V}; W, \tilde{W}; w)$, only the transpose $\mathcal{A}'$ is well-defined within the algebra. We can conjugate it by conjugate linear isomorphisms of bundles $\star$ (cf. Hodge's "star" operator), thus arriving at the "formal adjoint" operator $\mathcal{A}^* = \star^{-1} \mathcal{A}' \star$. But $\mathcal{A}^*$ is defined on the weighted Sobolev spaces of opposite weights, hence the compositions $\mathcal{A}^*\mathcal{A}$ and $\mathcal{A}\mathcal{A}^*$ are not defined unless $\mathcal{A}$ is of order zero. To cope with these difficulties, we invoke order-reducing isomorphisms $\mathcal{R}_i = \mathcal{R}^{\gamma_i}_{V_i,W_i}$ (cf. (0.0.9)):

$$\begin{array}{ccccccccc}
L^\cdot : \ 0 \longrightarrow & \begin{array}{c} H^{s,\gamma}(V^0) \\ \oplus \\ H^s(W^0) \end{array} & \xrightarrow{d_0} & \begin{array}{c} H^{s_1,\gamma_1}(V^1) \\ \oplus \\ H^{s_1}(W^1) \end{array} & \xrightarrow{d_1} \dots \xrightarrow{d_{N-1}} & \begin{array}{c} H^{s_N,\gamma_N}(V^N) \\ \oplus \\ H^{s_N}(W^N) \end{array} & \longrightarrow 0 \\
\big\downarrow \mathcal{R} & \big\downarrow \mathcal{R}_0 & & \big\downarrow \mathcal{R}_1 & & \big\downarrow \mathcal{R}_N & \\
\tilde{L}^\cdot : \ 0 \longrightarrow & \begin{array}{c} H^{s-\gamma,0}(V^0) \\ \oplus \\ H^{s-\gamma}(W^0) \end{array} & \xrightarrow{\tilde{d}_0} & \begin{array}{c} H^{s-\gamma,0}(V^1) \\ \oplus \\ H^{s-\gamma}(W^1) \end{array} & \xrightarrow{\tilde{d}_1} \dots \xrightarrow{\tilde{d}_{N-1}} & \begin{array}{c} H^{s-\gamma,0}(V^N) \\ \oplus \\ H^{s-\gamma}(W^N) \end{array} & \longrightarrow 0.
\end{array} \tag{0.0.13}$$

The operators $\tilde{d}_i$ are defined so as to make diagram (0.0.13) commutative.

Then, $\tilde{d}_i = \mathcal{R}_{i+1} d_i \mathcal{R}_i^{-1}$ belongs to $\mathrm{Alg}^0(V^i, V^{i+1}; W^i, W^{i+1}; \tilde{w}_i)$ and satisfies $\tilde{d}_{i+1}\tilde{d}_i = 0$, for each i. It follows that $\tilde{L}^{\cdot}$ is a complex of operators of order zero on the manifold M. Moreover, this complex is elliptic, which is clear from the ellipticity of the operators $\mathcal{R}_i$ and the fact that the cohomology of complex (0.0.12) is independent of s. We develop the Hodge theory for the complex $\tilde{L}^{\cdot}$, thus obtaining a very special parametrix $\left(\tilde{\mathcal{P}}_i\right)$ up to operators of finite rank being orthogonal projections onto spaces of harmonic sections. Then, $\mathcal{P}_i = \mathcal{R}_{i-1}^{-1}\tilde{\mathcal{P}}_i\mathcal{R}_i$ provides the desired parametrix to the original complex $L^{\cdot}$. We have

$$\mathcal{P}_{i+1} d_i u + d_{i-1}\mathcal{P}_i u = u - \mathcal{S}_i u \quad \text{for all} \quad u \in L^i, \tag{0.0.14}$$

$\mathcal{S}_i$ being a smoothing operator in the algebra improving also the weight exponent (*Green operator*). As but one consequence of this we conclude that the cohomology of an elliptic complex is finite-dimensional and independent of $s \in \boldsymbol{R}$.

In Chapter 6 we introduce the Lefschetz fixed point theorem for a manifold M with fibred boundary. Let $L^{\cdot}$ be an elliptic complex of boundary value problems on M as in (0.0.10). By an endomorphism of the complex $L^{\cdot}$ is meant a family $E = (E_i)$ of linear mappings $E_i : L^i \to L^i$ such that $d_i E_i = E_{i+1} d_i$. Then E induces an endomorphism $(HE)_i$ of the cohomology $H^i(L^{\cdot})$, for every $i = 0, 1, \ldots, n$. As described above, these are finite-dimensional vector spaces, and so the traces $\mathrm{tr}\,(HE)_i$ are well-defined. We introduce the Lefschetz number of E by $L(E) = \sum_{i=0}^N (-1)^i \mathrm{tr}\,(HE)_i$. If $\tilde{L}^{\cdot}$ is a complex homotopically equivalent to $L^{\cdot}$, then each endomorphism E of $L^{\cdot}$ induces a unique endomorphism $\tilde{E}$ of $\tilde{L}^{\cdot}$ via the mappings establishing the equivalence. Moreover, the Lefschetz number of $\tilde{E}$ is equal to that of E. In particular, the sequence $\tilde{E}_i = \mathcal{R}_i E_i \mathcal{R}_i^{-1}$ gives us an endomorphism of complex (0.0.13) and $L(\tilde{E}) = L(E)$. We are thus reduced to evaluating the Lefschetz number $L(E)$ for elliptic complexes of zero order operators but perhaps for "twisted" endomorphisms. This is disadvantageous in our case because we are interested in evaluating the Lefschetz number for geometric endomorphisms of $L^{\cdot}$. Suppose then that f is a smooth mapping of the underlying manifold M preserving the fibration $F : \partial M \to S$ of the boundary. This means that $f(S) \subset S$ and $f(F^{-1}(y)) \subset F^{-1}(f(y))$ for all $y \in S$. Let we be also given smooth bundle homomorphisms

$$\begin{array}{rcl} h_{V^i} : & f^* V^i & \to V^i, \\ h_{W^i} : & f^* W^i & \to W^i, \end{array}$$

and set $h_{V^i \oplus W^i} = h_{V^i} \oplus h_{W^i}$. Using the same notation for the induced mappings on sections, we can then define linear mappings $E_i : L^i \to L^i$ as the composition

$$\begin{matrix} H^{s_i,\gamma_i}(V^i) \\ \oplus \\ H^{s_i}(W^i) \end{matrix} \xrightarrow{f^*} \begin{matrix} H^{s_i,\gamma_i}(f^*V^i) \\ \oplus \\ H^{s_i}(f^*W^i) \end{matrix} \xrightarrow{h_{V^i\oplus W^i}} \begin{matrix} H^{s_i,\gamma_i}(V^i) \\ \oplus \\ H^{s_i}(W^i) \end{matrix} .$$

Thus, if $u = u_1 \oplus u_2 \in L^i$, then $E_i u$ is given by $h_{V^i} u_1(f(x)) \oplus h_{W^i} u_2(f(y))$. The point to note here is that $u_1(f(x)) \oplus u_2(f(y)) \in V^i_{f(x)} \oplus W^i_{f(y)}$, but $h_{V^i \oplus W^i}$ takes us back to $V^i_x \oplus W^i_y$. If further $d_i E_i = E_{i+1} d_i$, then the family (E_i) defines an endomorphism of the complex $L^{\cdot}$. An endomorphism of this type we call a *geometric* endomorphism. Returning now to the mapping f, we need further specifications of its fixed points on the boundary. Let p be such a point, i.e., $p \in \partial M$ and $f(p) = p$. Then $df(p)$ induces a mapping $d_{\partial M} f(p) \colon T_p \partial M \to T_p \partial M$ and hence a mapping

$$(df/d_{\partial M} f)\,(p) : \ N_p \partial M \to N_p \partial M \qquad (0.0.15)$$

of the quotient space $N_p \partial M = T_p M / T_p \partial M$. The latter can be identified with the normal space to the boundary at the point p. As $N_p \partial M$ is one-dimensional, (0.0.15) can be regarded as the multiplication by a number $q(p) \in \boldsymbol{R}$. It is clear that $q(p) \geq 0$, for M is invariant under f. Moreover, if p is a simple fixed point of f, then $q(p) \neq 1$. A simple fixed point $p \in \partial M$ is said to be *attracting*, if $q(p) < 1$, and *repulsing*, if $q(p) > 1$. Denote by $\mathrm{Fix}(f, M \setminus \partial M)$ the set of all interior fixed points of f, by $\mathrm{Fix}^{(a)}(f, \partial M)$ the set of all attracting boundary fixed points of f and by $\mathrm{Fix}(f, S)$ the set of all fixed points of f on S. The main result of Chapter 6 states that, if f is a smooth mapping of M with only simple fixed points, then

$$L(E) = \sum_{\substack{p \in \mathrm{Fix}(f, M \setminus \partial M) \\ \cup \mathrm{Fix}^{(a)}(f, \partial M)}} \frac{\sum\limits_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{V^i}(p)}{|\det(1 - df(p))|} + \sum_{p \in \mathrm{Fix}(f,S)} \frac{\sum\limits_{i=0}^{N} (-1)^i \mathrm{tr}\, h_{W^i}(p)}{|\det(1 - d_S f(p))|}. \qquad (0.0.16)$$

Note that if M is a closed compact manifold, then the second term in the right side of (0.0.16) is absent and this equality becomes the Atiyah-Bott formula (cf. (0.0.1)). The proof of (0.0.16) follows in the large the scheme suggested in [AB67], though we meet some difficulties caused by the boundary. To cope with them we use the material on pseudodifferential operators in the algebra $\mathrm{Alg}^m(V, \tilde{V}; W, \tilde{W}; w)$. The proof falls naturally into three parts. We first show the formula

$$\sum_{i=0}^{N} (-1)^i \mathrm{tr}\,(HE)_i = \sum_{i=0}^{N} (-1)^i \mathrm{tr}\, E_i \qquad (0.0.17)$$

for those endomorphisms E of the complex $L^{\cdot}$ which are given by Green operators in our algebra. Then, with the help of the parametrix of $L^{\cdot}$, we derive an approximation $E^{(\varepsilon)}$ of a geometric endomorphism E by Green endomorphisms in the strong operator topology. Such an approximation is constructed via approximations of the identity operators in $H^{s,\gamma}(V^i)$ and $H^s(W^i)$ by Green operators. For $H^{s,\gamma}(V^i)$, this is done locally in a neighbourhood of a boundary point by means of convolution operators $R^{(\varepsilon)}_{V^i}$ with a delta-like sequence consisting of functions with supports in the half-space $t < 0$, provided the coordinates (t, x, y) are so chosen that $M = \{t \geq 0\}$. Using such an approximation results in the cancellation of contributions of the repulsing boundary fixed points. This idea goes back at least as far as Brenner and Shubin [BS81]. Finally, since equality (0.0.17) holds for each endomorphism $E^{(\varepsilon)}$, $\varepsilon > 0$, and $\lim_{\varepsilon \to 0} L(E^{(\varepsilon)}) = L(E)$, we are reduced to evaluating the limit in the right-hand side of (0.0.17), i.e.

$$\lim_{\varepsilon \to 0} \sum_{i=0}^{N} (-1)^i \operatorname{tr} E_i^{(\varepsilon)}.$$

For this step we make essential use of the nature of pseudodifferential operators on M.

Chapter 7 presents a general approach to fixed point theory which applies to complexes on a manifold with fibred boundary whose differentials are local operators in the algebra $\mathrm{Alg}^m(V, \tilde{V}; W, \tilde{W}; w)$. Since the potential operators are non-local by the very nature, we consider elliptic complexes (0.0.10) with

$$d_i = \begin{pmatrix} -A_i & 0 \\ T_i & B_i \end{pmatrix}, \qquad (0.0.18)$$

where $A_i \in \mathrm{Diff}^{m_i}(V^i, V^{i+1})$, $B_i \in \mathrm{Diff}^{m_i}(W^i, W^{i+1})$ and T_i is the composition of a differential operator of order m_i along S with the restriction operator to S. In fact operators (0.0.18) go beyond the range of the algebra $\mathrm{Alg}^m(V, \tilde{V}; W, \tilde{W}; w)$, however they can be handled in much the same framework. Such complexes arise as cones of cochain mappings

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H^{s,\gamma}(V^0) & \xrightarrow{A_0} & H^{s_1,\gamma_1}(V^1) & \xrightarrow{A_1} \ldots \xrightarrow{A_{N-1}} & H^{s_N,\gamma_N}(V^N) \longrightarrow 0 \\
 & & \downarrow & & \downarrow T_0 & & \downarrow T_1 & & \downarrow \\
0 & \longrightarrow & H^s(W^0) & \xrightarrow{B_0} & H^{s_1}(W^1) & \xrightarrow{B_1} & H^{s_2}(W^2) & \xrightarrow{B_2} \ldots \longrightarrow & 0
\end{array}$$

(cf. Dold [Dol72]) and were first investigated by Dynin [Dyn72]. Suppose $f : M \to M$ be a smooth mapping possessing a lift E to the complex $L^{\cdot}$. Formula (0.0.16) expresses the Lefschetz number of E as $L(E) = \sum_p \iota(p)$, where the sum runs over the fixed points p of the mapping f and $\iota(p)$ are infinitesimal invariants of f at p. It is natural to ask whether their local index can be explained as a special case of a *cohomological* formula which always makes sense for isolated fixed points, as in the classical theorem where sign $\det(1 - df(p)) = \deg(1 - f, p)$. As mentioned, Toledo [Tol73] gave an exposition of the fixed point theory on a closed compact manifold which applied to isolated fixed points gives both the Atiyah-Bott formula and cohomological formulas. This method is based on a classical formula of de Rham [dR55, §33] which expresses intersection numbers in Riemannian manifolds in terms of the Green kernel. We develop the theory for elliptic complexes (0.0.18) on manifolds with fibred boundary. It leads to an integral representation for the Lefschetz number from which formula (0.0.16) can be derived by some delicate but quite elementary analysis. Moreover assuming that the Poincaré lemma holds, i.e., the local cohomology of the complex $L^\cdot$ is concentrated at the step 0, we derive a cohomological expression for the index of an isolated fixed point. For simple fixed points this reduces of course to the infinitesimal description. Finally, we apply the general theory to the classical elliptic complexes. For the de Rham complex it just gives the classical formula. But for the relative de Rham complex we arrive at a new formula which in the case of point fibrations (i.e., $S = \partial M$) was shown by Brenner and Shubin [BS81].

In the larger program of analysis on manifolds with singularities there always is a certain freedom in the choice of the algebra one intends to work with. Let us recall that, in 1967, Kondrat'ev published his paper [Kon67] on elliptic differential operators on manifolds with conical singularities. Although concrete examples had already been treated by other authors in the early '60s, he was the first to treat these questions systematically. The calculation of the asymptotics of solutions near conical points by means of the meromorphic inverses of the conormal symbols is often referred to today as Kondrat'ev technique. Another version of the analysis of operators near conical singularities in the boundaryless case was developed by Plamenevskii [Pla89]. He paid much attention to the point of view of C^*-algebras and considered also pseudodifferential operators with discontinuous symbols. This approach was extended by Derviz [Der90] to the case of boundary value problems. The desingularisation of closed compact manifolds with singularities leads to compact smooth manifolds with fibred boundary. The fibration of the boundary gives rise to a fibration of the manifold M itself close to the boundary via a collar neighbourhood of ∂M. The fibre of M is therefore the semicylinder over a smooth closed compact manifold. Then, the local algebra near the boundary is determined by the fibration of M, which is the only relevant point here. On the other hand, a compact manifold with an edge S on the boundary, when "desingularized" close to S, has the structure of a bundle over S whose fibre is the semicylinder over a compact manifold with boundary, X. Then, the local algebra near S is organized as the Fourier calculus along S with operator-valued symbols taking their values in the algebra of pseudodifferential operators over $\boldsymbol{R}_+ \times X$. In turn, this latter algebra is organized as the Mellin calculus along the semiaxis with operator-valued symbols taking their values in the Boutet de Monvel algebra on X (cf. Schrohe and Schulze [SS94a, SS95]). The strategy for obtaining the operator theories on manifolds with higher singularities is described in the books of the first author [Sch91, Sch94, Sch97b]; it relies on the concepts of "conification" and "edgification" of an operator algebra. The paper [Mel81] by Melrose has established another approach to the analysis of pseudodifferential and Fourier integral operators with totally characteristic symbols ('b'-calculus, 'b' indicating to 'boundary'). Applying these techniques, Melroze and Mendoza [MA83] established a Fredholm theory including ellipticity and parametrix construction for an algebra of pseudodifferential operators on closed compact manifolds with conical singularities. In recent years, Melrose *et al.* investigated some algebras of 'cusp' pseudodifferential operators on compact manifolds with boundary (cf. Mazzeo and Melrose [MM], Melrose and Nistor [MN96]). In [MN96], the Hochschild and cyclic homology groups are computed for the 'cusp' algebra. The index functional for this algebra is interpreted as a Hochschild 1-cocycle and evaluated in terms of extensions of the trace functional on the two natural ideals, corresponding to the two filtrations by interior order and vanishing degree at the boundary, together with the exterior derivations of the algebra. This leads to an index formula which is a pseudodifferential extension of that of Atiyah, Patodi and Singer for Dirac operators. It is to be expected that the Lefschetz fixed point theory makes sense also in these algebras. As above, the desingularisation of compact manifolds with singularities on the boundary leads to compact manifolds with piecewise smooth boundary. (Strictly speaking, by a *desingularisation* is just meant the resolution of an arbitrary singularity by *transversal intersections*.) Moreover, smooth pieces of the boundary inherit different fibrations induced by the desingularisation. This leads to operator algebras more intricate than (0.0.8); they carry function spaces over various fibred components of the boundary. The important point to note here is the proper definition of being attracting for boundary fixed point lying in the intersection of several boundary hypersurfaces, each of them being smooth. This follows automatically from the construction of the approximation of the identity operator in spaces of smooth functions on M by Green operators. Namely, a boundary fixed point p is said to be *attracting* if it is attracting with respect to each boundary hypersurface which meets p. Then, a kind of (0.0.16) is still true. In particular, for the

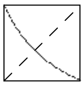
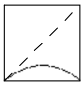
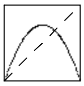
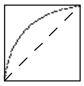
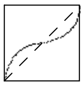
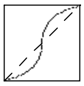
f						
$L(f)$	1	1	1	1	1	1
$\sum_{p\in \mathrm{Fix}(f,M\setminus\partial M)} \iota(p)$	1	0	1	0	1	-1
$\sum_{p\in \mathrm{Fix}^{(a)}(f,\partial M)} \iota(p)$	0	1	0	1	0	2

Fig. 0.1: The Lefschetz number of a mapping $f\colon [0,1] \to [0,1]$.

de Rham complex on a compact manifold with piecewise smooth boundary, this gives

$$L(f) = \sum_{\substack{p\in \mathrm{Fix}(f,M\setminus\partial M)\\ \cup \mathrm{Fix}^{(a)}(f,\partial M)}} \text{sign det}(1 - df(p)), \tag{0.0.19}$$

f being a smooth mapping of M of general position. Certainly, formula (0.0.19) can be deduced from the general Lefschetz fixed point theorem for CW-complexes (cf. Proposition 6.6 in Dold [Dol72, Ch.7]). However, this theorem gives no explicit description of the contribution of a non-interior fixed point while permitting f with arbitrary fixed sets. It is interesting to have look at the right-hand side of (0.0.19) in the one-dimensional case where $M = [0,1]$ (cf. Brenner and Shubin [BS81]). The identity boundary fibration we choose corresponds to the setting of a manifold with boundary. Figure 0.1 presents the graphs of 6 possible mappings $f : [0,1] \to [0,1]$ preserving the boundary. In fact these are different classes of mappings from the point of view of classification of boundary fixed points. In the first example there is no boundary fixed points, in the second example there is one attracting boundary fixed point, in the third example there is one repulsing boundary fixed point, in the fourth example there are one attracting boundary fixed point and one repulsing boundary fixed point, in the fifth example there are two repulsing boundary fixed points and in the sixth example there are two attracting fixed points. The table shows the summary contributions of interior fixed points and of attracting boundary fixed points. Formula (0.0.19) then means that the Lefschetz number of f (cf. the first line) is equal to the sum of the second and third lines. On the other hand, the boundary fibration with one-point base, $F : \partial M \to \{0\}$, corresponds to the setting of a stretched manifold with a conical point. In this case the first mapping in Fig. 0.1 is not compatible with the boundary fibration and is thus forbidden.

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