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Statement of Originality

This thesis contains no material which has been accepted for the award of any other degree or diploma at any other university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text.
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Dennis Koh
Potsdam, October 26, 2007

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Abstract

Orbits of charged particles under the effect of a magnetic field are mathematically described by magnetic geodesics. They appear as solutions to a system of ordinary differential equations of second order. But we are only interested in periodic solutions. To this end, we study the corresponding system of (nonlinear) parabolic equations for closed magnetic geodesics and, as a main result, eventually prove the existence of long time solutions.

Zusammenfassung

In der Mathematik werden Bahnen von geladenen Teilchen unter dem Einfluss eines Magnetfeldes durch magnetische Geodätische beschrieben. Diese Bahnen ergeben sich als Lösungen eines Systems von gewöhnlichen Differentialgleichungen zweiter Ordnung. Wir interessieren uns nur für periodische Lösungen. Dazu studieren wir ein System von parabolischen (nichtlinearen) Differentialgleichungen (die Evolutiongleichung für geschlossene magnetische Geodätische) und beweisen die Existenz von Langzeitlösungen.

Contents

Chapter 1

Introduction

In this work we investigate a certain evolution equation, which is motivated from String theory. Namely, let (Σ, g) and (M, G) be Riemannian manifolds, Σ be compact and oriented, $p = \dim(\Sigma)$. Furthermore let $Z \in \Gamma(\mathrm{Hom}(\Lambda^p TM, TM)) \cong \Gamma(\Lambda^p T^*M \otimes TM)$ be a tensor field such that $\Omega := G(\cdot, Z(\cdot))$ is a closed $(p+1)$-form. For a map $\varphi \in C^2(\Sigma, M)$, consider the nonlinear elliptic partial differential equation

$$\tau(\varphi) = Z((d\varphi)^{\underline{p}}(\mathrm{vol}_g^\sharp)). \tag{1.1}$$

In terms of a positively oriented local orthonormal frame $\{e_i\}$ of Σ, $\tau(\varphi)$ and $(d\varphi)^{\underline{p}}(\mathrm{vol}_g^\sharp)$ are given by $\tau(\varphi) = (\nabla_{e_i} d\varphi)(e_i)$ and $(d\varphi)^{\underline{p}}(\mathrm{vol}_g^\sharp) = d\varphi(e_1) \wedge \ldots \wedge d\varphi(e_p)$, respectively. $(d\varphi)^{\underline{p}}(\mathrm{vol}_g^\sharp)$ can be interpreted as vectorial volume element of Σ, being pushed forward to M. Now, if $p \geq 1$ is a positive integer and Σ is connected, then a solution to equation (1.1) describes the orbit of a closed $(p-1)$-brane under the effect of a field strength Ω. The tensor field $Z : M \to \mathrm{Hom}(\Lambda^p TM, TM)$ can be interpreted as a physical force influencing the motion of the closed $(p-1)$-brane. In String theory a p-brane is an "extended object" of dimension p. That is a 0-brane corresponds to a particle, a 1-brane to a string, 2-brane to a membrane etc. In the case $p = 1$, for a map $\gamma : \Sigma \cong S^1 \to M,\ s \mapsto \gamma(s)$, equation (1.1) reduces to the *equation for magnetic geodesics*

$$\frac{\nabla}{\partial s}\gamma' = Z(\gamma'). \tag{1.2}$$

Here, $\gamma'(s)$ denotes the tangent vector of the curve γ at the point $s \in S^1$. In this case the equation describes the orbit of a charged particle under the effect of a magnetic field. Z can be interpreted as Lorentz force.

A lot of people have investigated the following question: On which conditions can the existence of closed magnetic geodesics be shown? They used topological methods as Morse-Novikov theory and Lyusternik Shnirel'man theory for example; see [1], [4], [6], [22] and the references therein. In Chapter 2 we will derive that very one equation of motion (1.1) by the principle of stationary action from a certain $U(1)$-valued functional arising in String theory and discuss this equation at the end of that chapter by an example. In Chapter 3 we will derive the second variational formula of that String functional.

To the elliptic PDE (1.1) one can associate an evolution equation and study the long time behavior of its flow. Namely, we consider, for a map $\varphi : \Sigma \times [0,T) \to M$, setting $\varphi_t(x) = \varphi(x,t)$, the initial value problem of a system of nonlinear parabolic partial differential equations

$$(1.3) \qquad \begin{cases} \tau(\varphi_t)(x) = Z((d\varphi_t)^{\underline{p}})(x) + \frac{\partial \varphi_t}{\partial t}(x), & (x,t) \in \Sigma \times (0,T), \\ \varphi(x,0) = f(x), \end{cases}$$

where $\tau(\varphi_t) = \operatorname{trace}(\nabla d\varphi_t)$ and $f \in C^\infty(\Sigma, M)$ is a map given as initial condition. One hopes that this problem possesses a solution for $T = \infty$ and that the limit map $\varphi_\infty = \lim_{t\to\infty} \varphi_t : \Sigma \to M$, provided that it exists, is a solution to (1.1). We will show that it depends on the initial condition f whether the limit map φ_∞, provided that it exists, satisfies equation (1.1) or not. In $\dim(\Sigma) = p = 1$ the above parabolic PDE (1.3) is called the *Evolution Equation for Magnetic Geodesics*. A general introduction to nonlinear evolution equations and methods to prove existence of long time solutions are given in [12]. The method to find a solution to an elliptic PDE by solving an associated parabolic (evolution) equation has been applied by Eells and Sampson to prove the existence of harmonic maps. In the literature it is known as *heat flow method*. We discuss this method in Chapter 4 and provide some Bochner-type formulas for later purposes. Good references to this topic are [7], [19] and [23].

In Chapter 5 we will show short time existence of the flow. The main ingredient of the proof is the Inverse Function Theorem from functional analysis. Regardless of the dimension and the curvature of Σ and M, short time existence can always be guaranteed. For the long time existence the Bochner formulas come into play. We will use them in Chapter 6 to prove long time existence of the flow in $\dim(\Sigma) = 1$. Curvature assumptions on M and the maximum principle are used to obtain good a priori estimates from the Bochner formulas for the energy densities of a solution to the initial value problem (1.3). In this way the growth rate of the solutions, as time t increases, is controlled and blow ups are prevented.

In the Appendix we fix the notation, definitions and provide some basic facts about gerbes and important analytical tools that are used in this work.

General Assumptions. All appearing manifolds, maps and tensors are assumed to be smooth unless otherwise stated. Also we explicitely note that all manifolds are assumed to be without boundary. Furthermore we will frequently make use of "Einstein's sum convention": All sum signs are omitted if an index appears twice regardless of the position of the indices. Then one has to think of these sums to be performed. For example, $a_i b_i$ is to mean $\sum_i a_i b_i$ and $R^l_{kij} g_{ln} g^{km}$ is to mean $\sum_{k,l} R^l_{kij} g_{ln} g^{km}$. Deviations of this convention will be made explicit by writing out the sum signs.

Chapter 2

First variation formula

In this chapter we will derive a formula for the first variation of a certain String functional that involves the holonomy of a gerbe. It will be elucidating to move one step down the hierarchy of n-gerbes and recall the well-known formula of the variation of the holonomy of a complex line bundle with connection. Let $L \to M$ be a complex line bundle with connection ∇ over a smooth manifold M. By a smooth variation γ_t of a loop[1] $\gamma : \mathbb{R} \to M$ we mean a smooth map $F : (-\epsilon, \epsilon) \times \mathbb{R} \to M$, $(t, s) \mapsto \gamma_t(s)$ such that γ_t is a loop for all $t \in (-\epsilon, \epsilon)$ and $\gamma_0 = \gamma$. Putting $v := \frac{d\gamma_t}{dt}|_{t=0}$ and $\gamma' = \frac{\partial \gamma}{\partial s}$, denoting the holonomy of the line bundle along a loop $\gamma : \mathbb{R} \to M$ by $\mathrm{hol}(\gamma)$ and the curvature-2-form of ∇ by $\Omega = \Omega^\nabla$, then the formula (see [3], p. 234 ff.)

$$\frac{d}{dt}\mathrm{hol}(\gamma_t)\Big|_{t=0} = -\int_0^{2\pi} \Omega(v(s), \gamma'(s))ds \tag{2.1}$$

holds. If γ is a critical point of the function hol: $\mathcal{L}M \to U(1)$ on the loop space $\mathcal{L}M$ of M, then from the above formula we get

$$i_{\gamma'}\Omega = 0,$$

in which i is to mean the contraction between vectors and forms. Sometimes we will also use the notation $i(\xi_1, \dots, \xi_p)$ instead of $i_{(\xi_1,\dots,\xi_p)}$. Bearing this in mind, it is natural to expect an analogous expression for gerbes in which the curvature form of the gerbe should appear.

For the remainder of this chapter let (Σ^k, g) and (M^n, G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let $\mathcal{G} = [g, A^1, \dots, A^{k-1}, iB]$ be a $(k-1)$-gerbe over M (with respect to some good open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$) and Ω the globally defined $(k+1)$-curvature associated to the gerbe. We consider the $U(1)$-valued functional

$$C^\infty(\Sigma, M) \to U(1), \quad \varphi \mapsto \exp(i\mathcal{S}(\varphi)) := \exp(iE(\varphi)) \cdot \mathrm{hol}(\varphi), \tag{2.2}$$

where $E(\varphi)$ is the *Energy* of φ defined by

[1] By a loop we always mean a smooth map from the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ to M or what is amounting to the same thing: a smooth 2π-periodic map $\gamma : \mathbb{R} \to M$, $s \mapsto \gamma(s)$.

$$E(\varphi) = \frac{1}{2} \int_{\Sigma} \langle d\varphi, d\varphi \rangle \, \mathrm{dvol}_g \in \mathbb{R} \tag{2.3}$$

and $\mathrm{hol}(\varphi)$ is the holonomy of $\mathcal{G}$ along φ (see Appendix C.4). Here $\langle \cdot, \cdot \rangle$ denotes the induced metric on the bundle $T^*\Sigma \otimes \varphi^* TM$. The so-called *B-field action* $\mathcal{S}_B$ in physics is defined by

$$\mathcal{S}_B(\varphi) := -i \log(\mathrm{hol}(\varphi)) \in \mathbb{R}, \tag{2.4}$$

so that $\mathrm{hol}(\varphi) = \exp(i\mathcal{S}_B(\varphi))$.

Remark 2.1. In physics one loosely refers to the string action (2.2) as "$\mathcal{S}(\varphi) = E(\varphi) + \mathcal{S}_B(\varphi)$" and we also will repeatedly make use of this convention. We emphasize that we keep the metric g of Σ fixed. Actually in String physics the metric is also varied, but we are not going to do this.

We explicitly point out that the "functional" (2.4) in general is not well-defined (not even as a number) unless the gerbe is trivial, i.e. Ω is exact. For fixed φ the number $\mathcal{S}_B(\varphi)$ is only defined up to an integer multiple of 2π because the logarithm log is only defined mod $2\pi \mathrm{i}\mathbb{Z}$. However, we can define its variational derivative as the logarithmic derivative of $\mathrm{hol}(\varphi_t)$ times $(-i)$:

$$\frac{d}{dt}\Big|_{t=0} \mathcal{S}_B(\varphi_t) := -i \, \frac{d}{dt}\Big|_{t=0} \frac{\mathrm{hol}(\varphi_t)}{\mathrm{hol}(\varphi)} \, .$$

If the gerbe $\mathcal{G}$ is trivial, it can be represented in the form $\mathcal{G} = [1, \ldots, i(\pi^* B)]$, where B is a global defined ($\mathbb{R}$-valued) k-form on M and the map

$$\pi : \bigsqcup_{\alpha} U_\alpha \to M, \quad (x, \alpha) \mapsto x$$

forgets about the member $U_\alpha \in \mathcal{U}$ of the open good covering $\mathcal{U}$ of M from which a point $x \in U_\alpha \subset M$ stems. One can think of $\pi^* B$ as a family of k-forms obtained from B by restricting it to the neighborhoods U_α. The holonomy of the gerbe $\mathcal{G}$ along φ then reduces to

$$\mathrm{hol}(\varphi) = \exp(i \int_{\Sigma} \varphi^* B). \tag{2.5}$$

Therefore, in the trivial case one defines $\mathcal{S}_B$ by

$$\mathcal{S}_B(\varphi) := \int_{\Sigma} \varphi^* B. \tag{2.6}$$

We are interested in the Euler-Lagrange equations belonging to the functional (2.2). Let φ_t be a smooth variation of φ, i.e. a smooth map $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t, p) \mapsto \varphi_t(p)$ such that $\varphi_0 = \varphi$. Then:

$$\frac{d}{dt}\Big|_{t=0} \exp(i\mathcal{S}(\varphi_t)) = 0 \;\Leftrightarrow\; \frac{d}{dt}\Big|_{t=0} E(\varphi_t) + \frac{d}{dt}\Big|_{t=0} \mathcal{S}_B(\varphi_t) = 0. \tag{2.7}$$

The first variation of the energy is well-known from the theory of harmonic maps (see [8], [24])

$$\frac{d}{dt}\Big|_{t=0} E(\varphi_t) = -\int_\Sigma \langle \tau(\varphi), v\rangle \, d\mathrm{vol}_g, \tag{2.8}$$

where $\tau(\varphi) := \operatorname{trace} \nabla d\varphi$, $v := \frac{\partial \varphi_t}{\partial t}\big|_{t=0} \in \Gamma(\varphi^* TM)$ and ∇ denotes the induced connection on $T^*\Sigma \otimes \varphi^* TM$. Here we have used the symbol $\langle \cdot, \cdot \rangle$ for the metric G of M. As we have done here, we will denote all connections and metrics on the various bundles by the same letter, for it will always be clear from the context with respect to which connection we are differentiating and which metric is meant. To accomplish (2.7), we just have to compute $\frac{d}{dt}\big|_{t=0} \mathcal{S}_B(\varphi_t)$.

The local case: Let $(W, x = (x^1, \ldots, x^k) : W \to U)$ be a coordinate neighborhood of Σ and τ be a triangulation of Σ as in Appendix C.4. Shrinking the domain of the chart, if necessary, we may assume that W is contained the interior of some k-face of τ. We denote this face by σ^k. Choose a variation $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t, p) \mapsto \varphi_t(p)$ of $\varphi \in C^\infty(\Sigma, M)$ such that $\varphi_t(p) = \varphi(p)$ for all p outside of a compact subset $K \subset W$, and let $v = \frac{\partial \varphi_t}{\partial t}\big|_{t=0}$ denote the corresponding variational field. Plugging φ_t into the holonomy formula (C.1) of Appendix C.4 and taking its logarithmic derivative (times $-i$) with respect to the variational parameter at $t = 0$, only the integral over σ^k survives, i.e.

$$\frac{d}{dt}\Big|_{t=0} \mathcal{S}_B(\varphi_t) = \int_{\sigma^k} \frac{d}{dt}\Big|_{t=0} \varphi_t^* B_{\rho(\sigma^k)} = \int_W \frac{d}{dt}\Big|_{t=0} \varphi_t^* B_{\rho(\sigma^k)}.$$

Let $\xi_1, \ldots, \xi_k \in T_p\Sigma$. For notational convenience we omit the subscript $\rho(\sigma^k)$ of $B_{\rho(\sigma^k)}$ in the following calculations and put $\frac{\partial \varphi_t}{\partial x^i} := d\varphi_t(\xi_i)$ and $\frac{\nabla v}{\partial x^i} := \nabla_{\xi_i} v$. Then differentiation of

$$(\varphi_t^* B)_p(\xi_1, \ldots, \xi_k) = B_{\varphi_t(p)}(\frac{\partial \varphi_t}{\partial x^1}(p), \ldots, \frac{\partial \varphi_t}{\partial x^k}(p))$$

at $t = 0$ and $p \in W$ yields

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} &(\varphi_t^* B)(\xi_1, \ldots, \xi_k) = \\
&(\nabla_v B)(\frac{\partial \varphi}{\partial x^1}, \ldots, \frac{\partial \varphi}{\partial x^k})\Big|_p + \sum_{r=1}^k B(\frac{\partial \varphi}{\partial x^1}, \ldots, \frac{\nabla v}{\partial x^r}, \ldots, \frac{\partial \varphi}{\partial x^k})\Big|_p \\
= \;& \Omega(v, \frac{\partial \varphi}{\partial x^1}, \ldots, \frac{\partial \varphi}{\partial x^k})\Big|_p + \sum_{i=1}^k (-1)^{i-1} \xi_i \{B(v, \frac{\partial \varphi}{\partial x^1}, \ldots, \frac{\partial \varphi}{\partial x^k})\}\Big|_p \\
&+ \sum_{i<j} (-1)^{i+j} B(v, d\varphi([\xi_i, \xi_j]), \frac{\partial \varphi}{\partial x^1}, \ldots, \widehat{\frac{\partial \varphi}{\partial x^i}}, \ldots, \widehat{\frac{\partial \varphi}{\partial x^j}}, \ldots, \frac{\partial \varphi}{\partial x^k})\Big|_p.
\end{aligned} \tag{2.9}$$

For the first equals sign we have used that ∇ is torsion-free. The second equals sign follows from $dB = \Omega$ and the general fact that for any k-form $\omega \in \Gamma(\Lambda^k T^* M)$ and any vector fields $X_0, \ldots, X_k \in \Gamma(TM)$ on a Riemannian manifold with Levi-Civita connection M, we have the identity (see [21], Chapter II)

$$d\omega(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i (\nabla_{X_i} \omega)(X_0, \ldots, \hat{X}_i, \ldots, X_k) \tag{2.10}$$

at our disposal, in which the hat on $\hat{X}_i$ indicates that this entry has to be omitted. The result of course holds for any differentiable manifold, because the LHS and the RHS of (2.9) do not depend on any metric structure. Integrating (2.9) with $\xi_i = \frac{\partial}{\partial x^i}$ yields

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\mathcal{S}_B(\varphi_t) &= \int_\Sigma \frac{d}{dt}\Big|_{t=0}\varphi_t^* B = \int_W \frac{d}{dt}\Big|_{t=0}\varphi_t^* B \\
&= \int_U \Omega(v(x^{-1}(z)), \frac{\partial \varphi}{\partial x^1}(x^{-1}(z)), \ldots, \frac{\partial \varphi}{\partial x^k}(x^{-1}(z)))\, dz^1 dz^2 \ldots dz^k \\
&= \int_W \Omega(v, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\mathrm{vol}_g \\
&= \int_\Sigma \Omega(v, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g
\end{aligned} \tag{2.11}$$

since $\mathrm{supp}(v)$ lies in W. For the definition of $(d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)$ see Appendix A(a). In terms of a positively oriented local orthonormal frame $\{e_i\}$ of Σ this is simply given by $(d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp) = d\varphi(e_1) \wedge \ldots \wedge d\varphi(e_k)$.

Now, we consider the general case: Let $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t,p) \mapsto \varphi_t(p)$ be an arbitrary smooth deformation of $\varphi : \Sigma \to M$ with variational field v. Here the support of v do not necessarily has to lie in a chart. Because the variational derivative of $\mathcal{S}_B(\varphi_t)$ at $t = 0$ only depends on $\varphi_0 = \varphi$ and on the t-derivative v of φ_t at $t = 0$, the map φ is critical for the "functional" $\mathcal{S}_B$ among all variations iff it is critical among those variations that are of type

$$F : (-\epsilon, \epsilon) \times \Sigma \to M,\ (t,p) \mapsto \exp_{\varphi(p)}(t \cdot v(p)) =: \varphi_t(p), \tag{2.12}$$

in which exp is the exponential map of (M, G) and $v \in \Gamma(\varphi^* TM)$. Thus, we consider WLOG only variations of type (2.12). We choose a finite number of charts $(W_1, x_1), \ldots, (W_m, x_m)$ covering Σ and a subordinate partition of unity, i.e. smooth functions $\rho_1, \ldots, \rho_m : \Sigma \to \mathbb{R}$ with $0 \le \rho_r \le 1$, $\sum_{r=1}^m \rho_r = 1$ and $\mathrm{supp}(\rho_r) \subset W_r$ for all $r = 1, \ldots, m$. We put $v_r := \rho_r \cdot v$ so that $\mathrm{supp}(v_r)$ lies in W_r for all r. Furthermore we define a smooth multi-parameter variation $\phi : (-\epsilon, \epsilon)^m \times \Sigma \to M$, $(t_1, \ldots, t_m, p) \mapsto \phi_{(t_1,\ldots,t_m)}(p)$ by

$$\phi_{(t_1,\ldots,t_m)}(p) := \exp_{\varphi(p)}(\sum_{r=1}^m t_r \cdot v_r(p))$$

such that $\phi_{(t,\ldots,t)}(p) = \varphi_t(p)$ and

$$\frac{\partial}{\partial t_r}\phi_{(t_1,\ldots,t_m)}(p)\Big|_{(t_1,\ldots,t_m)=(0,\ldots,0)} = v_r(p)$$

for all r. Applying the previous result (2.11), we obtain

$$\frac{\partial}{\partial t_r}\mathcal{S}_B(\phi_{(t_1,\ldots,t_m)})\Big|_{(t_1,\ldots,t_m)=(0,\ldots,0)} = \int_\Sigma \Omega(v_r, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g.$$

Due to linearity and by virtue of the chain rule we get

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathcal{S}_B(\varphi_t) &= \frac{d}{dt}\Big|_{t=0} \mathcal{S}_B(\phi_{(t,\dots,t)}) \\
&= \sum_{r=1}^{m} \frac{\partial}{\partial t_r} \mathcal{S}_B(\phi_{(t_1,\dots,t_m)})\Big|_{(t_1,\dots,t_m)=(0,\dots,0)} \\
&= \sum_{r=1}^{m} \int_\Sigma \Omega(v_r, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g \\
&= \int_\Sigma \Omega(v, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g. \qquad (2.13)
\end{aligned}
$$

Now, we use the musical isomorphism to associate to a $(k+1)$-form $\Omega \in \Gamma(\Lambda^{k+1}T^*M)$ a smooth vector bundle homomorphism $Z : \Lambda^k TM \to TM$, i.e. a smooth section of $\mathrm{Hom}(\Lambda^k TM, TM)$ defined by the equation

$$
\langle \eta, Z_x(\xi_1 \wedge \cdots \wedge \xi_k)\rangle = \Omega_x(\eta, \xi_1, \dots, \xi_k) \qquad (2.14)
$$

for all $x \in M$ and all $\eta, \xi_1, \dots, \xi_k \in T_xM$. Here $\langle\cdot,\cdot\rangle$ denotes the metric G of M.

Remark 2.2. *Equation* (2.13) *says that $\varphi \in C^\infty(\Sigma, M)$ is a critical point of* $\mathrm{hol}_{\mathcal{G}}$ *iff one of the following equivalent conditions is satisfied:*

a) $i_{(d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)}\Omega = 0,$

b) $Z((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)) = 0.$

Definition 2.3. Let (M, G) be a Riemannian manifold and $k \in \mathbb{N}$. Then an element $Z \in \Gamma(\mathrm{Hom}(\Lambda^k TM, TM))$, determined by a closed $(k+1)$-form Ω as in (2.14), is called a *k-force*. We will mark the dependence on Ω by $Z = Z^\Omega$. For $k = 1$ we call a one-force $Z = Z^\Omega$ a *Lorentz force*.

In the sequel we will also denote the metric G of M by $\langle\cdot,\cdot\rangle$. We summarize the previous calculations in the following.

Proposition 2.4 (First variation formula)**.** *Let (Σ^k, g) and (M^n, G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let $Z = Z^\Omega$ be a k-force coming from a $(k-1)$-gerbe over M. Then for any deformation φ_t of $\varphi \in C^\infty(\Sigma, M)$ the first variation of $\mathcal{S}$ is given by*

$$
\frac{d}{dt}\Big|_{t=0} \mathcal{S}(\varphi_t) = \int_\Sigma \langle Z((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)) - \tau(\varphi), v\rangle\, d\mathrm{vol}_g. \qquad \text{(V1)}
$$

Here Ω is the curvature of the gerbe.

Proof. Let $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t, p) \mapsto \varphi_t(p)$ be any variation of φ. Put $v := \frac{\partial \varphi_t}{\partial t}\Big|_{t=0}$. By combining (2.8) and (2.13) we see that the first variation of $\mathcal{S}$ is given by

$$
-\int_\Sigma \langle \tau(\varphi), v\rangle\, d\mathrm{vol}_g + \int_\Sigma \Omega(v, (d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g.
$$

Rewriting this with (2.14) we immediately arrive at (V1) so that we are done. ☐

Definition 2.5. We call a critical point $\varphi \in C^\infty(\Sigma, M)$ of the String action $\mathcal{S}$ a *generalized harmonic map.*

Corollary 2.6. *Let* (Σ, g)*,* (M, G) *and* Z *be given as above in Proposition 2.4. Then the Euler-Lagrange equation associated to the String action* $C^\infty(\Sigma^k, M^n) \to U(1)$*,* $\varphi \mapsto \mathcal{S}(\varphi) = E(\varphi) + \mathcal{S}_B(\varphi)$ *is given by*

$$\tau(\varphi) = Z((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)). \tag{2.15}$$

Proof. This follows from (2.7) and the first variation formula. □

Remark 2.7. Let (Σ^k, g) and (M^n, G) be compact orientable Riemannian manifolds. If a map $\varphi : \Sigma \to M$ satisfies (2.15) and in addition is an isometric imbedding, then from the definition of $\tau(\varphi)$ we have

$$\mathcal{H}(\varphi) = \frac{1}{k}\, Z((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)). \tag{2.16}$$

Here $\mathcal{H}(\varphi)$ denotes the mean curvature vector of the isometric imbedding φ,

$$\mathcal{H}(\varphi) = \frac{1}{k}\, \mathrm{trace}\, \nabla d\varphi = \frac{1}{k}\, \tau(\varphi).$$

In particular, if $n = k+1$, any smooth n-form Ω on M must be a multiple of the volume form vol_G of M, i.e. we can express $\Omega = f \mathrm{vol}_G$, where $f : M \to \mathbb{R}$ is a smooth real-valued function. If we denote the outward unit normal field of $\varphi(\Sigma) \subset M$ by ν, then we get $\mathcal{H}(\varphi) = (H\nu) \circ \varphi$ and $Z^\Omega((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp)) = \pm(f\nu) \circ \varphi$, where $H = H(\varphi) : \varphi(\Sigma) \to \mathbb{R}$ denotes the mean curvature of $\varphi(\Sigma) \subset M$. Hence, (2.16) becomes

$$H = \pm \frac{f}{k} \quad \text{in} \quad \varphi(\Sigma). \tag{2.17}$$

The sign depends on whether $\varphi : \Sigma \to \varphi(\Sigma) \subset M$ is orientation preserving (+) or orientation reversing (-) with respect to the orientation on Σ, and the orientation on $\varphi(\Sigma)$ induced by M.

Example 2.8. For $k \neq 2$ let $M = S^{k+1} \subset \mathbb{R}^{k+2}$ be the $(k+1)$-dimensional standard sphere and G its canonical metric. Let $d : M \times M \to [0, \infty)$ denote the Riemannian distance function on M. Fix a point $p \in S^{k+1}$ and set $f(x) = d(p, x)$. Denote the geodesic distance sphere of radius r by $S(r) = f^{-1}(r)$ and its induced metric by $g(r)$, $r \in [0, \infty)$. For $r \in (0, \pi)$ the sphere $S(r)$ with the induced metric is a k-dimensional Riemannian submanifold of $M = S^{k+1}$ which is diffeomorphic to the standard sphere S^k. Its mean curvature $H = H(S(r))$ is given by $H(r) = -\cot(r)$. Let $V_{k+1} = \mathrm{vol}(S^{k+1})$ denote the volume of the standard sphere S^{k+1}. Moreover let $n \in \mathbb{Z}$ be an integer and

$$\Omega_n = \frac{2\pi n}{V_{k+1}} \mathrm{vol}_G.$$

Here vol_G is the volume form of $M = S^{k+1}$. Since

$$\int_M \Omega_n = \frac{2\pi n}{V_{k+1}} \int_M \mathrm{vol}_G = 2\pi n,$$

we may regard Ω_n as curvature form of a k-Deligne class $\mathcal{G}_n \in H^k(M, \mathcal{D}^k)$ such that the corresponding Diximier-Douady class realizes $[\Omega_n]/2\pi = n \in H^{k+1}(M, \mathbb{Z}) \cong \mathbb{Z}$ (see Appendix C.2). By means of (2.17) we see that for a given $n \in \mathbb{Z}$ we have to solve the equation

$$H(r) = -\cot(r) = \frac{2\pi n}{kV_{k+1}}.$$

Noting that $\cot : (0, \pi) \to \mathbb{R}$ is bijective, we can find for any integer $n \in \mathbb{Z}$ a real number $r_n \in (0, \pi)$ such that $\iota_n : S(r_n) \hookrightarrow S^{k+1}$ satisfies

$$\tau(\iota_n) = Z_n((d\iota_n)^{\underline{k}}(\mathrm{vol}^\sharp_{g(r_n)})).$$

Here $Z_n = Z^{\Omega_n}$ and $\iota_n : S(r_n) \hookrightarrow S^{k+1}$ is the natural inclusion which, by definition, is an orientation preserving isometric imbedding.

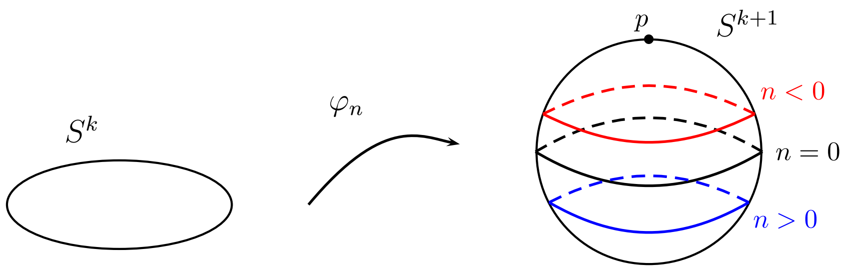

FIGURE 2.1. Geodesic distance spheres

Exchanging two coordinates if necessary, we may assume WLOG that the diffeomorphism $\psi_r : S(r) \to S^k \subset \mathbb{R}^{k+1}$, realizing $S(r) \cong S^k$, is orientation preserving. Equipping S^k with the metric $g_n := (\psi_{r_n}^{-1})^* g(r_n)$ obtained by pulling back the metric of $S(r_n)$ by means of the diffeomorphism $\psi_{r_n}^{-1}$, we see that for any $n \in \mathbb{Z}$ we can find a metric g_n on $\Sigma := S^k$ and an orientation preserving imbedding $\varphi_n := \iota_n \circ \psi_{r_n}^{-1} : \Sigma \to M$ such that the following conditions are satisfied:

$$\tau(\varphi_n) = Z_n((d\varphi_n)^{\underline{k}}(\mathrm{vol}^\sharp_{g_n})) \quad \text{and} \quad g_n = \varphi_n^* G.$$

This is the pair of Euler-Lagrange equations that one obtains if one looks for critical points (φ, g) of the functional $(\varphi, g) \mapsto E_g(\varphi) + \mathcal{S}_B(\varphi)$, where also the metric is varied.

Remark 2.9. In the special case $p = \dim(\Sigma) = 1$, locally we can parametrize Σ by arc length, that is, we can always find local coordinates $\Phi : (-\epsilon, \epsilon) \to U \subset \Sigma, s \mapsto \Phi(s)$ of Σ such that for the norm of the corresponding coordinate vector field $g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}) = 1$ holds. With respect to such coordinates, for $\varphi = \gamma : \Sigma \to M, s \mapsto \gamma(s)$, putting $\gamma' = \frac{\partial \gamma}{\partial s} = d\gamma(\frac{\partial}{\partial s})$, equation (2.15) reduces to

$$\frac{\nabla}{\partial s}\gamma' = Z(\gamma'), \tag{2.18}$$

and we recover the *equation for magnetic geodesics* whose solutions are known as closed *magnetic geodesics*. The reason for this is that Z can be interpreted as Lorentz force and the solutions as the orbits of a charged particle under the effect of a magnetic field. From now on, whenever $\Sigma \cong S^1$, equations like (2.18) and expression like $\gamma' = \frac{\partial \gamma}{\partial s} = d\gamma(\frac{\partial}{\partial s})$ are to be understood with respect to arc length parametrization.

Chapter 3

Second variation formula

Throughout the entire chapter let (Σ^k, g) and (M^n, G) be Riemannian manifolds, let Σ be compact and oriented, Ω be the $(k+1)$ curvature of a fixed $(k-1)$-gerbe $\mathcal{G} = [g, A^1, \ldots, A^{k-1}, iB]$ and $Z = Z^\Omega$ be the corresponding k-force. Moreover let $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t, p) \mapsto \varphi_t(p)$ be a smooth variation of a map $\varphi \in C^\infty(\Sigma, M)$ with $v_t := \frac{\partial \varphi_t}{\partial t}$. As before, sometimes we will write $\langle \cdot, \cdot \rangle$ instead of G. We recall the formula

$$\frac{d}{ds}\mathcal{S}_B(\varphi_t) = \int_\Sigma i_{\frac{\partial}{\partial t}}(\varphi_t^*\Omega) = \int_\Sigma \Omega(v_t, (d\varphi_t)^{\underline{k}}(\mathrm{vol}_g^\sharp))\, d\mathrm{vol}_g. \tag{3.1}$$

Let $e_1, \ldots, e_k$ be a positively oriented orthonormal frame field near $p \in \Sigma$ and w_t be a vector field along φ_t. Set $v := v_0$, $w := w_0$, $a := \frac{\nabla w_t}{\partial t}\Big|_{t=0}$. Differentiating $\Omega(w_t, (d\varphi_t)^{\underline{k}}(\mathrm{vol}_g^\sharp))$ at $t = 0$ yields

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \Omega(w_t, d\varphi_t(e_1), \ldots, d\varphi_t(e_k)) = \\ (\nabla_v \Omega)(w, d\varphi(e_1), \ldots, d\varphi(e_k)) + \Omega(a, d\varphi(e_1), \ldots, d\varphi(e_k)) \\ + (\varphi^* \alpha^{(w,v)})(e_1, \ldots, e_k), \end{aligned} \tag{3.2}$$

where and $\alpha^{(w,v)}$ is a k-form depending on Ω defined as follows. Let v, w be two smooth vector fields on M, then $\alpha^{(w,v)} \in \Gamma(\Lambda^k T^*M)$ is given by

$$\alpha^{(w,v)}(\eta_1, \ldots, \eta_k) := \sum_{r=1}^{k} \Omega(w, \eta_1, \ldots, \nabla_{\eta_r} v, \ldots, \eta_k) \tag{3.3}$$

for all $\eta_1, \ldots, \eta_k \in \Gamma(TM)$.

For two smooth vector bundles E_1, E_2 over M we denote by $\mathcal{D}_k(E_1, E_2)$ the space of differential operators of (at most) degree k mapping smooth sections from E_1 to smooth ones of E_2. For $k = 0$ we have $\mathcal{D}_0(E_1, E_2) = \mathrm{Hom}_{C^\infty(M)}(\Gamma(E_1), \Gamma(E_2)) \cong \Gamma(\mathrm{Hom}(E_1, E_2))$. Again we use the musical isomorphism to introduce the following two mappings

$$L^0 : \Gamma(\Lambda^k TM) \to \mathcal{D}_0(TM, TM) \quad \text{and} \quad L^1 : \Gamma(\Lambda^k TM) \to \mathcal{D}_1(TM, TM)$$

by

$$\langle w_1, L^0_{\xi_1 \wedge \ldots \wedge \xi_k}(v_1) \rangle := (\nabla_{v_1} \Omega)(w_1, \xi_1, \ldots, \xi_k) \tag{3.4}$$

and

$$\langle w_2, L^1_{\eta_1\wedge\ldots\wedge\eta_k}(v_2)\rangle := \alpha^{(w_2,v_2)}(\eta_1,\ldots,\eta_k) \tag{3.5}$$

for all $v_1, v_2, w_1, w_2, \xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k \in \Gamma(TM)$. For a smooth map $\varphi : \Sigma \to M$ the composition of $(d\varphi)^{\underline{k}}$ with $L^\Omega := L^0 + L^1$ yields a mapping

$$\Gamma(\Lambda^k T\Sigma) \to \mathcal{D}_1(\varphi^*TM, \varphi^*TM)$$
$$\xi \mapsto L^\Omega_{(d\varphi)^{\underline{k}}(\xi)}.$$

For $\xi = \mathrm{vol}^\sharp_g$ we set $L^\Omega_\varphi := L^\Omega_{(d\varphi)^{\underline{k}}(\mathrm{vol}^\sharp_g)}$. In the sequel we will write $L_\varphi = L^\Omega_\varphi$ for convenience.

Remark 3.1. Using the identity $\Omega(\eta, \xi_1, \ldots, \xi_k) = \langle \eta, Z(\xi_1 \wedge \ldots \wedge \xi_k)\rangle$ and the relations (3.3),(3.4),(3.5), we see that

$$L^0_\xi(v) = (\nabla_v Z)(\xi) \tag{3.6}$$

and

$$L^1_{\eta_1\wedge\ldots\wedge\eta_k}(w) = Z(\sum_{r=1}^{k} \eta_1 \wedge \ldots \wedge \nabla_{\eta_r} w \wedge \ldots \wedge \eta_k) \tag{3.7}$$

hold for all $v, w, \eta_1, \ldots, \eta_k \in \Gamma(TM)$ and $\xi \in \Gamma(\Lambda^k TM)$. Furthermore $L_\varphi(v)$ can be computed by

$$L_\varphi(v) = \frac{\nabla}{\partial t}\Big|_{t=0} Z((d\varphi_t)^{\underline{k}}(\mathrm{vol}^\sharp_g)), \tag{3.8}$$

where $F : (-\epsilon, \epsilon) \times \Sigma \to M$, $(t, p) \mapsto \varphi_t(p)$ is any variation of φ with $\varphi_0 = \varphi$ and $\frac{\partial\varphi_t}{\partial t}\big|_{t=0} = v$.

Now, integrating (3.2) with $w_t = v_t = \frac{\partial\varphi_t}{\partial t}$, $a = \frac{\nabla v_t}{\partial t}\big|_{t=0}$ and using (3.3),(3.4),(3.5) yields the following.

Lemma 3.2. *For any variation φ_t of $\varphi \in C^\infty(\Sigma, M)$ the second variational derivative of $\mathcal{S}_B$ is given by*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{S}_B(\varphi_t) = \int_\Sigma \Big\{\langle a, Z((\,d\varphi)^{\underline{k}}(\mathrm{vol}^\sharp_g))\rangle + \langle v, L_\varphi(v)\rangle\Big\}\, d\mathrm{vol}_g. \tag{3.9}$$

From the theory of harmonic maps the second variation formula of E (see [8], [24]) is known to be

$$\frac{d^2}{dt^2}\Big|_{t=0} E(\varphi_t) \;=\; \int_\Sigma \Big\{\langle J_\varphi(v), v\rangle + \langle -\tau(\varphi), a\rangle\Big\}\, d\mathrm{vol}_g. \tag{3.10}$$

Here J_φ is the *Jacobi operator* associated to φ. It is a formally selfadjoint second-order elliptic differential operator. For a local orthonormal frame field $e_1, \ldots, e_k$ of Σ it acts on $v \in \Gamma(\varphi^*TM)$ as follows:

$$J_\varphi(v) := -\sum_{r=1}^{k} \{\nabla^\varphi_{e_r}\nabla^\varphi_{e_r} - \nabla^\varphi_{\nabla_{e_r} e_r}\} v - \sum_{r=1}^{k} R^M(v,\, d\varphi(e_r))\, d\varphi(e_r), \tag{3.11}$$

where the Levi-Civita connection of Σ is denoted by ∇, the induced connection of φ^*TM by ∇^φ and the curvature tensor of M by R^M.

Definition 3.3. The first sum in (3.11) is a second-order elliptic differential operator called *rough Laplacian* and is denoted by $\bar{\Delta}_\varphi \in \mathcal{D}_2(\varphi^*TM, \varphi^*TM)$:

$$\bar{\Delta}_\varphi v := \sum_{r=1}^{k} \{\nabla^\varphi_{e_r} \nabla^\varphi_{e_r} - \nabla^\varphi_{\nabla_{e_r} e_r}\} v, \qquad v \in \Gamma(\varphi^*TM). \tag{3.12}$$

The second sum in (3.11) denoted by $\mathcal{R}_\varphi$ is an element of $\mathcal{D}_0(\varphi^*TM, \varphi^*TM)$ and given by

$$\mathcal{R}_\varphi(v) := \sum_{r=1}^{k} R^M(v,\, d\varphi(e_r))\, d\varphi(e_r), \qquad v \in \Gamma(\varphi^*TM). \tag{3.13}$$

The formally selfadjointness of $J_\varphi = -\bar{\Delta}_\varphi - \mathcal{R}_\varphi$ follows from the formally selfadjointness of the rough Laplacian and the symmetries of R^M.

Adding (3.9) and (3.10) provides us with the following.

Proposition 3.4 (Second variation formula)**.** *Let $\varphi \in C^\infty(\Sigma, M)$ be a generalized harmonic map. Then for any deformation φ_t of φ the formula for the second variation of $\mathcal{S}$ is given by*

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{S}(\varphi_t) = \int_\Sigma \langle \mathcal{J}_\varphi(v), v\rangle \, \mathrm{dvol}_g. \tag{V2}$$

Here $\mathcal{J}_\varphi : \Gamma(\varphi^*TM) \to \Gamma(\varphi^*TM)$ is a formally selfadjoint second-order elliptic differential operator defined by

$$\mathcal{J}_\varphi := J_\varphi + L_\varphi, \tag{3.14}$$

and $L_\varphi : \Gamma(\varphi^*TM) \to \Gamma(\varphi^*TM)$ is a formally selfadjoint first-order differential operator.

Proof. The terms involving the acceleration field $a = \frac{\nabla v_t}{\partial t}\big|_{t=0}$ are canceling each other, because φ is a generalized harmonic map. The formally selfadjointness of $\mathcal{J}_\varphi$ is equivalent to the fact that the Hessian I_φ (see Definition 3.6 below) is a well-defined symmetric bilinear form. The formally selfadjointness of L_φ then follows from that of $\mathcal{J}_\varphi$ and J_φ. ☐

Definition 3.5. The differential operator $\mathcal{J}_\varphi$ defined by equation (3.14) is called *generalized Jacobi operator* associated to φ.

Definition 3.6. For a generalized harmonic map $\varphi \in C^\infty(\Sigma, M)$ we define the *index form* (or*Hessian*) of φ by

$$I_\varphi(\xi, \eta) := \int_M \langle \mathcal{J}_\varphi(\xi), \eta\rangle \, \mathrm{dvol}_g \tag{3.15}$$

for $\xi, \eta \in \Gamma(\varphi^*TM)$. A generalized harmonic map φ is called *stable* if $I_\varphi(\xi, \xi) \geq 0$ for all $\xi \in \Gamma(\varphi^*TM)$.

The equation $\mathcal{J}_\varphi(v) = 0$ as generalization of the *Jacobi field equation* reads:

$$\bar{\Delta}_\varphi v + \mathcal{R}_\varphi(v) = L_\varphi(v). \tag{3.16}$$

We call a vector field along φ satisfying (3.16) a *generalized Jacobi field.*

Definition 3.7. A smooth map $F : (-\epsilon, \epsilon) \times \Sigma \to M$ is called a *harmonic variation* of φ if for any $t \in (-\epsilon, \epsilon)$ the map $p \mapsto \varphi_t(p) := F(t, p)$ is a generalized harmonic map.

Remark 3.8. Let φ_t be a harmonic variation with $\varphi := \varphi_0$ and $v := \frac{\partial \varphi_t}{\partial t}\big|_{t=0}$ be the corresponding variational field. We will show that variational fields of harmonic variations satisfy the generalized Jacobi field equation (3.16). Choose a positively oriented local orthonormal frame field $\{e_j\}$ near $p \in \Sigma$ with $\nabla_{e_i} e_j\big|_p = 0$. Then we have at p

$$\begin{aligned}
\nabla^{\varphi_t}_{e_r} \nabla^{\varphi_t}_{e_r} v\big|_{t=0} &= \nabla^{\varphi_t}_{e_r} \nabla^{\varphi_t}_{e_r} \frac{\partial \varphi_t}{\partial t}\big|_{t=0} \\
&= \nabla^{\varphi_t}_{e_r} \frac{\nabla^{\varphi_t}}{\partial t} d\varphi_t(e_r)\big|_{t=0} \\
&= \frac{\nabla^{\varphi_t}}{\partial t} \nabla^{\varphi_t}_{e_r} d\varphi_t(e_r)\big|_{t=0} + R^M\left(d\varphi_t(e_r), \frac{\partial \varphi_t}{\partial t} \right) d\varphi_t(e_r)\big|_{t=0} \\
&\overset{(2.15)}{=} \frac{\nabla^{\varphi_t}}{\partial t} Z((\, d\varphi_t)^{\underline{k}}(\mathrm{vol}_g^\sharp))\big|_{t=0} - \mathcal{R}_\varphi(v) \\
&\overset{(3.8)}{=} L_\varphi(v) - \mathcal{R}_\varphi(v).
\end{aligned}$$

For $k = 1$ and a smooth curve $\gamma : S^1 \to M, s \mapsto \gamma(s)$, putting $\gamma' = \frac{\partial \gamma}{\partial s}$, the generalized Jacobi field equation reduces to

$$\frac{\nabla^2 v}{ds^2} + R^M(v, \gamma')\gamma' = L_\gamma(v). \tag{3.17}$$

We call a vector field v along γ satisfying (3.17) a *magnetic Jacobi field*. Here equation (3.17) and γ' are to be understood as in Remark 2.9 with respect to the canonical coordinates.

Chapter 4

The heat flow method

Notational convention. Throughout the whole chapter let (Σ^k, g) and (M^n, G) be Riemannian manifolds. Furthermore let Σ be compact and oriented and let $Z \in \Gamma(\mathrm{Hom}(\Lambda^k TM, TM))$ be a k-force determined by some closed $(k+1)$-form Ω as in (2.14). Henceforth, we abbreviate $Z((d\varphi)^{\underline{k}}) = Z((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))$ and $Z_\varphi((d\varphi)^{\underline{k}}) = Z_\varphi((d\varphi)^{\underline{k}}(\mathrm{vol}_g^\sharp))$. For the sake of simplicity all appearing metrics and covariant derivatives are denoted by $\langle \cdot, \cdot \rangle$ and ∇, respectively.

In 1964 Eells and Sampson proved the existence of harmonic maps (see [7]) by the heat flow method, that is, they demonstrated that the time limit of the solution to an associated evolution equation is a harmonic map. We would like to use this technique to prove the existence of a solution to (4.2). It turns out that in general this method does not yield a solution to our problem. On the contrary, we will see that the solvability rather depends on the initial value for the associated evolution equation. However, short time existence of solutions to the associated evolution equation can always be shown, regardless of the dimension of (Σ, g) and (M, G) and without making any further assumptions, excepting that Σ is required to be compact and oriented. On the other hand, only if $\dim(\Sigma) = 1$ and imposing nonpositive curvature on M, i.e. $K^M \leq 0$, we are able to verify existence of long time solutions. So, we consider for a map $\varphi : \Sigma \times [0, T) \to M$, setting $\varphi_t(x) = \varphi(x, t)$, the initial value problem (IVP) for the system of nonlinear parabolic partial differential equations

$$\begin{cases} \tau(\varphi_t)(x) = Z((d\varphi_t)^{\underline{k}})(x) + \frac{\partial \varphi_t}{\partial t}(x), & (x,t) \in \Sigma \times (0,T), \\ \varphi(x, 0) = f(x), \end{cases} \tag{4.1}$$

where $\tau(\varphi_t) = \mathrm{trace}\,(\nabla d\varphi_t)$ and $f \in C^\infty(\Sigma, M)$ is a map given as initial condition. We assume that

$$\varphi \in C^0(\Sigma \times [0,T), M) \cap C^\infty(\Sigma \times (0,T), M).$$

Firstly we note the following.

Theorem 4.1. *If a C^2 differentiable map $\varphi : \Sigma \to M$ satisfies the equation for generalized harmonic maps*

$$\tau(\varphi) = Z((d\varphi)^{\underline{k}}), \tag{4.2}$$

then φ is a C^∞ differentiable map.

Proof. Since differentiability is a local property, we may verify it locally at each point $p \in \Sigma$. Choose coordinate neighborhoods $(U, (x^i))$ around p and $(V, (y^\alpha))$ around $\varphi(p)$ such that $\varphi(U) \subset V$. In these coordinates (4.2) reads, for each $\varphi^\alpha = y^\alpha \circ \varphi$,

$$\Delta\varphi^\alpha = -g^{ij}\tilde{\Gamma}^\alpha_{\beta\gamma}(\varphi)\frac{\partial\varphi^\beta}{\partial x^i}\frac{\partial\varphi^\gamma}{\partial x^j} + |g|^{\frac{3}{2}}\frac{\partial\varphi^{\mu_1}}{\partial x^1}\cdots\frac{\partial\varphi^{\mu_k}}{\partial x^k}Z^\alpha_{\mu_1\ldots\mu_k}, \tag{4.3}$$

where $\Delta f = -\delta df = \mathrm{div}(\mathrm{grad} f)$ is the Hodge-Laplacian on functions, $f \in C^2(\Sigma)$. By $\tilde{\Gamma}^\alpha_{\beta\gamma}$ we have denoted the Christoffel symbols of the Levi-Civita connection on M and by $|g| = \det(g_{ij})$ the determinant of the metric (g_{ij}) of Σ. Suppose that a C^2 map φ satisfies (4.3). Since the right hand side is a C^1 function, in particular, it is σ-Hölder continuous for $0 < \sigma < 1$. From the theorem on differentiability for solutions to linear elliptic partial differential equations (see Appendix B.1) it follows that φ is $C^{2+\sigma}$ and therefore the RHS of (4.3) becomes $C^{1+\sigma}$. From the same theorem we obtain that φ is $C^{3+\sigma}$. Iterating this argument, we see that φ must be C^∞. ☐

Example 4.2. Let $\Sigma = S^1$ the unit circle and $M = T^2 = S^1 \times S^1$ the two-dimensional standard torus with the natural induced metrics. Recall the conventions of Remark 2.9 concerning canonical coordinates on S^1. Then for a map $\gamma : S^1 \times [0, \infty) \to M$, setting $\gamma_t(s) = \gamma(s, t)$, the IVP (4.1) takes the form

$$(*) \qquad \begin{cases} \frac{\nabla}{\partial s}\gamma_t'(s) = Z(\gamma_t')(s) + \frac{\partial\gamma_t}{\partial t}(s), & (s,t) \in S^1 \times (0, \infty), \\ \gamma(s, 0) = c(s), \end{cases}$$

where $\gamma_t'(s) = \frac{\partial\gamma}{\partial s}(s, t)$ and $c : S^1 \to T^2$ is a smooth initial curve. Let $\hat{M} = S^1 \times \mathbb{R} \subset \mathbb{R}^3$ be the standard cylinder with metric induced from $\mathbb{R}^3$ and, denoting the standard coordinates of $\mathbb{R}^3$ by (x, y, z), let the z-axis be the axis of symmetry. For the radial vector field $\hat{B} : \mathbb{R}^3 \to \mathbb{R}^3$, given by

$$\hat{B} : (x, y, z)^t \mapsto (x, y, 0)^t,$$

we define a skew-symmetric bundle homomorphism $\hat{Z} : T\hat{M} \to T\hat{M}$ by $\hat{Z}(v) = v \times \hat{B}$ by means of the vector product of $\mathbb{R}^3$, (all tangent spaces of $\mathbb{R}^3$ are identified by parallel transport). We note that $\nabla\hat{Z} = 0$, implying that $\hat{Z}$ defines a closed 2-form $\hat{\Omega}$ via (2.14), and consider for a map $\gamma : S^1 \times [0, \infty) \to \hat{M} \subset \mathbb{R}^3$ the initial value problem

$$(**) \qquad \begin{cases} \frac{\nabla}{\partial s}\gamma_t'(s) = \hat{Z}(\gamma_t')(s) + \frac{\partial\gamma_t}{\partial t}(s), & (s,t) \in S^1 \times (0, \infty), \\ \gamma(s, 0) = c(s). \end{cases}$$

Since $\hat{B}$ is invariant under z-translations, $\hat{Z}$ descends to a well-defined parallel skew-symmetric bundle homomorphism $Z : TM \to TM$ on the Torus $M = \hat{M}/\!\sim\, = S^1 \times S^1$, regarded as quotient of $\hat{M}$ by moding out the $\mathbb{Z}$-action on the second factor of $\hat{M} = S^1 \times \mathbb{R}$. Hence, the entire initial value problem $(**)$ on the cylinder $\hat{M}$ descends to a corresponding initial value problem $(*)$ on the torus $M = T^2$. So, for simplicity we will do all our computations on the cylinder $\hat{M}$. Passing to the quotient $M = \hat{M}/\!\sim$ then yields a corresponding result for the torus. Expressing $\gamma_t(s)$ and $\hat{B}$ in cylindrical coordinates

$$\gamma_t(s) = \begin{pmatrix} \cos(\varphi(s,t)) \\ \sin(\varphi(s,t)) \\ z(s,t) \end{pmatrix} \quad \text{and} \quad \hat{B}(r, \varphi, z) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ 0 \end{pmatrix},$$

$r \in (0,\infty)$, $\varphi \in (-\pi,\pi)$, $z \in (-\infty,\infty)$, a straightforward computation shows that, for functions $\varphi, z : S^1 \times [0,\infty) \to \mathbb{R}$, $(**)$ is equivalent to the following system of partial differential equations

$$(+) \qquad \begin{cases} \varphi''(s,t) = z'(s,t) + \dot{\varphi}(s,t), & (s,t) \in [0,2\pi] \times (0,\infty), \\ z''(s,t) = -\varphi'(s,t) + \dot{z}(s,t), & (s,t) \in [0,2\pi] \times (0,\infty), \\ \varphi(s,0) = \varphi_0(s), \\ z(s,0) = z_0(s). \end{cases}$$

Here we identify $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$, i.e. we regard φ and z as functions defined on $\mathbb{R} \times [0,\infty)$, which are 2π-periodic in the first argument. Furthermore we abbreviate $\varphi'' = \frac{\partial^2\varphi}{\partial s^2}$, $\varphi' = \frac{\partial\varphi}{\partial s}$ and $\dot{\varphi} = \frac{\partial\varphi}{\partial t}$ (in the same way for z) and φ_0, z_0 are initial conditions. Now, let us explicitely calculate the flow for the initial conditions

$$\text{a)} \begin{cases} \varphi_0(s) = A\cos(s) \\ z_0(s) = B\sin(s) \end{cases} \qquad \text{and} \qquad \text{b)} \begin{cases} \varphi_0(s) = s \\ z_0(s) = \mu\cos(s), \end{cases}$$

where $\mu, A, B \geq 0$ are nonnegative numbers and the function φ_0 from initial condition b) is to be understood as being defined on $[0,2\pi]$; in terms of $\gamma_0(s) = (\cos(s), \sin(s), \mu\cos(s))$ we see that b) is a well-defined smooth initial condition $\gamma_0 : S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \to S^1 \times \mathbb{R}$. To this end, let us introduce the complex variable $\xi = \varphi + iz$. Here i denotes the imaginary unit. Then system $(+)$ reduces to a single partial differential equation

$$(++) \qquad \begin{cases} \dot{\xi}(s,t) = \xi''(s,t) + i\xi'(s,t), & (s,t) \in [0,2\pi] \times (0,\infty), \\ \xi(s,0) = \varphi_0(s) + iz_0(s). \end{cases}$$

To solve this we try a power series ansatz

$$\xi(s,t) = \sum_{n=0}^{\infty} a_n(s) t^n.$$

Plugging this into $(++)$ yields the following recursion formula for the coefficients a_n for all $n \geq 1$:

$$(\mathrm{R}) \qquad a_n = \frac{a''_{n-1} + i a'_{n-1}}{n}, \qquad a_0 = \varphi_0 + iz_0.$$

ad a): If $a_0(s,t) = A\cos(s) + iB\sin(s)$, for $n \geq 1$ we get

$$a_n(s) = \frac{(A+B)}{2}\frac{(-2)^n}{n!}\exp(is),$$

and consequently,

$$\begin{aligned} \xi(s,t) &= a_0(s) + \sum_{n=1}^{\infty} \frac{(A+B)}{2}\frac{(-2)^n}{n!}\exp(is) \\ &= A\cos(s) + iB\sin(s) - \frac{(A+B)}{2}\exp(is) + \frac{(A+B)}{2}\exp(is)\exp(-2t) \\ &= \frac{(A-B)}{2}\exp(-is) + \frac{(A+B)}{2}\exp(is)\exp(-2t). \end{aligned}$$

We see that the limit as $t \to \infty$ exists, namely

$$\xi_\infty(s) = \lim_{t\to\infty} \xi(s,t) = \frac{(A-B)}{2} \exp(-is).$$

Also one readily verifies that $\xi''_\infty + i\xi'_\infty = 0$ holds, i.e. on the torus $T^2 = \hat{M}/\sim$ the corresponding loop $\gamma_\infty = \lim_{t\to\infty} \gamma_t : S^1 \to M = T^2$ satisfies the equation for magnetic geodesics

$$\frac{\nabla}{\partial s}\gamma'_\infty = Z(\gamma'_\infty).$$

ad b): If $a_0(s,t) = s + i\mu\cos(s)$, we get $a_1(s) = i(1-\mu\exp(is))$, and for $n \geq 2$

$$a_n(s) = \frac{i\mu}{2}\frac{(-2)^n}{n!}\exp(is),$$

and thus,

$$\xi(s,t) = s + it + i\mu\cos(s) + \frac{i\mu}{2}\exp(is)\Big[\exp(-2t) - 1\Big].$$

On the torus $T^2 = \hat{M}/\sim$ the subsequence $\{\xi(s, 2\pi n)\}_{n\geq 0}$ corresponds to a constant sequence, namely to a loop $\gamma_\infty : S^1 \to T^2$, surrounding the neck of the torus. (see Figure 4.1) The limit of any other convergent subsequence is just a translation of that loop γ_∞ along the "soul" of the torus, i.e. a translation in t-direction. However, since $\xi'' + i\xi' = i \neq 0$, we see that a limit loop γ_∞ can never satisfy the equation for magnetic geodesics in contrast to case a).

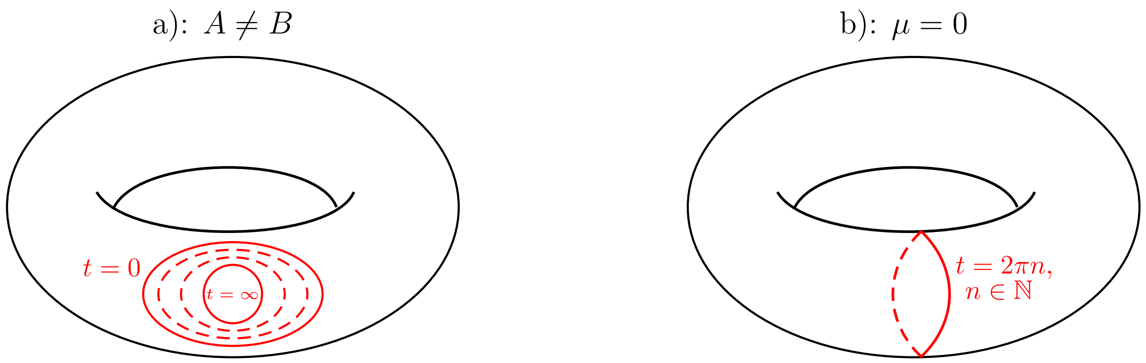

Figure 4.1. The flow of the evolution equation

We may summarize as follows:
On the torus we have computed the flow of the parabolic equation for magnetic geodesics for two families of initial conditions. For an ellipse $c : S^1 \to T^2$ as initial condition (case a)) not enclosing the neck of the torus, the limit loop γ_∞, as $t \to \infty$, exists and is a magnetic geodesic. In the case b) when the initial curve $c : S^1 \to T^2$ forms an ellipse enclosing the neck of the torus, there exist convergent subsequences; but then a limit loop can not be a magnetic geodesic. Hence, we see that the existence of a convergent subsequence such that its limit curve satisfies the equation for magnetic geodesics depends on the initial condition. However, for the cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$, respectively, long time existence of the flow is guaranteed for any initial condition by Theorem 6.8 and Theorem 6.6, respectively.

In general, to show existence of solutions to the equation (4.2) one has to verify the steps of the following program:

1. Show existence of short time solutions to the parabolic initial value problem (4.1).
2. Rule out occurrence of blow ups in finite time, i.e. show existence of long time solutions to the initial value problem (4.1).
3. Show convergence $\varphi_t \to \varphi_\infty$ as $t \to \infty$.
4. If the limit φ_∞ exists, show that φ_∞ satisfies (4.2).

As seen from the above example, it depends on the initial condition whether a limit map φ_∞, provided that it exists, is a solution to (4.2) or not. Consequently one cannot expect a general existence result for generalized harmonic maps in the sense of Eells and Sampson. So, we restrict ourselves to tackle the long time existence problem, i.e. in the following chapters we are going to carry out 1) and 2) of the previous program. The strategy is to derive some Bochner-type formulas and to use the maximum principle for parabolic equations to get a priori estimates which allow to control the growth rate of solutions to the IVP (4.1).
The estimates for the energy densities will show that in $\dim(\Sigma) = k = 1$ everything is fine. For $\dim(\Sigma) > 1$ we would have to deal with "bad" terms that possibly could destroy the long time behavior of our solutions whereas short time existence can be guaranteed without any restrictions on the dimension and the curvature of Σ and M.
For a given solution φ of (4.1) we set $\varphi_t(x) = \varphi(x,t)$ and define

$$\begin{aligned} e(\varphi_t) &:= \frac{1}{2}|d\varphi_t|^2, \quad \text{(energy density)} \\ E(\varphi_t) &:= \int_\Sigma e(\varphi_t)\, d\mathrm{vol}_g, \quad \text{(energy)} \\ \kappa(\varphi_t) &:= \frac{1}{2}\Big|\frac{\partial \varphi_t}{\partial t}\Big|^2, \quad \text{(kinetic energy density)} \\ K(\varphi_t) &:= \int_\Sigma \kappa(\varphi_t)\, d\mathrm{vol}_g. \quad \text{(kinetic energy)} \end{aligned}$$

Now, we state a Weitzenböck formula for vector bundle valued 1-forms.

Proposition 4.3 (Weitzenböck formula). *Let ω be a 1-form on a Riemannian manifold (M,g) with values in a Riemannian vector bundle (E, ∇^E, h). Then*

$$\Delta\omega = \bar{\Delta}\omega + S_\omega.$$

Here $S_\omega \in \Gamma(T^*M \otimes E)$ is given by

$$S_\omega(X) = (R(X, e_i)\omega)(e_i), \tag{4.4}$$

where $\{e_i\}$ is a local orthonormal frame on M, $X \in \Gamma(TM)$ and R is the curvature tensor corresponding to the connection on $T^*M \otimes E$ which is induced by the connections of T^*M and E, respectively.

A proof can be found in ([26], p. 21).

Proposition 4.4 (Bochner-type formulas)**.** *Let* $\varphi \in C^0(\Sigma\times[0,T), M)\cap C^\infty(\Sigma\times(0,T), M)$ *be a solution to the parabolic* IVP (4.1)*, and let* $\varphi_t(x) = \varphi(x,t)$*. In* $\Sigma \times (0,T)$ *we have,*
(1) (*Bochner formula for* $e(\varphi_t)$)

$$\frac{\partial e(\varphi_t)}{\partial t} = \Delta e(\varphi_t) - |\nabla d\varphi_t|^2 + \langle R^M(d\varphi_t(e_i), d\varphi_t(e_k))d\varphi_t(e_k), d\varphi_t(e_i)\rangle - \langle d\varphi_t(Ric^\Sigma(e_i)), d\varphi_t(e_i)\rangle - \langle \nabla Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle. \tag{4.5}$$

(2) (*Bochner formula for* $\kappa(\varphi_t)$)

$$\frac{\partial \kappa(\varphi_t)}{\partial t} = \Delta\kappa(\varphi_t) - |\nabla \frac{\partial \varphi_t}{\partial t}|^2 + \langle R^M(\frac{\partial\varphi_t}{\partial t}, d\varphi_t(e_i))d\varphi_t(e_i), \frac{\partial\varphi_t}{\partial t}\rangle - \langle \frac{\nabla}{\partial t} Z((d\varphi_t)^{\underline{k}}), \frac{\partial \varphi_t}{\partial t}\rangle. \tag{4.6}$$

Here $\Delta = -\delta d$ *is the Hodge-Laplacian on* $C^2(\Sigma)$*,* $\nabla d\varphi_t(X,Y) = (\nabla_X d\varphi_t)(Y)$*, for* $X, Y \in T_x\Sigma$*, is the second fundamental form of* φ_t*, and* Ric^Σ *and* R^M *denote, respectively, the Ricci tensor of* Σ *and the curvature tensor of* M*. The family* $\{e_i\}$ *represents a positively oriented orthonormal basis for the tangent space at each* $x \in \Sigma$*. The covariant derivatives and the metrics are the natural induced ones.*

Proof. We will only show the validity of the Bochner formula for the energy density $e(\varphi_t)$. The proof for $\kappa(\varphi_t)$ is similar. Choose a positively oriented local orthonormal frame field $\{e_i\}$ near $x \in \Sigma$ with $\nabla_{e_i} e_j\big|_x = 0$. From the Weitzenböck formula (4.4) we get at point x,

$$\begin{aligned}
\Delta e(\varphi_t) &= \partial_{e_i}\partial_{e_i} e(\varphi_t) \\
&= \partial_{e_i}\langle \nabla_{e_i} d\varphi_t(e_k), d\varphi_t(e_k)\rangle \\
&= \langle \nabla_{e_i}\nabla_{e_i} d\varphi_t(e_k), d\varphi_t(e_k)\rangle + \langle \nabla_{e_i} d\varphi_t(e_k), \nabla_{e_i} d\varphi_t(e_k)\rangle \\
&= \langle \bar{\Delta} d\varphi_t, d\varphi_t\rangle + |\nabla d\varphi_t|^2 \\
&= \langle \Delta d\varphi_t, d\varphi_t\rangle - \langle S_{d\varphi_t}, d\varphi_t\rangle + |\nabla d\varphi_t|^2 \\
&= \langle \Delta d\varphi_t, d\varphi_t\rangle + |\nabla d\varphi_t|^2 - \langle (R(e_k, e_i)d\varphi_t)(e_i), d\varphi_t(e_k)\rangle \\
&= \langle \Delta d\varphi_t, d\varphi_t\rangle + |\nabla d\varphi_t|^2 - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle \\
&\quad - \langle d\varphi_t(R^\Sigma(e_i, e_k)e_i), d\varphi_t(e_k)\rangle \\
&= \langle \Delta d\varphi_t, d\varphi_t\rangle + |\nabla d\varphi_t|^2 - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle \\
&\quad - \langle d\varphi_t(Ric^\Sigma(e_k), d\varphi_t(e_k)\rangle.
\end{aligned}$$

Noting $\Delta d\varphi_t = -d\delta d\varphi_t = \nabla\tau(\varphi_t)$ and that φ_t satisfies the IVP (4.1), we arrive at

$$\begin{aligned}
\Delta e(\varphi_t) &= \langle \nabla\tau(\varphi_t), d\varphi_t\rangle + |\nabla d\varphi_t|^2 - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle \\
&\quad - \langle d\varphi_t(Ric^\Sigma(e_k), d\varphi_t(e_k)\rangle \\
&= \langle \nabla\frac{\partial\varphi_t}{\partial t}, d\varphi_t\rangle + \langle \nabla Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle + |\nabla d\varphi_t|^2 \\
&\quad - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle - \langle d\varphi_t(Ric^\Sigma(e_k), d\varphi_t(e_k)\rangle \\
&= \langle \frac{\nabla}{\partial t} d\varphi_t, d\varphi_t\rangle + \langle \nabla Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle + |\nabla d\varphi_t|^2 \\
&\quad - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle - \langle d\varphi_t(Ric^\Sigma(e_k), d\varphi_t(e_k)\rangle \\
&= \frac{\partial e(\varphi_t)}{\partial t} + \langle \nabla Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle + |\nabla d\varphi_t|^2 \\
&\quad - \langle R^M(d\varphi_t(e_k), d\varphi_t(e_i))d\varphi_t(e_i), d\varphi_t(e_k)\rangle - \langle d\varphi_t(Ric^\Sigma(e_k), d\varphi_t(e_k)\rangle.
\end{aligned}$$

We have made use of the relation $\nabla\frac{\partial\varphi_t}{\partial t} = \frac{\nabla}{\partial t}d\varphi_t$. This holds, because the covariant derivative along mappings is torsion free. □

Remark 4.5. Since Σ is compact, the unit sphere bundle $S\Sigma$ is also compact. Being a smooth function on $S\Sigma$, Ric^Σ achieves its minimum on it. Consequently there exists a constant C such that $Ric^\Sigma \geq -Cg$. Namely, we can take $C := -\min_{v\in S\Sigma} Ric^\Sigma(v,v)$.

Now, set $E = \mathrm{Hom}(\Lambda^k TM, TM)$.

Corollary 4.6. *Let $\varphi : \Sigma \times [0,T) \to M$ be a solution to the* IVP (4.1) *and set $\varphi_t(s) = \varphi(s,t)$. Let $Z = Z^\Omega$ be some k-force determined by some closed $(k+1)$-form $\Omega \in \Gamma(\Lambda^{k+1}T^*M)$ as in* (2.14)*, with $|Z|_{L^\infty(M,E)}, |\nabla Z|_{L^\infty(M,E)} < \infty$. The following holds in $\Sigma \times (0,T)$:*

(1) *Let C be a real number such that $Ric^\Sigma \geq -Cg$. If M is of nonpositive curvature $K^M \leq 0$, then*

$$\frac{\partial e(\varphi_t)}{\partial t} \leq \Delta e(\varphi_t) + 2Ce(\varphi_t) + 2^{k-2}k|Z|^2_{L^\infty(M,E)}\, e(\varphi_t)^k. \tag{4.7}$$

(2) *If M is of nonpositive curvature $K^M \leq 0$, then*

$$\begin{aligned}\frac{\partial \kappa(\varphi_t)}{\partial t} \leq \Delta\kappa(\varphi_t) &+ 2^{k-2}k^2|Z|^2_{L^\infty(M,E)}\, e(\varphi_t)^{k-1}\kappa(\varphi_t) \\ &+ 2^{1+k/2}|\nabla Z|_{L^\infty(M,E)}\, e(\varphi_t)^{k/2}\kappa(\varphi_t).\end{aligned} \tag{4.8}$$

The norms are given by $|Z|_{L^\infty(M,E)} = \sup_M \langle Z, Z\rangle^{1/2}$ and $|\nabla Z|_{L^\infty(M,E)} = \sup_M\langle\nabla Z, \nabla Z\rangle^{1/2}$. All covariant derivatives, metrics and norms used here are the natural ones induced by the metrics g and G.

Proof. Firstly recall the definition of $(d\varphi)^{\underline{k}}$ and the $\tilde{\wedge}$-product in Appendix A(a). For simplicity we will denote all appearing metrics by $\langle\cdot,\cdot\rangle$.

ad (1): Firstly we note that, for an orthonormal frame with $\nabla_{e_i} e_j\big|_x = 0$, at x

$$\langle \nabla\, Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle = \partial_{e_i} \underbrace{\langle Z((d\varphi_t)^{\underline{k}}), d\varphi_t(e_i)\rangle}_{=0} - \langle Z((d\varphi_t)^{\underline{k}}), \nabla_{e_i} d\varphi_t(e_i)\rangle$$
$$= -\langle Z((d\varphi_t)^{\underline{k}}), \operatorname{trace} \nabla d\varphi_t\rangle$$

holds due to the skew-symmetry of Ω. From this we get

$$|\langle \nabla\, Z((d\varphi_t)^{\underline{k}}), d\varphi_t\rangle| \le k^{1/2}|Z||d\varphi_t|^k|\nabla d\varphi_t|$$
$$\le |\nabla d\varphi_t|^2 + \frac{k}{4}|Z|^2_{L^\infty(M,E)}|d\varphi_t|^{2k}.$$

Using this estimate, the curvature assumptions $K^M \le 0$ and $Ric^\Sigma \ge -Cg$, and the Bochner formula for the energy density $e(\varphi_t)$, inequality (1) readily follows.

ad (2): From

$$\frac{\nabla}{\partial t} Z((d\varphi_t)^{\underline{k}}) = \Big(\nabla_{\frac{\partial \varphi_t}{\partial t}} Z\Big)((d\varphi_t)^{\underline{k}}) + Z\Big((\nabla \frac{\partial \varphi_t}{\partial t})\tilde{\wedge}(d\varphi_t)^{\underline{k-1}}\Big),$$

we see

$$\Big|\Big\langle \frac{\nabla}{\partial t} Z((d\varphi_t)^{\underline{k}}), \frac{\partial \varphi_t}{\partial t}\Big\rangle\Big| \le |\nabla Z||d\varphi_t|^k\Big|\frac{\partial \varphi_t}{\partial t}\Big|^2 + k|Z||d\varphi_t|^{k-1}\Big|\frac{\partial \varphi_t}{\partial t}\Big|\Big|\nabla\frac{\partial \varphi_t}{\partial t}\Big|$$
$$\le \Big|\nabla\frac{\partial \varphi_t}{\partial t}\Big|^2 + \frac{k^2}{4}|Z|^2_{L^\infty(M,E)}\,|d\varphi_t|^{2k-2}\Big|\frac{\partial \varphi_t}{\partial t}\Big|^2$$
$$+ |\nabla Z|_{L^\infty(M,E)}|d\varphi_t|^k\Big|\frac{\partial \varphi_t}{\partial t}\Big|^2.$$

From this estimate, the curvature assumption $K^M \le 0$ and the Bochner formula for $\kappa(\varphi_t)$ we obtain the desired inequality (2). ☐

As a special case of Corollary 4.6, for $k = 1$ we have the following.

Corollary 4.7. *Assume that $\Sigma = S^1$ and Z is a Lorentz force. Let $\varphi = \gamma : S^1 \times [0,T) \to M$ be a solution to the* IVP (4.1)*, and set $\gamma_t(s) = \gamma(s,t)$. The following hold in $S^1 \times (0,T)$:*

(1') *If $|Z|_{L^\infty(M,E)} < \infty$, then*

$$\frac{\partial e(\gamma_t)}{\partial t} \le \Delta e(\gamma_t) + \lambda\, e(\gamma_t). \tag{4.9}$$

(2') *If $K^M \le 0$ and $|Z|_{L^\infty(M,E)}, |\nabla Z|_{L^\infty(M,E)} < \infty$, then*

$$\frac{\partial \kappa(\gamma_t)}{\partial t} \le \Delta\kappa(\gamma_t) + \lambda\,\kappa(\gamma_t) + \mu\, e(\gamma_t)^{1/2}\kappa(\gamma_t), \tag{4.10}$$

where $\lambda = \lambda(M,Z) = \frac{1}{2}|Z|^2_{L^\infty(M,E)}$ and $\mu = \mu(M,\nabla Z) = 2^{3/2}|\nabla Z|_{L^\infty(M,E)}$ are constants only depending on M, Z and ∇Z. All metrics and norms used here are the natural ones induced by the metrics g and G.

Chapter 5

Short time existence

Now, let us carry out step 1) of our program and show the short time existence of solutions to the IVP (4.1). To this end, we cast the parabolic initial value problem in a form that is analytically easier to handle with. As before let (Σ^k, g) and (M^n, G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let Z be a smooth section of $\mathrm{Hom}(\Lambda^k TM, TM) \cong \Lambda^k T^*M \otimes TM$ and $f \in C^\infty(\Sigma, M)$ be the initial condition from (4.1). We use Nash's imbedding theorem, which says that any Riemannian manifold can be isometrically imbedded into an Euclidean space of sufficient high dimension, in order to isometrically imbedd M into a certain $\mathbb{R}^q$. Let

$$\iota : M \hookrightarrow \mathbb{R}^q$$

denote the isometric imbedding, and let $\tilde{M}$ be a tubular neighborhood of the submanifold $\iota(M) \subset \mathbb{R}^q$. It can be defined as an open subset of $\mathbb{R}^q$ by

$$\tilde{M} = \{(x, v) \mid x \in \iota(M),\ v \in T_x\iota(M)^\perp,\ |v| < \epsilon(x)\}.$$

Here $\epsilon : M \to (0, \infty)$ is a positive smooth function on M. By

$$\pi : \tilde{M} \to \iota(M)$$

we denote the canonical projection which assigns to each $z \in \tilde{M}$ the closest point in $\iota(M)$ from z. We extend this projection to a smooth map $\pi : \mathbb{R}^q \to \mathbb{R}^q$ that vanishes outside $\tilde{M}$. This can be done by choosing the positive function ϵ small enough. Also the bundle homomorphism Z can be extended to a bundle homomorphism $\tilde{Z} : \Lambda^k T\mathbb{R}^q \to T\mathbb{R}^q$, meaning that $d\iota \circ Z = \tilde{Z} \circ (d\iota)^{\underline{k}}$ holds; and we do this as follows: Denote by $\tilde{M}_1, \tilde{M}_2$ smaller tubular neighborhoods of M such that $M \subset \tilde{M}_1 \subset \tilde{M}_2 \subset \tilde{M}$ holds. For example, as $\tilde{M}_1$ and $\tilde{M}_2$ we can take the $\epsilon/4$-tubular neighborhood and the $\epsilon/2$-tubular neighborhood, respectively, both contained in the above defined ϵ-tubular neighborhood $\tilde{M}$. In $\tilde{M}_2$ we define $\tilde{Z}$ by

$$\tilde{Z}_x(\xi) := d\iota\Big(Z_{\pi(x)}((d\pi)^{\underline{k}}(\xi))\Big),$$

for all $\xi \in \Lambda^k T_x\mathbb{R}^q$ and all $x \in \tilde{M}_2$. Here we have identified all tangent spaces $T_x\mathbb{R}^q \cong T_y\mathbb{R}^q \cong \mathbb{R}^q$ by parallel translation. Then choose a smooth function $\psi : \mathbb{R}^q \to \mathbb{R}$ with support in $\tilde{M}_2$ such that $\psi \equiv 1$ in the closure of $\tilde{M}_1$ and $0 \leq \psi \leq 1$ in $\mathbb{R}^q$ hold. Multiplying the above $\tilde{Z}$ defined in $\tilde{M}_2$ by this cut-off function ψ, yields a smooth bundle map $\tilde{Z} : \Lambda^k T\mathbb{R}^q \to T\mathbb{R}^q$ which is globally defined in $\mathbb{R}^q$ and vanishes outside $\tilde{M}_2$.

Now, let $u : \Sigma \times [0,T) \to \tilde{M}$ be a map from $\Sigma \times [0,T)$ into $\tilde{M} \subset \mathbb{R}^q$. Regarding u as a function with values in $\mathbb{R}^q$, we may consider the following initial value problem (IVP) for the system of parabolic partial differential equations:

$$
(5.1) \quad \begin{cases} (\Delta - \frac{\partial}{\partial t})\, u(x,t) = \Pi_u(du,du)(x,t) + \tilde{Z}_u((du)^{\underline{k}})(x,t), & (x,t) \in \Sigma \times (0,T), \\ u(x,0) = \iota \circ f(x). \end{cases}
$$

Here $\Delta = -\delta d$ is the Hodge Laplacian of Σ componentwise applied to u and f is the map given as initial condition of the IVP (4.1). $\tilde{Z}$ is the extension of the k-force as described above and $\Pi(du,du)$ is a vector in $\mathbb{R}^q$ defined as follows. Let $\{e_i\}$ be a local orthonormal frame field on Σ regarded, by canonically extension, as a local frame field on $\Sigma \times (0,T)$. Then

$$
(5.2) \qquad \Pi(du,du) := \operatorname{trace} \nabla d\pi(du,du) = (\nabla_{du(e_i)} d\pi)(du(e_i)).
$$

We consider only those solutions $u : \Sigma \times [0,T) \to \tilde{M}$ to the IVP (5.1) which are continuous on $\Sigma \times [0,T)$, C^2 differentiable in Σ and of class C^1 in $(0,T)$. In symbols this means

$$
u \in C^0(\Sigma \times [0,T), \tilde{M}) \cap C^{2,1}(\Sigma \times (0,T), \tilde{M}).
$$

The relation between the two initial value problems is ruled by the following.

Proposition 5.1. *Let $u \in C^0(\Sigma \times [0,T), \tilde{M}) \cap C^{2,1}(\Sigma \times (0,T), \tilde{M})$. If u is a solution to the initial value problem* (5.1)*, then $u(\Sigma \times [0,T)) \subset \iota(M)$ holds true and $\varphi = \iota^{-1} \circ u$ is a solution to the* IVP (4.1)*. The converse also holds true.*

Proof. Suppose that $u \in C^0(\Sigma \times [0,T), \tilde{M}) \cap C^{2,1}(\Sigma \times (0,T), \tilde{M})$ is a solution to the IVP (5.1) and let $\tilde{Z}$ be the extension of $Z \in \Gamma(\operatorname{Hom}(\Lambda^k TM, TM))$ constructed above. At first we will show that $u(\Sigma \times [0,T)) \subset \iota(M)$ holds. For this we define a map $\rho : \tilde{M} \to \mathbb{R}^q$ by

$$
\rho(z) = z - \pi(z), \quad z \in \tilde{M},
$$

and a function $h : \Sigma \times [0,T) \to \mathbb{R}^q$ by

$$
h(x,t) = |\rho(u(x,t))|^2, \quad (x,t) \in \Sigma \times [0,T).
$$

We see, by definition, that $\rho(z) = 0$ iff $z \in \iota(M)$. Thus, we only have to verify $h \equiv 0$. Since $u(x,0) = \iota(f(x)) \in \iota(M)$, we see $h(x,0) = 0$. As u is a solution to the IVP (5.1), we obtain with $\rho(u) = \rho \circ u$

$$
\begin{aligned}
\frac{\partial h}{\partial t} &= \frac{\partial}{\partial t}\langle \rho(u), \rho(u)\rangle = 2\big\langle d\rho\big(\frac{\partial u}{\partial t}\big), \rho(u)\big\rangle \\
&= 2\big\langle d\rho(\Delta u - \Pi(du,du) - Z((du)^{\underline{k}})), \rho(u)\big\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\Delta h &= \Delta\langle \rho(u), \rho(u)\rangle \\
&= 2\langle \Delta\rho(u), \rho(u)\rangle + 2|d\rho(u)|^2,
\end{aligned}
$$

where $\langle\ ,\ \rangle$ is the scalar product in $\mathbb{R}^q$. The formula for the second fundamental form of composite maps (see Lemma 5.2 below) says

$$
\Delta\rho(u) = d\rho(\Delta u) + \operatorname{trace} \nabla d\rho(du,du),
$$

where Δ is the Hodge-Laplacian of Σ. Since, by definition, $\pi(z) + \rho(z) = z$, we have $d\pi + d\rho = id$ and $\nabla d\pi + \nabla d\rho = 0$. This together with the fact that the images of $d\pi$ and ρ are orthogonal to each other yields

$$\begin{aligned}\Delta h &= 2\langle d\rho(\Delta u) - \operatorname{trace} \nabla d\pi(du, du), \rho(u)\rangle + 2|d\rho(u)|^2 \\ &= 2\langle d\rho(\Delta u - \Pi(du, du)), \rho(u)\rangle + 2|d\rho(u)|^2,\end{aligned}$$

and hence,

$$\begin{aligned}\frac{\partial h}{\partial t} &= \Delta h - 2|d\rho(u)|^2 - 2\langle d\rho(\tilde{Z}((du)^{\underline{k}})), \rho(u)\rangle \\ &= \Delta h - 2|d\rho(u)|^2 - 2\langle \tilde{Z}((du)^{\underline{k}}), \rho(u)\rangle \\ &= \Delta h - 2|d\rho(u)|^2.\end{aligned} \tag{5.3}$$

The term $\langle \tilde{Z}((du)^{\underline{k}}), \rho(u)\rangle$ vanishes since $\tilde{Z}((du)^{\underline{k}}) \perp \rho(u)$ by construction of $\tilde{Z}$. Then by the Divergence Theorem (see Appendix B.6) we have for each $t \in (0, T)$,

$$\frac{d}{dt}\int_\Sigma h(\cdot, t)\, d\mathrm{vol}_g = \int_\Sigma \frac{\partial h}{\partial t}(\cdot, t)\, d\mathrm{vol}_g = -2\int_\Sigma |d\rho(u)|^2\, d\mathrm{vol}_g \leq 0.$$

Since $h(x, 0) = 0$ from the assumption, we have

$$\int_\Sigma h(\cdot, t)\, d\mathrm{vol}_g \leq \int_\Sigma h(\cdot, 0)\, d\mathrm{vol}_g = 0$$

and consequently $h \equiv 0$.

Now, we turn to the second half of the assertion. Therefore, let $u : \Sigma \times [0, T) \to \tilde{M}$ be a solution to the IVP (5.1). From the previous assertion we know that $u(\Sigma \times [0, T)) \subset \iota(M)$. Hence, we can write $u = \iota \circ \varphi$, where φ is a map from $\Sigma \times [0, T)$ to M. We will show that φ is a solution to the IVP (4.1). Due to the formula (see Lemma 5.2) for the second fundamental form of composition maps for $u = \iota \circ \varphi$ and for $\iota = \pi \circ \iota$ we get

$$\begin{aligned}\Delta u &= \operatorname{trace} \nabla d\iota(d\varphi, d\varphi) + d\iota(\tau(\varphi)), \\ \nabla d\iota &= \nabla d\pi(d\iota, d\iota) + d\pi(\nabla d\iota).\end{aligned}$$

Since $\iota : M \to \mathbb{R}^q$ is an isometric imbedding, the second fundamental $\nabla d\iota$ of ι is orthogonal to $\iota(M)$ at each point, and thus $d\pi(\nabla d\iota) = 0$. Combining this and the preceding equations, we obtain

$$d\iota(\tau(\varphi)) = \Delta u - \operatorname{trace} \nabla d\pi(du, du).$$

Bearing in mind that $d\iota \circ Z = \tilde{Z} \circ (d\iota)^{\underline{k}}$ and $d\iota(\frac{\partial \varphi}{\partial t}) = \frac{\partial u}{\partial t}$ hold, we finally arrive at

$$d\iota\Big(\tau(\varphi) - \frac{\partial \varphi}{\partial t} - Z((d\varphi)^{\underline{k}})\Big) = (\Delta - \frac{\partial}{\partial t})\, u - \tilde{Z}((du)^{\underline{k}}) - \Pi(du, du).$$

From this one reads off that φ is a solution to the IVP (4.1) if u is a solution to the initial value problem (5.1). Analogously the converse can easily be verified. $\square$

In the proof of the preceding proposition we have made use of the

Lemma 5.2. *Let (Σ, g), (M, G) and (N, h) be Riemannian manifolds. Given maps $\Sigma \xrightarrow{\varphi} M \xrightarrow{\psi} N$, we have $\nabla d(\psi \circ \varphi) = d\psi(\nabla d\varphi) + \nabla d\psi(d\varphi, d\varphi)$; and $\tau(\psi \circ \varphi) = d\psi(\tau(\varphi)) + \operatorname{trace} \nabla d\psi(d\varphi, d\varphi)$.*

Proof. For $X, Y \in \Gamma(T\Sigma)$ we compute

$$\begin{aligned}\nabla d(\psi \circ \varphi)(X, Y) &= \nabla_X(d\psi \circ d\varphi(Y)) - d(\psi \circ \varphi)(\nabla_X Y)\\ &= (\nabla_{d\varphi(X)} d\psi)(d\varphi(Y)) + d\psi(\nabla_X d\varphi(Y)) - d\psi \circ d\varphi(\nabla_X Y)\\ &= \nabla d\psi(d\varphi(X), d\varphi(Y)) + d\psi(\nabla d\varphi(X, Y)).\end{aligned}$$

Since $\tau(\cdot) = \operatorname{trace} \nabla d(\cdot)$, the formula for the tension field of composite maps follows immediately from that for the second fundamental form by taking the trace. ☐

From Proposition 5.1 we see that we can prove short time existence for solutions to the IVP (4.1) by establishing short time existence for IVP (5.1). For the latter IVP one can set up a function space which is well adapted to our problem. To this end, we follow Ladyženskaya, Solonnikov and Ural'ceva ([15], p. 7). Given $T > 0$, set $Q = \Sigma \times [0, T]$. Let $0 < \alpha < 1$. Given a vector valued function $u : Q \to \mathbb{R}^q$, set

$$|u|_Q = \sup_{(x,t)\in Q} |u(x, t)|,$$

$$\langle u \rangle_x^{(\alpha)} = \sup_{\substack{(x,t),(x',t)\in Q\\ x\neq x'}} \frac{|u(x, t) - u(x', t)|}{d(x, x')^\alpha},$$

$$\langle u \rangle_t^{(\alpha)} = \sup_{\substack{(x,t),(x,t')\in Q\\ t\neq t'}} \frac{|u(x, t) - u(x, t')|}{|t - t'|^\alpha},$$

and define the norms $|u|_Q^{(\alpha,\alpha/2)}$, $|u|_Q^{(2+\alpha,1+\alpha/2)}$ by

$$\begin{aligned}|u|_Q^{(\alpha,\alpha/2)} &= |u|_Q + \langle u \rangle_x^{(\alpha)} + \langle u \rangle_t^{(\alpha/2)},\\ |u|_Q^{(2+\alpha,1+\alpha/2)} &= |u|_Q + |\partial_t u|_Q + |D_x u|_Q + |D_x^2 u|_Q\\ &\quad + \langle \partial_t u \rangle_t^{(\alpha/2)} + \langle D_x u \rangle_t^{(1/2+\alpha/2)} + \langle D_x^2 u \rangle_t^{(\alpha/2)}\\ &\quad + \langle \partial_t u \rangle_x^{(\alpha)} + \langle D_x^2 u \rangle_x^{(\alpha)}.\end{aligned} \tag{5.4}$$

Here $d(x, x')$ is the Riemannian distance between x and x' in Σ and $\partial_t u$ represents $\partial u/\partial t$. Also $D_x u$ and $D_x^2 u$ represent the first order derivative of u in Σ and its covariant derivative, respectively. In terms of a local coordinate system (x^i) in Σ and the standard coordinates (y^α) of $\mathbb{R}^q$, $D_x u$ and $D_x^2 u$ are, respectively, given by

$$D_x u = du = \partial_i u^\alpha \cdot dx^i \otimes \frac{\partial}{\partial y^\alpha},$$

$$D_x^2 u = \nabla du = \nabla_i \partial_j u^\alpha \cdot dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^\alpha},$$

and $|D_x u|_Q^2$ and $|D_x^2 u|_Q^2$ are, respectively, given as

$$|D_x u|_Q^2 = \sup_{(x,t)\in Q} g^{ij}\partial_i u^\alpha \partial_j u^\alpha,$$
$$|D_x^2 u|_Q^2 = \sup_{(x,t)\in Q} g^{ik}g^{jl}\nabla_i\partial_j u^\alpha \nabla_k \partial_l u^\alpha,$$

where $\partial_i = \partial/\partial x^i$. With respect to these norms we define the function spaces $C^{\alpha,\alpha/2}(Q,\mathbb{R}^q)$ and $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$, respectively, by

$$C^{\alpha,\alpha/2}(Q,\mathbb{R}^q) = \{u \in C^0(\Sigma\times[0,T]) \mid |u|_Q^{(\alpha,\alpha/2)} < \infty\},$$
$$C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q) = \{u \in C^{2,1}(\Sigma\times[0,T]) \mid |u|_Q^{(2+\alpha,1+\alpha/2)} < \infty\},$$

and set
$$C^{2+\alpha,1+\alpha/2}(Q,M) = \{u \in C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q) \mid u(Q)\subset M\},$$
where we have naturally identified M with $\iota(M)\subset\mathbb{R}^q$. One can show that $C^{\alpha,\alpha/2}(Q,\mathbb{R}^q)$ and $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$ are Banach spaces with norms $|u|_Q^{(\alpha,\alpha/2)}$, $|u|_Q^{(2+\alpha,1+\alpha/2)}$, respectively. They are called *Hölder spaces* on $Q\times[0,T]$. See [9], [11] for example. $C^{2+\alpha,1+\alpha/2}(Q,M)$ is a closed subset of $C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$. This follows immediately because M, as a compact subset, is closed in $\mathbb{R}^q$.

Now, we prove the following.

Theorem 5.3. *Let (Σ,g) and (M,G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let $Z\in\Gamma(\mathrm{Hom}(\Lambda^k TM,TM))$. For any $C^{2+\alpha}$ map $f\in C^{2+\alpha}(\Sigma,M)$ there exists a positive number $\epsilon=\epsilon(\Sigma,M,Z,f,\alpha)>0$ and a map $u\in C^{2+\alpha,1+\alpha/2}(\Sigma\times[0,\epsilon],\tilde{M})$ such that u is a solution in $\Sigma\times[0,\epsilon)$ to the* IVP (5.1). *Here, $\epsilon=\epsilon(\Sigma,M,Z,f,\alpha)$ is a constant depending on Σ,M,Z,f, and α.*

The main tool that we use to prove this theorem is the Inverse Function Theorem (see Appendix B.9) for Banach spaces. It says that a C^1 map is locally invertible at a point iff its linearization is invertible at this point. The idea is to apply the Inverse Function Theorem to reduce the solvability of a nonlinear differential equation to the solvability of its linearized version. However, before it we review the following classically well known result about existence and uniqueness of solutions to linear parabolic partial differential equations. (see [15], p. 320) or ([9], p. 350 ff.)

Theorem 5.4. *Let (Σ,g) be a compact Riemannian manifold of dimension k, and set $Q=\Sigma\times[0,T]$. Given a vector valued function $u:Q\to\mathbb{R}^q$, let*

$$Lu = \Delta u + \mathbf{a}\cdot\nabla u + \mathbf{b}\cdot u - \partial_t u$$

be a linear parabolic partial differential operator, and consider the initial value problem

$$\begin{cases} Lu(x,t) = F(x,t), & (x,t)\in\Sigma\times(0,T), \\ u(x,0)=f(x). \end{cases} \tag{5.5}$$

Here the components of Δu, $\mathbf{a}\cdot\nabla$, $\mathbf{b}\cdot u$, $\partial_t u$ are, respectively, defined by

$$\Delta u^A, \quad \mathbf{a}_B^{iA}(x,t)\frac{\partial u^B}{\partial x^i}, \quad \mathbf{b}_B^A(x,t)u^B, \quad \frac{\partial u^A}{\partial t}, \quad 1\le A\le q.$$

If

$$\mathbf{a}^{iA}_B,\ \mathbf{b}^A_B \in C^{\alpha,\alpha/2}(Q,\mathbb{R}), \quad 1 \le i \le k, \quad 1 \le A, B \le q,$$

for some $0 < \alpha < 1$, *then for any*

$$F \in C^{\alpha,\alpha/2}(Q,\mathbb{R}^q), \quad f \in C^{2+\alpha}(\Sigma,\mathbb{R}^q),$$

there exist a unique solution $u \in C^{2+\alpha,1+\alpha/2}(Q,\mathbb{R}^q)$ *to* (5.5) *such that*

$$|u|_Q^{(2+\alpha,1+\alpha/2)} \le C(|F|_Q^{(\alpha,\alpha/2)} + |f|_\Sigma^{(2+\alpha)})$$

holds. Here the constant $C = C(\Sigma, L, q, T, \alpha)$ *only depends on* Σ, L, q, T, α.

To prove Theorem 5.3 we need a technical lemma.

Lemma 5.5. *Let* (Σ, g) *be a compact Riemannian manifold and* $\alpha', \alpha, \epsilon \in \mathbb{R}$ *such that* $0 < \alpha' < \alpha < 1$ *and* $0 < \epsilon < 1$, *respectively. Set* $Q = \Sigma \times [0,1]$. *Let* $w \in C^{\alpha,\alpha/2}(Q,\mathbb{R}^q) \subset C^{\alpha',\alpha'/2}(Q,\mathbb{R}^q)$ *with* $w(x,0) = 0$ *and* $\zeta : \mathbb{R} \to \mathbb{R}$ *be a* C^∞ *function satisfying* $\zeta(t) = 1$ $(t \le \epsilon)$, $\zeta(t) = 0$ $(t \ge 2\epsilon)$, $0 \le \zeta(t) \le 1$, $|\zeta'(t)| \le 2/\epsilon$ $(t \in \mathbb{R})$. *Then the following estimate holds:*

$$|\zeta w|_Q^{(\alpha',\alpha'/2)} \le C\epsilon^{(\alpha-\alpha')}|w|_Q^{(\alpha,\alpha/2)}.$$

Here, C *is a constant independent of* w *and* ϵ.

Proof. Since, by definition, $|\zeta w|_Q^{(\alpha',\alpha'/2)} = |\zeta w|_Q + \langle \zeta w\rangle_x^{(\alpha')} + \langle \zeta w\rangle_t^{(\alpha'/2)}$, it suffices to estimate each term of the sum individually. Regarding that $w(x,0) = 0$ and $\zeta(t) = 0$ for $t \ge 2\epsilon$, we get

$$\begin{aligned}|\zeta(t)w(x,t)| &\le |\zeta(t)w(x,t) - \zeta(t)w(x,0)| \le |\zeta(t)|\langle w\rangle_t^{(\alpha/2)}|t|^{\alpha/2} \\ &\le \langle w\rangle_t^{(\alpha/2)}(2\epsilon)^{\alpha/2} \le 2^{\alpha/2}|w|_Q^{(\alpha,\alpha/2)}\epsilon^{\alpha/2}.\end{aligned}$$

From this and $\epsilon^{\alpha'/2} \le 1$ we see that $|\zeta w|_Q \le C_1\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}$. To estimate $\langle \zeta w\rangle_x^{(\alpha')}$ we distinguish the two cases where $d(x,x') \le \epsilon^{1/2}$ and $d(x,x') \ge \epsilon^{1/2}$, respectively.

For $d(x,x') \le \epsilon^{1/2}$ and $x \ne x'$ we have:

$$\begin{aligned}|\zeta(t)w(x,t) - \zeta(t)w(x',t)| &\le |w(x,t) - w(x',t)| \\ &\le \frac{|w(x,t) - w(x',t)|}{d(x,x')^\alpha}\, d(x,x')^{\alpha'}\epsilon^{(\alpha-\alpha')/2} \\ &\le \langle w\rangle_x^{(\alpha)}\, d(x,x')^{\alpha'}\epsilon^{(\alpha-\alpha')/2} \\ &\le |w|_Q^{(\alpha,\alpha/2)}\, d(x,x')^{\alpha'}\epsilon^{(\alpha-\alpha')/2}.\end{aligned}$$

For $d(x,x') \ge \epsilon^{1/2}$ we get:

$$\begin{aligned}|\zeta(t)w(x,t) - \zeta(t)w(x',t)| &\le |\zeta(t)w(x,t) - \zeta(t)w(x,0)| + |\zeta(t)w(x',0) - \zeta(t)w(x',t)| \\ &\le \langle w\rangle_t^{(\alpha/2)}(2\epsilon)^{\alpha/2} + \langle w\rangle_t^{(\alpha/2)}(2\epsilon)^{\alpha/2} \\ &\le 2^{1+\alpha/2}\,|w|_Q^{(\alpha,\alpha/2)}\epsilon^{\alpha/2}.\end{aligned}$$

Dividing both sides of this inequality by $d(x,x')^{\alpha'}$ and noting that $d(x,x')^{\alpha'} \geq \epsilon^{\alpha'/2}$ yields

$$\frac{|\zeta(t)w(x,t)-\zeta(t)w(x',t)|}{d(x,x')^{\alpha'}} \leq 2^{1+\alpha/2}\,|w|_Q^{(\alpha,\alpha/2)}\epsilon^{(\alpha-\alpha')/2}.$$

Consequently we obtain

$$\begin{aligned}
\langle\zeta w\rangle_x^{(\alpha')} &= \sup_{\substack{(x,t),(x',t)\in Q\\ x\neq x'}} \frac{|\zeta(t)w(x,t)-\zeta(t)w(x',t)|}{d(x,x')^{\alpha'}}\\
&\leq \sup_{d(x,x')\leq\epsilon^{1/2}}\Big\{\dots\Big\} + \sup_{d(x,x')\geq\epsilon^{1/2}}\Big\{\dots\Big\}\\
&\leq C_2\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}.
\end{aligned}$$

After all we discuss the term $\langle\zeta w\rangle_t^{(\alpha'/2)}$. There are only three nontrivial cases:

Case i) For $t,t'\in[0,2\epsilon]$, $t\neq t'$, we get:

$$\begin{aligned}
&|\zeta(t)w(x,t)-\zeta(t')w(x,t')|\\
&\quad\leq |\zeta(t)(w(x,t)-w(x,t'))| + |(\zeta(t)-\zeta(t'))(w(x,t')-w(x,0))|\\
&\quad\leq |\zeta(t)||w|_Q^{(\alpha,\alpha/2)}|t-t'|^{\alpha/2} + 2\epsilon^{-1}|t-t'||w|_Q^{(\alpha,\alpha/2)}|t'|^{\alpha/2}.
\end{aligned}$$

Dividing both sides of the above inequality by $|t-t'|^{\alpha'/2}$, we get

$$\begin{aligned}
&|\zeta(t)w(x,t)-\zeta(t')w(x,t')||t-t'|^{-\alpha'/2}\\
&\quad\leq |w|_Q^{(\alpha,\alpha/2)}(2\epsilon)^{(\alpha-\alpha')/2} + 2\epsilon^{-1}(2\epsilon)^{1-\alpha'/2}|w|_Q^{(\alpha,\alpha/2)}(2\epsilon)^{\alpha/2}\\
&\quad\leq \widetilde{C}_3\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}.
\end{aligned}$$

Case ii) For $t\in[0,2\epsilon]$, $t'>2\epsilon$ and $|t-t'|>2\epsilon$, we see

$$|\zeta(t)w(x,t)-\zeta(t')w(x,t')| = |\zeta(t)||w(x,t)-w(x,0)| \leq |w|_Q^{(\alpha,\alpha/2)}(2\epsilon)^{\alpha/2}.$$

Dividing both sides by $|t-t'|^{\alpha'/2}$ and noting that $|t-t'|^{\alpha'/2}\geq(2\epsilon)^{\alpha'/2}$, we obtain

$$|\zeta(t)w(x,t)-\zeta(t')w(x,t')||t-t'|^{-\alpha'/2} \leq \widetilde{\widetilde{C}}_3|w|_Q^{(\alpha,\alpha/2)}\epsilon^{(\alpha-\alpha')/2}$$

Case iii) For $t\in[0,2\epsilon]$, $t'>2\epsilon$ and $|t-t'|\leq2\epsilon$, one computes

$$\begin{aligned}
|\zeta(t)w(x,t)-\zeta(t')w(x,t')| &= |\zeta(t)-\zeta(t')||w(x,t)-w(x,0)|\\
&\leq |w|_Q^{(\alpha,\alpha/2)}(2\epsilon)^{\alpha/2}\left|\frac{\zeta(t)-\zeta(t')}{t-t'}\right|^{\alpha'/2}|t-t'|^{\alpha'/2}|\zeta(t)-\zeta(t')|^{1-\alpha'/2}\\
&\leq 2^{(\alpha+\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}\epsilon^{(\alpha-\alpha')/2}|t-t'|^{\alpha'/2}.
\end{aligned}$$

However, at all events it follows that $\langle\zeta w\rangle_t^{(\alpha'/2)} \leq C_3\epsilon^{(\alpha-\alpha')/2}|w|_Q^{(\alpha,\alpha/2)}$. Hence, the inequalities for the three terms $|\zeta w|_Q$, $\langle\zeta w\rangle_x^{(\alpha')}$, and $\langle\zeta w\rangle_t^{(\alpha'/2)}$ yield the desired estimate for $|\zeta w|_Q^{(\alpha',\alpha'/2)}$ with a constant $C=C_1+C_2+C_3$.

□

Now, we turn to the proof of Theorem 5.3.

Proof. At first let $\tilde{Z}$ be the smooth extension of Z constructed at the beginning of this chapter. We choose an α' such that $0 < \alpha' < \alpha < 1$ and use the abbreviation $\partial_t = \partial/\partial t$.

Step 1 (Construction of an approximate solution). Consider the following initial value problem of a system of linear parabolic partial differential equations:

$$(5.6) \qquad \begin{cases} (\Delta - \frac{\partial}{\partial t})\, v(x,t) = \Pi_f(df, df)(x,t) + \tilde{Z}_f((df)^{\underline{k}})(x,t), & (x,t) \in \Sigma \times (0,1), \\ v(x,0) = f(x), \end{cases}$$

where we have identified f with $\iota \circ f$. From the assumption $f \in C^{2+\alpha}(\Sigma, \mathbb{R}^q)$ we get

$$\Pi_f(df, df),\ \tilde{Z}_f((df)^{\underline{k}}) \in C^{\alpha}(\Sigma, \mathbb{R}^q) \subset C^{\alpha,\alpha/2}(\Sigma \times [0,1], \mathbb{R}^q),$$

and consequently by virtue of the previous Theorem 5.4 the existence of a unique solution

$$v \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,1], \mathbb{R}^q)$$

to the IVP (5.6). If we denote the desired solution by u, then v approximates u at $t = 0$ in the following sense,

$$v(x,0) = u(x,0), \qquad \partial_t v(x,0) = \partial_t u(x,0).$$

Step 2 (Application of the Inverse Function Theorem). Now, putting $Q = \Sigma \times [0,1]$, we consider the differential operator

$$P(u) = \Delta u - \partial_t u - \Pi_u(du, du) - \tilde{Z}_u((du)^{\underline{k}})$$

and note that an $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,\epsilon], \mathbb{R}^q)$ satisfying $P(u) = 0$ is our desired solution.

For $0 < \alpha' < 1$ we introduce the subspaces X and Y in $C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ and $C^{\alpha',\alpha'/2}(Q, \mathbb{R}^q)$, respectively, by

$$\begin{aligned} X &= \{h \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q) \mid h(x,0) = 0,\ \partial_t h(x,0) = 0\}, \\ Y &= \{k \in C^{\alpha',\alpha'/2}(Q, \mathbb{R}^q) \mid k(x,0) = 0\}. \end{aligned}$$

The spaces X and Y are, by definition, closed subspaces; and hence Banach spaces. We define a map $\mathcal{P} : X \to Y$ by

$$\mathcal{P}(h) = P(v + h) - P(v), \quad \text{for } h \in X.$$

From the definition of P and X we see that $\mathcal{P}(h) \in C^{\alpha',\alpha'/2}(Q, \mathbb{R}^q)$ and $\mathcal{P}(h)(x,0) = 0$ for $h \in X$ so that in fact $\mathcal{P}(h) \in Y$ holds true. In particular, $\mathcal{P}(0) = 0$. $\mathcal{P}$ is Fréchet differentiable in a neighborhood of $h = 0$. A direct computation using the definition of $\mathcal{P}$ shows that the Fréchet derivative $\mathcal{P}'(0) : X \to Y$, for $h \in X$, is given by

$$\begin{aligned} \mathcal{P}'(0)(h) = \Delta h - \partial_t h - (d\Pi)\big|_v(h)(dv, dv) - 2\Pi_v(dv, dh) \\ - (d\tilde{Z})\big|_v(h)((dv)^{\underline{k}}) - \tilde{Z}_v(dh \tilde{\wedge} (dv)^{\underline{k-1}}). \end{aligned}$$

Here, $\tilde{Z}(dh\tilde{\wedge}(dv)^{\underline{k-1}}) = \tilde{Z}((dh\tilde{\wedge}(dv)^{\underline{k-1}})(\mathrm{vol}_g^\sharp))$ and $(d\tilde{Z})(h)((dv)^{\underline{k}}) = (d\tilde{Z})(h)((dv)^{\underline{k}}(\mathrm{vol}_g^\sharp))$, respectively. (For the definition of the $\tilde{\wedge}$-product, see Appendix A.) From this it can readily be verified that $\mathcal{P}'(0) : X \to Y$ is an isomorphism of Banach spaces. In fact, since $v \in C^{2+\alpha,1+\alpha/2}(Q, \mathbb{R}^q)$, from the definition of $\mathcal{P}'(0)$ and Theorem 5.4 we see that for any $K \in Y$ there exists a unique $H \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ satisfying

$$\begin{cases} \mathcal{P}'(0)(H)(x,t) = K(x,t), & (x,t) \in \Sigma \times (0,1), \\ H(x,0) = 0. \end{cases}$$

We also see that for such a H the following estimate holds:

$$|H|_Q^{(2+\alpha',1+\alpha'/2)} \leq C|K|_Q^{(\alpha',\alpha'/2)}. \tag{5.7}$$

Since $K(x,0) = 0$ and $H(x,0) = 0$ hold, we obtain $\partial_t H(x,0) = 0$; and thus $H \in X$. From this and the definition of X, Y and the expression for $\mathcal{P}'(0)$ we know that $\mathcal{P}'(0)$ is a bounded and surjective linear mapping of Banach spaces. Equation (5.7) tells us that $\mathcal{P}'(0)$ is injective and the Open Mapping Theorem from functional analysis that also the inverse $\mathcal{P}'(0)^{-1}$ is bounded. Hence, $\mathcal{P}'(0)$ is an isomorphism.

Applying the Inverse Function Theorem (see Appendix B.9) for Banach spaces, $\mathcal{P} : X \to Y$ is a homeomorphism between a sufficiently small neighborhood $\mathcal{U}$ of $0 \in X$ and a neighborhood $\mathcal{P}(\mathcal{U})$ of $0 \in Y$. This means that we can find a positive number $\delta = \delta(\Sigma, M, Z, f) > 0$, depending only on Σ, M, Z and f, such that the following holds: For any $k \in C^{\alpha',\alpha'/2}(Q, \mathbb{R}^q)$ with $k(x,0) = 0$ and $|k|_Q^{(\alpha',\alpha'/2)} < \delta$, there exists a $h \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ satisfying

$$\mathcal{P}(h) = k, \quad h(x,0) = 0, \quad \partial_t h(x,0) = 0. \tag{5.8}$$

Here $\delta = \delta(\Sigma, M, Z, f)$ is a positive number determined by Σ, M, Z and f. Setting $u = v + h$ and $w = P(v)$, from (5.8) we see that there exists a $u \in C^{2+\alpha',1+\alpha'/2}(Q, \mathbb{R}^q)$ satisfying

$$\begin{cases} P(u)(x,t) = (w+k)(x,t), & (x,t) \in \Sigma \times (0,1), \\ u(x,0) = f(x). \end{cases} \tag{5.9}$$

Step 3 (Short time existence). For a given real number $\epsilon > 0$ consider a C^∞ function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying $\zeta(t) = 1$ $(t \leq \epsilon)$, $\zeta(t) = 0$ $(t \geq 2\epsilon)$, $0 \leq \zeta(t) \leq 1$, $|\zeta'(t)| \leq 2/\epsilon$ $(t \in \mathbb{R})$. We note that $w = P(v) \in C^{\alpha,\alpha/2}(Q, \mathbb{R}^q) \subset C^{\alpha',\alpha'/2}(Q, \mathbb{R}^q)$ and that $w(x,0) = 0$ holds from the definition of $P(v)$, $v \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,1], \mathbb{R}^q)$ and $v(x,0) = f(x)$. From Lemma 5.5 we get a constant $C > 0$ independent of ϵ and w such that the estimate

$$|\zeta w|_Q^{(\alpha',\alpha'/2)} \leq C\epsilon^{(\alpha-\alpha')}|w|_Q^{(\alpha,\alpha/2)} \tag{5.10}$$

holds. Set $k = -\zeta w$. Then $k(x,0) = 0$. From (5.10) we have $|k|_Q^{(\alpha',\alpha'/2)} < \delta$ for sufficiently small ϵ. Thus, there exists a $u \in C^{2+\alpha',1+\alpha'/2}(\Sigma \times [0,\epsilon], \mathbb{R}^q)$ such that the following special case of (5.9) holds:

$$\begin{cases} P(u)(x,t) = 0, & (x,t) \in \Sigma \times (0,\epsilon), \\ u(x,0) = f(x). \end{cases}$$

In other words, we have obtained a solution $u \in C^{2+\alpha',1+\alpha'/2}(\Sigma \times [0,\epsilon],\mathbb{R}^q)$ to the initial value problem

$$\begin{cases} (\Delta - \partial_t)\, u(x,t) = \Pi_u(du,du)(x,t) + \tilde{Z}_u((du)^{\underline{k}})(x,t), & (x,t) \in \Sigma \times (0,\epsilon), \\ u(x,0) = f(x). \end{cases}$$

As we have

$$f \in C^{2+\alpha}(\Sigma,\mathbb{R}^q), \quad \Pi_u(du,du),\, \tilde{Z}_u((du)^{\underline{k}}) \in C^{\alpha,\alpha/2}(\Sigma \times [0,\epsilon],\mathbb{R}^q),$$

we see by Theorem 5.4 that

$$u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,\epsilon],\mathbb{R}^q).$$

Due to compactness of Σ and continuity of u we always can reach that $u(\Sigma \times [0,\epsilon']) \subset \tilde{M}$ holds true if we choose $0 < \epsilon' < \epsilon$ small enough . Replacing ϵ by ϵ' if necessary, we may assume that $u(\Sigma \times [0,\epsilon]) \subset \tilde{M}$ holds true. Thus, u is a solution to the IVP (5.1) in $\Sigma \times [0,\epsilon]$. It is also clear from the above proof that $\epsilon > 0$ is a positive number only depending on Σ, M, Z, f and α. ☐

As a result of combining Proposition 5.1 and Theorem 5.3, we obtain the following.

Corollary 5.6. *Let (Σ,g) and (M,G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let $Z \in \Gamma(\mathrm{Hom}(\Lambda^k TM, TM))$. For a given $C^{2+\alpha}$ map $f \in C^{2+\alpha}(\Sigma, M)$ there exist a positive number $T = T(\Sigma,M,Z,f,\alpha) > 0$ and a map $\varphi \in C^{2+\alpha,1+\alpha}(\Sigma \times [0,T], M)$ such that*

$$\begin{cases} \tau(\varphi_t)(x) = Z((d\varphi_t)^{\underline{k}})(x) + \frac{\partial \varphi_t}{\partial t}(x), & (x,t) \in \Sigma \times (0,T), \\ \varphi(x,0) = f(x) \end{cases} \tag{5.11}$$

holds. Here, $T = T(\Sigma,M,Z,f,\alpha) > 0$ is a constant depending on Σ, M, Z, f and α alone.

Noting the result concerning differentiability of the solutions to a linear parabolic partial differential equation, we obtain the following.

Theorem 5.7 (Short time existence)**.** *Let (Σ,g) and (M,G) be Riemannian manifolds, and Σ be compact and oriented. Furthermore let $Z \in \Gamma(\mathrm{Hom}(\Lambda^k TM, TM))$. For a given $C^{2+\alpha}$ map $f \in C^{2+\alpha}(\Sigma, M)$ there exist a positive number $T = T(\Sigma,M,Z,f,\alpha) > 0$ and a map $\varphi \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,T], M) \cap C^\infty(\Sigma \times (0,T), M)$ such that*

$$\begin{cases} \tau(\varphi_t)(x) = Z((d\varphi_t)^{\underline{k}})(x) + \frac{\partial \varphi_t}{\partial t}(x), & (x,t) \in \Sigma \times (0,T), \\ \varphi(x,0) = f(x) \end{cases} \tag{5.12}$$

holds. Here, $T = T(\Sigma,M,Z,f,\alpha) > 0$ is a constant depending on Σ, M, Z, f and α alone.

Proof. Let $u \in C^{2+\alpha,1+\alpha/2}(\Sigma \times [0,T], M)$ be the solution in Corollary 5.6. Since differentiability is a local property, we may verify it locally at each point $(p,t) \in \Sigma$. As in the proof of Theorem 4.1 denote by (x^i) and (y^α) the local coordinate systems near p and $\varphi(p,t)$, respectively. With respect to these coordinates (5.12) reads, for each $\varphi^\alpha = y^\alpha \circ \varphi$,

$$(\Delta - \frac{\partial}{\partial t})\, \varphi^\alpha = -g^{ij}\tilde{\Gamma}^\alpha_{\beta\gamma}(\varphi)\frac{\partial \varphi^\beta}{\partial x^i}\frac{\partial \varphi^\gamma}{\partial x^j} + |g|^{\frac{3}{2}}\frac{\partial \varphi^{\mu_1}}{\partial x^1}\cdots\frac{\partial \varphi^{\mu_k}}{\partial x^k} Z^\alpha_{\mu_1\ldots\mu_k}, \tag{5.13}$$

where $k = \dim(\Sigma)$. Noting that the right hand side is $C^{1+\alpha,\alpha/2}$ from the assumption on u, we see that the theorem (see Appendix B.2) concerning differentiability of solutions to linear parabolic partial differential equations implies that φ is $C^{3+\alpha,1+\alpha/2}$ differentiable. Thus, the right hand side is of $C^{2+\alpha,1+\alpha/2}$. Then again we see that u is $C^{4+\alpha,1+\alpha/2}$. Iterating this argument we see that φ must be C^∞. ☐

Chapter 6

Long time existence

Let (Σ, g) and (M, G) be Riemannian manifolds of dimension k and n, respectively, and Σ be compact and oriented. Let $Z \in \Gamma(\mathrm{Hom}(\Lambda^k TM, TM))$ be a k-force determined by some closed $(k+1)$-form $\Omega \in \Gamma(\Lambda^{k+1}T^*M)$ via (2.14).

To prove long time existence of a solution $\varphi : \Sigma \times [0, T) \to M$ to the initial value problem (IVP) for the system of nonlinear parabolic partial differential equations

$$\begin{cases} \tau(\varphi_t)(x) = Z((d\varphi_t)^{\underline{k}})(x) + \frac{\partial \varphi_t}{\partial t}(x), & (x,t) \in \Sigma \times (0,T), \\ \varphi(x,0) = f(x), \end{cases} \tag{6.1}$$

one has to show that it exists when $T = \infty$. Short time existence of a solution to (6.1) can be guaranteed by Theorem 5.7 in contrast to long time existence. As already mentioned in Chapter 4 it becomes an essential matter to control the growth rate of the solution $\varphi(x, t)$ in time t. In order to get a grip on the "blowing up" effects of the nonlinear terms of the equation, the dimension of Σ and the curvature of M play a crucial role in this game. In fact, in $\dim(\Sigma) > 1$ the nonlinear terms possibly may destroy the long time behavior of our solutions. In this section we will reveal the relationship between the existence of long time solutions to our problem and the curvature of M. The main ingredients are the energy estimates and the maximum principle for parabolic equations. Both are typical tools in the theory of linear partial differential equations to get a priori estimates that allow to show e.g. uniqueness and stability of solutions. For an introduction to this topic see [9], [20]. Here we state a version of the maximum principle that will suffice our needs.

Lemma 6.1 (Maximum principle)**.** *Let (Σ, g) be a compact Riemannian manifold. Furthermore let Δ be the Hodge-Laplacian of Σ and $L = \Delta - \frac{\partial}{\partial t}$ be the heat operator. Let $u \in C^0(\Sigma \times [0,T)) \cap C^{2,1}(\Sigma \times (0,T))$ be a real valued function in $\Sigma \times [0,T)$, which is C^2 in Σ and C^1 in $(0,T)$. If u satisfies $Lu \geq 0$ in $\Sigma \times (0,T)$, then*

$$\max_{\Sigma \times [0,T)} u = \max_{\Sigma \times \{0\}} u$$

holds. Said in words, the maximum of u in the "cylinder" $\Sigma \times [0,T)$ is achieved at the bottom of the cylinder, i.e. in $\Sigma \times \{0\}$.

Proof. Let $\delta, \epsilon > 0$ be positive numbers and set

$$\tilde{u}(x,t) = u(x,t) - \delta t, \quad Q = \Sigma \times [0, T - \epsilon]$$

We verify that for $\tilde{u}$, we have

$$\max_{Q} \tilde{u} = \max_{\Sigma\times\{0\}} \tilde{u}. \tag{6.2}$$

In fact, since $\tilde{u}$ is continuous in Q, it attains the maximum at a point $(x_0, t_0) \in Q$. We suppose that $t_0 > 0$ and derive a contradiction from it. Since $Lu \geq 0$ in $\Sigma \times (0,T)$ from the assumption, $\tilde{u}$ satisfies at (x_0, t_0)

$$\frac{\partial \tilde{u}}{\partial t} \leq \Delta \tilde{u} - \delta.$$

In other words, if (x^i) are local coordinates near x_0,

$$\frac{\partial \tilde{u}}{\partial t} \leq g^{ij}\Big\{\frac{\partial^2 \tilde{u}}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial \tilde{u}}{\partial x^k}\Big\} - \delta$$

holds at (x_0, t_0). Here, Γ_{ij}^k's are the Christoffel symbols of the Levi-Civita connection in Σ. Since $\tilde{u}(x_0, t_0)$ is the maximum of $\tilde{u}$ in Q, we get

$$\frac{\partial \tilde{u}}{\partial t}(x_0,t_0) \geq 0, \quad \frac{\partial \tilde{u}}{\partial x^i}(x_0, t_0) = 0$$

and the matrix $\left(\frac{\partial^2 \tilde{u}}{\partial x^i \partial x^j}\right)$ is nonpositive definite at the point (x_0, t_0). But this contradicts $\delta > 0$. Hence, (6.2) has been proved. The lemma follows by letting $\delta, \epsilon \to 0$. ☐

In the sequel we denote by S^1 the unit circle in $\mathbb{R}^2$, carrying the induced metric by $\mathbb{R}^2$. Set $E = \mathrm{Hom}(\Lambda^k TM, TM)$. From Corollary 4.7 in Chapter 4 and the maximum principle we gain the following estimates for a solution to the IVP (6.1).

Proposition 6.2 (Energy estimates)**.** *Assume that $\Sigma = S^1$ and Z is a Lorentz force. Let $\varphi = \gamma \in C^{2,1}(S^1 \times [0,T), M) \cap C^\infty(S^1 \times (0,T), M)$ be a solution to the* IVP (6.1) *and set $\gamma_t(s) = \gamma(s,t)$. Then the following hold:*

(1) *If $|Z|_{L^\infty(M,E)} < \infty$, then for all $(s,t) \in S^1 \times [0,T)$,*

$$e(\gamma_t)(s) \leq e^{\lambda T} \sup_{s\in\Sigma} e(f)(s).$$

(2) *If $K^M \leq 0$ and $|Z|_{L^\infty(M,E)}, |\nabla Z|_{L^\infty(M,E)} < \infty$, then for all $(s,t) \in S^1 \times [0,T)$,*

$$\Big|\frac{\partial \gamma}{\partial t}(s,t)\Big| \leq e^{CT} \sup_{s\in\Sigma} \Big|\frac{\partial \gamma}{\partial t}(s,0)\Big|.$$

Here, $\lambda = \lambda(M,Z)$ and $\mu = \mu(M, \nabla Z)$ are the constants defined in Corollary 4.7 *and $C = C(\Sigma, M, Z, \nabla Z, f, T) = \lambda + \mu\, e^{\frac{\lambda T}{2}} \max_{\Sigma} e(f)^{1/2}$ is a constant depending on $\Sigma, M, Z, \nabla Z, f$ and T alone.*

Proof. ad (1): From (1') of Corollary 4.7 we see

$$Le(\gamma_t) = \Big(\Delta - \frac{\partial}{\partial t}\Big)\, e(\gamma_t) \geq -\lambda e(\gamma_t).$$

Putting $v(s,t) = e^{-\lambda t}\, e(\gamma_t)(s)$, a straightforward computation shows that v satisfies $Lv \geq 0$ in $S^1 \times (0,T)$. Hence, from the maximum principle and the definition of the energy density $e(\gamma_t)$

$$e^{-\lambda t}\, e(\gamma_t)(s) = v(s,t) \leq \max_{s \in S^1} v(s,0) = \max_{s \in S^1} e(f)(s)$$

holds at any $(s,t) \in S^1 \times [0,T)$.

ad (2): Let C be the constant defined as above. From (1) of Proposition 6.2 and (2') of Corollary 4.7 we see that for $v(s,t) := e^{-Ct}\, \kappa(\gamma_t)(s)$, we have $L\kappa(\gamma_t) \geq 0$ in $S^1 \times (0,T)$. Hence, from the maximum principle and the definition of the energy density $\kappa(\gamma_t)$

$$e^{-Ct}\Big|\frac{\partial \gamma}{\partial t}(s,t)\Big|^2 = 2\, v(s,t) \leq 2 \max_{s \in S^1} v(s,0) = \max_{s \in S^1} \Big|\frac{\partial \gamma}{\partial t}(s,0)\Big|^2$$

holds at any $(s,t) \in S^1 \times [0,T)$. $\square$

Proposition 6.2 implies that the growth rate of a solution γ to the initial value problem (6.1) is uniformly bounded on $S^1 \times [0,T)$ with respect to the time variable $t \in [0,T)$, if $K^M \leq 0$ and $|Z|_{L^\infty(M,E)}, |\nabla Z|_{L^\infty(M,E)} < \infty$. More precisely we state the following.

Proposition 6.3. *Assume that $\Sigma = S^1$. Furthermore let (M,G) be a compact Riemannian manifold and $\varphi = \gamma \in C^{2,1}(S^1 \times [0,T), M) \cap C^\infty(S^1 \times (0,T), M)$ be a solution to the* IVP *(6.1). Set $\gamma_t(s) = \gamma(s,t)$. Let $K^M \leq 0$ and Z be a Lorentz force. Then for any $0 < \alpha < 1$ there exists a positive number $C = C(\Sigma, M, Z, f, \alpha, T) > 0$ such that*

$$|\gamma(\cdot,t)|_{C^{2+\alpha}(S^1,M)} + \Big|\frac{\partial \gamma}{\partial t}(\cdot,t)\Big|_{C^\alpha(S^1,M)} \leq C$$

holds at any $t \in [0,T)$. Here, $C = C(\Sigma, M, Z, \nabla Z, f, \alpha, T)$ is a constant only depending on $\Sigma, M, Z, \nabla Z, f, \alpha$ and T.

Proof. We set $\gamma_t'(s) = \frac{\partial \gamma}{\partial s}(s,t)$. All metrics and norms here are the natural induced ones. As in the proof of Proposition 5.3, we assume the (M,G) is realized as a Riemannian submanifold in a q-dimensional Euclidean space $\mathbb{R}^q$ via an isometric imbedding $\iota : M \hookrightarrow \mathbb{R}^q$ and that the vector valued function $\gamma : S^1 \times [0,T) \to \mathbb{R}^q$ is a solution to the IVP (5.1). Furthermore let $\tilde{Z}$ be the smooth extension of Z, constructed at the beginning of Chapter 5. However, since γ, from the assumption, is a solution to the IVP (6.1), the solution stays inside $M \subset \mathbb{R}^q$ and therefore all expressions, terms and constants c_i, appearing in the course of the proof will only depend on Z and its covariant derivatives, but not on $\tilde{Z}$ and its covariant derivatives. Thus, for simplicity we denote $\tilde{Z}$ by Z.

Now, depending on the point of view, γ satisfies an elliptic and, on the other hand, a parabolic partial differential equation. We will exploit both positions in order to attain our result. Taking the first view, γ satisfies the system of elliptic partial differential equations

$$\Delta \gamma = \Pi_\gamma(d\gamma, d\gamma) + Z_\gamma(d\gamma) + \frac{\partial \gamma}{\partial t},$$

where Δ is the Hodge-Laplacian in Σ. Noting Proposition 6.2, we see that the right hand side of the above equation is bounded independent of $t \in [0, T)$, i.e. we have

$$\left|\Pi_\gamma(d\gamma, d\gamma)(\cdot, t) + Z_\gamma(d\gamma)(\cdot, t) + \frac{\partial\gamma}{\partial t}(\cdot, t)\right|_{L^\infty(S^1,\mathbb{R}^q)} \leq c_1(\Sigma, M, Z, f, T). \tag{6.3}$$

In fact, for all $(s, t) \in S^1 \times [0, T)$ we have

$$\begin{aligned}\left|\Pi_\gamma(d\gamma, d\gamma)(s, t) + Z_\gamma(d\gamma)(\cdot, t) + \frac{\partial\gamma}{\partial t}(s, t)\right| &= \left|(\nabla_{\gamma_t'} d\pi)(\gamma_t')(s) + Z_\gamma(\gamma_t')(s) + \frac{\partial\gamma_t}{\partial t}(s)\right| \\ &\leq |\nabla d\pi|_{L^\infty(M,E)}\, |\gamma_t'(s)|^2 + \frac{1}{2}|Z|^2_{L^\infty(M,E)} + \frac{1}{2}|\gamma_t'(s)|^2 + \left|\frac{\partial\gamma_t}{\partial t}(s)\right|.\end{aligned}$$

The right hand side of this inequality can be estimated from above by Proposition 6.2 with a constant c_1 only depending on $\Sigma, M, Z, \nabla Z, f$ and T (actually c_1 also depends on $|\nabla d\pi|_{L^\infty(M,E)}$, but we won't pick this up in our notation). Here, $|Z|_{L^\infty(M,E)} = \sup_M \langle \nabla Z, \nabla Z\rangle^{1/2}$ and $|\nabla d\pi|_{L^\infty(M,E)} = \sup_M \langle \nabla d\pi, \nabla d\pi\rangle^{1/2}$. This shows (6.3).

Since the image of γ is always contained in the bounded set $M \subset \mathbb{R}^q$, at any $t \in [0, T)$ we have

$$|\gamma(\cdot, t)|_{L^\infty(S^1,\mathbb{R}^q)} \leq c_2(M). \tag{6.4}$$

Hence, by the Schauder estimate (see Appendix B.3 and B.4) for the solutions to an elliptic partial differential equation, at any $t \in [0, T)$ we have

$$\begin{aligned}|\gamma(\cdot, t)|_{C^{1+\alpha}(S^1,\mathbb{R}^q)} &\leq c_3(\Sigma, \alpha)\Big(\sup_{t\in[0,T)} |\Delta\gamma(\cdot, t)|_{L^\infty(S^1,\mathbb{R}^q)} + \sup_{t\in[0,T)} |\gamma(\cdot, t)|_{L^\infty(S^1,\mathbb{R}^q)}\Big) \\ &\leq c_4(\Sigma, M, Z, \nabla Z, f, \alpha, T).\end{aligned} \tag{6.5}$$

Taking the second view, γ is also a solution to the system of parabolic partial differential equations

$$L\gamma = \Pi_\gamma(d\gamma, d\gamma) + Z_\gamma(d\gamma),$$

where $L = \Delta - \frac{\partial}{\partial t}$ is the heat operator in S^1. Regarding (6.5) we see that

$$|\Pi_\gamma(d\gamma, d\gamma)(\cdot, t) + Z_\gamma(d\gamma)(\cdot, t)|_{C^\alpha(S^1,\mathbb{R}^q)} \leq c_5(\Sigma, M, Z, \nabla Z, f, \alpha, T)$$

holds. Using the Schauder estimate for linear parabolic partial differential equations (see Appendix B.3 and B.4), we get for any $t \in [0, T)$

$$\begin{aligned}&|\gamma(\cdot, t)|_{C^{2+\alpha}(S^1,\mathbb{R}^q)} + \left|\frac{\partial\gamma}{\partial t}(\cdot, t)\right|_{C^\alpha(S^1,\mathbb{R}^q)} \\ &\qquad\leq c_6(\Sigma, \alpha)\Big(\sup_{t\in[0,T)} |L\gamma(\cdot, t)|_{C^\alpha(S^1,\mathbb{R}^q)} + \sup_{t\in[0,T)} |\gamma(\cdot, t)|_{L^\infty(S^1,\mathbb{R}^q)}\Big) \\ &\qquad\leq c_7(\Sigma, M, Z, \nabla Z, f, \alpha, T).\end{aligned}$$

□

So far we have nothing said about stability and uniqueness of solutions to our problem. This will be done now.

Theorem 6.4 (Stability and uniqueness of solutions)**.** *Assume that $\Sigma = S^1$. Let (M^n, G) be a Riemannian manifold and $Z, Z' \in \Gamma(\mathrm{Hom}(TM, TM))$ be Lorentz forces. Let $u, v \in C^0(S^1 \times [0,T), M) \cap C^{2,1}(S^1 \times (0,T), M)$. Setting $u_t(s) = u(s,t)$ and $v_t(s) = v(s,t)$, assume that u satisfies the evolution equation for magnetic geodesics*

$$\frac{\nabla}{\partial s}\frac{\partial u_t}{\partial s}(s) = Z(\frac{\partial u_t}{\partial s})(s) + \frac{\partial u_t}{\partial t}(s), \qquad (s,t) \in S^1 \times (0,T), \tag{6.6}$$

and similarly that v satisfies (6.6) *with Z' instead of Z. Furthermore assume that Z and Z' are bounded, i.e. $|Z|_{L^\infty(M,E)}, |Z'|_{L^\infty(M,E)} < \infty$. Then for any $0 < T_0 < T$ there exists a constant $C = C(T_0) \geq 0$ such that*

$$|u_t - v_t|^2_{L^2(\Sigma,M)} \leq 2\pi e^{Ct}\Big(|u_0 - v_0|^2_{L^\infty(\Sigma,M)} + t|Z - Z'|^2_{L^\infty(M,E)}\Big) \tag{6.7}$$

holds for all $t \in [0, T_0]$. Here, $E = \mathrm{Hom}(\Lambda^k TM, TM)$, $|Z|_{L^\infty(M,E)} = \sup_M \langle Z, Z\rangle^{1/2}$ and $C = C(T_0) \geq 0$ is a nonnegative constant depending on T_0 and other parameters. The dependence is clarified in the course of the proof. In particular, $u_0 = v_0$ and $Z = Z'$ imply $u = v$ throughout $\Sigma \times [0,T)$.

Proof. As in the proof of Proposition 5.3 we regard u, v as vector valued functions $u, v : S^1 \times [0,T) \to \iota(M) \subset \mathbb{R}^q$, and consider u, v as solutions to the system of nonlinear parabolic differential equations (5.1). Let $\tilde{Z}$ and $\tilde{Z}'$ be the smooth extensions of Z and Z', respectively, constructed as at the beginning of Chapter 5. However, since the solution must stay in $M \cong \iota(M) \subset \mathbb{R}^q$, the majority of appearing expressions, involving $\tilde{Z}$ and $\tilde{Z}'$, only depend on Z and Z'. Define a function $h : \Sigma \times [0,T) \to \mathbb{R}$ by

$$h(s,t) = |u(s,t) - v(s,t)|^2, \quad (s,t) \in S^1 \times [0,T).$$

For $u_1, u_2 \in C^2(S^1, \mathbb{R}^q)$, one computes

$$\Delta\langle u_1, u_2\rangle = \langle \Delta u_1, u_2\rangle + 2\langle du_1, du_2\rangle + \langle u_1, \Delta u_2\rangle,$$

and hence for $u_1 = u_2 = u - v$ we get

$$\Delta h = \Delta\langle u - v, u - v\rangle = 2\langle \Delta u - \Delta v, u - v\rangle + 2|du - dv|^2.$$

On the other hand, one has

$$\frac{\partial h}{\partial t} = 2\langle \Delta u - \Delta v - \big(\Pi_u(du, du) - \Pi_v(dv, dv) + Z_u(du) - Z'_v(dv)\big), u - v\rangle.$$

Then for $L = \Delta - \frac{\partial}{\partial t}$ it follows

$$\begin{aligned} Lh = {} & \langle \Pi_u(du, du) - \Pi_v(dv, dv), u - v\rangle + \langle Z_u(du) - Z'_v(dv), u - v\rangle \\ & + 2|du - dv|^2. \end{aligned} \tag{6.8}$$

Now, for $0 < T_0 < T$ we choose a number $r = r(T_0)$ such that $u(S^1 \times [0,T_0]) \cup v(S^1 \times [0,T_0])$ is contained in the open ball $B(0,r) = \{x \in \mathbb{R}^q \mid |x| < r\}$. Rewriting

$$Z_u(du) - Z'_v(dv) = (Z_u - Z_v)(du) + (Z_v - Z'_v)(du) + Z'_v(du - dv)$$

and applying the Mean Value Theorem to $Z_u - Z_v$, we get for any $(s,t) \in S^1 \times [0, T_0]$

$$\begin{aligned}
(6.9) \qquad & |\langle Z_u(du) - Z_v(dv), u - v\rangle| \\
&\le c_1\,|u-v|^2 + 2^{1/2}|Z - Z'|_{L^\infty(M,E)}\, e(u_t)^{1/2}|u-v| \\
&\quad + c_3\,|du - dv||u - v| \\
&\le c_1\,|u-v|^2 + |Z - Z'|^2_{L^\infty(M,E)} + c_2\,|u-v|^2 \\
&\quad + c_3\,|du - dv||u - v|.
\end{aligned}$$

Here, $c_1, c_2, c_3 \ge 0$ are nonnegative constants. c_1 only depends on $\Sigma, M, \nabla\tilde{Z}, T_0$ and on the maximum value of the energy density $e(u_t)$ on $\Sigma \times [0, T_0]$, c_2 only on the maximum value of the energy density $e(u_t)$ on $\Sigma \times [0, T_0]$, whereas c_3 only depends on $\overline{B(0,r)}$ and Z', i.e. on T_0 and Z'. Note that the energy densities can be globally estimated independent of T_0 by virtue of Proposition 6.2. In fact, noting $\sup_{\overline{B(0,r)}} |\nabla\tilde{Z}| < \infty$ (here $|\nabla\tilde{Z}| = \langle \nabla\tilde{Z}, \nabla\tilde{Z}\rangle^{1/2}$ as usual) and applying the Mean Value Theorem (see Appendix B.5) yields Lipschitz continuity, namely

$$|\tilde{Z}_x(\xi) - \tilde{Z}_y(\xi)| \le (\sup_{\overline{B(0,r)}} |\nabla\tilde{Z}|)|\xi||x - y|$$

holds, for all $x, y \in \overline{B(0,r)} \subset \mathbb{R}^q$ and all $\xi \in \Lambda^k\mathbb{R}^q$. Here we have identified $\Lambda^k T_x\mathbb{R}^q \cong \Lambda^k T_y\mathbb{R}^q \cong \Lambda^k\mathbb{R}^q$ by parallel transport. From this, (6.9) can readily be verified. Similarly rewriting

$$\begin{aligned}
&\Pi_u(du, du) - \Pi_v(dv, dv) \\
&\qquad = (\Pi_u - \Pi_v)(du, du) + \Pi_v(du - dv, du) + \Pi_v(dv, du - dv)
\end{aligned}$$

and applying the Mean Value Theorem to $\Pi_u - \Pi_v$, we get for any $(s,t) \in S^1 \times [0, T_0]$

$$\begin{aligned}
(6.10) \qquad & |\langle \Pi_u(du, du) - \Pi_v(dv, dv), u - v\rangle| \\
&\qquad \le c_4\,|u - v|^2 + c_5\,|du - dv||u - v|,
\end{aligned}$$

where $c_4, c_5 \ge 0$ are constants only depending on Σ, M, on the maximum values of the energy densities $e(u_t)$ and $e(v_t)$ on $S^1 \times [0, T_0]$, and on derivatives of the canonical projection $\pi : \tilde{M} \to M$ up to third order. Using Cauchy's inequality $ab \le \epsilon a^2 + (4\epsilon)^{-1}b^2$ $(a, b \ge 0,\ \epsilon > 0)$ for the terms

$$\mathit{constant} \cdot |du - dv||u - v|,$$

we obtain from (6.8), (6.9) and (6.10) for any $(s,t) \in S^1 \times [0, T_0]$

$$\begin{aligned}
Lh &\ge -|\langle \Pi_u(du, du) - \Pi_v(dv, dv), u - v\rangle| - |\langle Z_u(du) - Z'_v(dv), u - v\rangle| \\
&\quad + 2|du - dv|^2 \\
&\ge -C|u - v|^2 - |Z - Z'|^2_{L^\infty(M,E)} = -Ch - |Z - Z'|^2_{L^\infty(M,E)},
\end{aligned}$$

where $C \ge 0$ is a constant only depending on $\Sigma, M, Z, \nabla\tilde{Z}, Z', T_0$, on the maximum values of the energy densities $e(u_t)$ and $e(v_t)$ on $S^1 \times [0, T_0]$, and on derivatives of the canonical projection $\pi : \tilde{M} \to M$ up to third order. Integrating and using the Divergence Theorem (see Appendix B.6) yields for any $t \in [0, T_0]$

$$\frac{d}{dt}\int_\Sigma h(\cdot, t)\, d\mathrm{vol}_g \le C\int_\Sigma h(\cdot, t)\, d\mathrm{vol}_g + 2\pi|Z - Z'|^2_{L^\infty(M,E)}.$$

Here, g denotes the canonical metric of $\Sigma = S^1 \subset \mathbb{R}^2$ induced by $\mathbb{R}^2$. Applying Gronwall's Lemma (see Appendix B.7) to the function $H : [0,T) \to \mathbb{R}$ defined by $H(t) = \int_\Sigma h(\cdot,t)\, d\mathrm{vol}_g$, we get for any $t \in [0,T_0]$

$$H(t) \le e^{Ct}\big(H(0) + 2\pi t|Z - Z'|^2_{L^\infty(M,E)}\big).$$

$\square$

Corollary 6.5. *Let Σ, M, Z, Z', u, v and the assumptions on them as above in Theorem 6.4. If in addition M is compact, then* (6.7) *holds for all $t \in [0,T)$.*

Proof. Since M is compact, the ball $B(0,r)$ in the above proof can be chosen such that $M \subset B(0,r) \subset \mathbb{R}^q$. The boundedness of Z and Z' (need not to be assumed, but follows from the compactness of M) implies that the energy densities $e(u_t)$ and $e(v_t)$ can be globally estimated on $[0,T)$ by Proposition 6.2. Consequently the constant $C \ge 0$ from the above proof can be chosen to be independent of T_0.

$\square$

Finally after this preliminary work we can prove the existence of long time solutions to the initial value problem of the parabolic equation for magnetic geodesics.

Theorem 6.6 (Long time existence)**.** *Let $\Sigma = S^1$ and (M,G) be a compact Riemannian manifold of nonpositive curvature, $K^M \le 0$. Moreover let $Z \in \Gamma(\mathrm{Hom}(TM,TM))$ be a Lorentz force. Set $\gamma_t(s) = \gamma(s,t)$. Then for any $C^{2+\alpha}$ map $f \in C^{2+\alpha}(S^1, M)$, there exists a unique $\gamma \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0,\infty), M) \cap C^\infty(S^1 \times (0,\infty), M)$ such that*

$$\begin{cases} \frac{\nabla}{\partial s}\gamma_t'(s) = Z(\gamma_t')(s) + \frac{\partial \gamma_t}{\partial t}(s), & (s,t) \in S^1 \times (0,\infty), \\ \gamma(s,0) = f(s), & \end{cases} \tag{6.11}$$

holds.

Proof. Short time existence is guaranteed by Theorem 5.7, namely there exists a positive number $T = T(\Sigma, M, Z, f, \alpha) > 0$ such that, without making any curvature assumptions, the initial value problem (6.11) has a solution $\gamma \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0,T], M) \cap C^\infty(S^1 \times (0,T), M)$ in $S^1 \times [0,T]$. We have to demonstrate now that our solution can not blow up in finite time if M is compact and of nonpositive curvature, $K^M \le 0$, i.e. that our solution γ can be extended to $S^1 \times [0,\infty)$. Setting

$$T_0 = \sup\{t \in [0,\infty) \,|\, (6.11) \text{ has a solution in } S^1 \times [0,t]\},$$

we must show that $T_0 = \infty$ holds. Assume that this would not be the case. Then choose any sequence of numbers $\{t_i\} \subset [0,T_0)$ such that $t_i \to T_0$ as i tends to ∞. As in the proof of Proposition 6.3 we regard M to be an isometrically imbedded submanifold in some Euclidean space $\mathbb{R}^q$ and each $\gamma(\cdot,t_i) \in C^\infty(S^1, M)$ as a $\mathbb{R}^q$-valued function. We set $\gamma_t(s) = \gamma(s,t)$, $\gamma' = \gamma_t' = \frac{\partial \gamma}{\partial s}$, $\partial_t = \frac{\partial}{\partial t}$ and choose a positive number α' such that $0 < \alpha < \alpha' < 1$. Since S^1 is compact, it follows that the imbedding $C^{k+\alpha'}(S^1, \mathbb{R}^q) \hookrightarrow C^{k+\alpha}(S^1, \mathbb{R}^q)$ is compact. By Proposition 6.3 the sequences

$$\{\gamma(\cdot,t_i)\} \quad \text{and} \quad \{\partial_t \gamma(\cdot,t_i)\},$$

respectively, are bounded in $C^{2+\alpha'}(S^1, \mathbb{R}^q)$ and in $C^{\alpha'}(S^1, \mathbb{R}^q)$. Thus, there exist a subsequence $\{t_{i_k}\}$ of $\{t_i\}$ and functions

$$\gamma(\cdot, T_0) \in C^{2+\alpha}(S^1, \mathbb{R}^q) \quad \text{and} \quad \partial_t \gamma(\cdot, T_0) \in C^{\alpha}(S^1, \mathbb{R}^q)$$

such that the subsequences

$$\{\gamma(\cdot, t_{i_k})\} \quad \text{and} \quad \{\partial_t \gamma(\cdot, t_{i_k})\},$$

respectively, converge uniformly to $\gamma(\cdot, T_0)$ and $\partial_t \gamma(\cdot, T_0)$, as $t_{i_k} \to T_0$. Since for each t_{i_k} we have

$$\partial_t \gamma(\cdot, t_{i_k}) = \frac{\nabla}{\partial s}\gamma'(\cdot, t_{i_k}) - Z(\gamma')(\cdot, t_{i_k}),$$

we also get at T_0

$$\partial_t \gamma(\cdot, T_0) = \frac{\nabla}{\partial s}\gamma'(\cdot, T_0) - Z(\gamma')(\cdot, T_0).$$

Consequently, we see that (6.11) has a solution in $S^1 \times [0, T_0]$. Application of Theorem 5.7 with $\gamma(\cdot, T_0)$ as initial value, yields an positive number $\epsilon > 0$ such that the IVP

$$\text{(6.12)} \qquad \begin{cases} \frac{\nabla}{\partial s}\gamma_t'(s) = Z(\gamma_t')(s) + \frac{\partial \gamma_t}{\partial t}(s), & (s,t) \in S^1 \times (T_0, T_0 + \epsilon), \\ \gamma(s, 0) = \gamma(s, T_0) \end{cases}$$

has a solution $\gamma \in C^{2+\alpha, 1+\alpha/2}(S^1 \times [T_0, T_0 + \epsilon], M)$ in $S^1 \times [T_0, T_0 + \epsilon]$. Noting that this and the previous solution coincide on $S^1 \times \{0\}$, we can patch them together to a solution $\gamma \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0, T_0 + \epsilon], M)$ to the IVP (6.11). From the arguments concerning the differentiability of the solutions in Theorem 5.7 we see that γ is C^∞ in $S^1 \times (0, T_0 + \epsilon)$. Hence, (6.11) has a solution in $S^1 \times [0, T_0 + \epsilon]$ which contradicts the definition of T_0. Consequently $T_0 = \infty$. The uniqueness of γ immediately follows from Theorem 6.4. □

Remark 6.7. The compactness of Σ cannot be dropped. In general, if Σ is non-compact the lifetime T of a solution to the IVP (6.11) may be finite. For example, let $\Sigma = M = \mathbb{R}$ and $T > 0$ be a positive number. Consider the function $u : \mathbb{R} \times [0, T) \to \mathbb{R}$ defined by

$$u(s,t) = \frac{s}{T - t}.$$

This is a smooth function on $\mathbb{R} \times [0, T)$ which blows up as $t \to T$. Let $Z : T\mathbb{R} \to T\mathbb{R}$ be the bundle homomorphism defined by $Z_s(v) := -sv$, $(s, v) \in \mathbb{R} \times \mathbb{R}$. The function u solves the IVP (6.11) on $\mathbb{R} \times (0, T)$, with initial condition $u(s, 0) = s/T$ and the above defined Z. In this case the parabolic equation just reads

$$v' = -uv + \dot{u}, \quad \text{on } \mathbb{R} \times (0, T),$$

where $\dot{u} = \frac{\partial u}{\partial t}$, $v = \frac{\partial u}{\partial s}$ and $v' = \frac{\partial^2 u}{\partial s^2}$. This demonstrates that the lifetime of solutions to the IVP (6.11) can be finite for non-compact Σ.

Corollary 6.8. *Let (M, G) be a Riemannian manifold of nonpositive curvature, $K^M \leq 0$, and $Z \in \Gamma(\mathrm{Hom}(TM, TM))$ be a Lorentz force. Furthermore, let H be a discrete group of isometries of (M, G) acting properly discontinuously on M. If Z is H-invariant, i.e. $dh \circ Z = Z \circ dh$ for all $h \in H$, and the quotient M/H is compact, then for any $C^{2+\alpha}$ map $f \in C^{2+\alpha}(S^1, M)$, there exists a unique long time solution $\gamma \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0, \infty), M) \cap C^\infty(S^1 \times (0, \infty), M)$ to the* IVP (6.11) *in M.*

Proof. Let Z be a H-invariant Lorentz force and $\tilde{M} := M/H$ be compact. Equipping $\tilde{M}$ with the unique structure of a smooth manifold and the unique Riemannian metric $\tilde{G}$ such that the canonical projection $\pi : M \to \tilde{M}$ becomes a Riemannian covering, we have $K^{\tilde{M}} \leq 0$. The H-invariance of Z implies that it descends to a well-defined Lorentz force $\tilde{Z}$ on $\tilde{M}$. Now, projecting everything down to $\tilde{M}$, we consider the corresponding IVP (6.11) in $\tilde{M}$ with $\tilde{Z}$ and initial condition $\tilde{f} = \pi \circ f$. From Theorem 6.6 we get a long time solution $\tilde{\gamma} \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0,\infty), \tilde{M}) \cap C^\infty(S^1 \times (0,\infty), \tilde{M})$ with $\tilde{\gamma}(s,0) = \tilde{f}(s)$. By the Homotopy Lifting Property of coverings there exists a lifting $\gamma : S^1 \times [0,\infty) \to M$ of $\tilde{\gamma}$ with $\gamma(s,0) = f(s)$. Concerning differentiability properties everything is preserved under the lifting since the covering π is smooth, i.e. $\gamma \in C^{2+\alpha,1+\alpha/2}(S^1 \times [0,\infty), M) \cap C^\infty(S^1 \times (0,\infty), M)$. Moreover one readily verifies that γ is a solution to the original IVP (6.11) in M; in fact, one just has to keep the following in mind: H- invariance of Z, the evolution equation is a geometric equation, satisfying a PDE is a local property, π is a local isometry.
The uniqueness can be similarly seen. Assume that $\gamma_1, \gamma_2 : S^1 \times [0,\infty) \to M$ are solutions to the IVP (6.11) in M with $\gamma_1(\cdot,0) = \gamma_2(\cdot,0)$, then $\tilde{\gamma}_i = \pi \circ \gamma_i$ $(i = 1,2)$ are solutions to the projected IVP (6.11) in $\tilde{M}$ which coincide on $S^1 \times \{0\}$. Consequently, by the uniqueness statement of Theorem 6.11, they must agree throughout $S^1 \times [0,\infty)$. Since S^1 is connected and γ_1, γ_2 are both liftings of $\tilde{\gamma}_1$ which agree on $S^1 \times \{0\}$, they must agree throughout $S^1 \times [0,\infty)$. By the way, the uniqueness can also be seen by directly applying Theorem 6.4. This is possible since the H-invariance of Z and the compactness of the quotient $\tilde{M} = M/H$ imply that Z is bounded. ☐

Example 6.9. Let (M,G) be the three-dimensional Euclidean space $\mathbb{R}^3$ and $B \in \mathbb{R}^3$ be a parallel vector field in $\mathbb{R}^3$, (all tangent spaces of $\mathbb{R}^3$ are identified by parallel transport). We define a skew-symmetric bundle homomorphism $Z : T\mathbb{R}^3 \to T\mathbb{R}^3$, $Z(v) = v \times B$ for all $v \in \mathbb{R}^3$, by means of the vector product. From $\nabla Z = 0$ we see that, in fact, Z comes from a closed two-form Ω via (2.14). Since Z is translation-invariant and the three-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ is compact, we deduce long time existence of solutions to the IVP (6.11) from Corollary 6.8. This holds more generally for any $\mathbb{Z}^3$-invariant Lorentz force Z.

Corollary 6.10. *Let $\Sigma = S^1$ and (M,G) be a Riemannian manifold of nonpositive curvature, $K^M \leq 0$. Furthermore let $Z \in \Gamma(\mathrm{Hom}(TM,TM))$ be a Lorentz force and $\gamma \in C^{2,1}(S^1 \times [0,T), M) \cap C^\infty(S^1 \times (0,T), M)$ be a solution to the* IVP (6.11)*, where $T = \sup\{t \in [0,\infty) \,|\, (6.11)$ has a solution in $S^1 \times [0,t]\}$. Set $\gamma_t(s) = \gamma(s,t)$. If $T < \infty$, then for any compact subset $K \subset M$ and any $0 < T_0 < T$, there exists a $t \in (T_0,T)$ such that $\gamma_t(S^1) \cap (M - K) \neq \emptyset$. Said in words: If the lifetime T of a solution γ is finite, then it leaves any compact subset of M, or equivalently, if a solution γ stays its entire life in a compact set, then its lifetime $T = \infty$.*

Proof. Let $T < \infty$ and assume that the conclusion is false. Then there exist a compact subset $K \subset M$ such that $\gamma(S^1 \times [0,T)) \subset K$ holds. Set $E = \mathrm{Hom}(\Lambda^k TM, TM)$. Now, we proceed quite literally as in the proof of Proposition 6.3 and obtain

$$|\gamma(\cdot,t)|_{C^{2+\alpha}(S^1,M)} + \Big|\frac{\partial\gamma}{\partial t}(\cdot,t)\Big|_{C^\alpha(S^1,M)} \leq C.$$

Here, $C = C(\Sigma,K,M,Z,f,\alpha,T)$ is a constant only depending on $\Sigma, K, M, Z, f, \alpha$ and T. The only difference is that in all estimates (energy estimates etc.) one has to replace all appearances of $|\cdot|_{L^\infty(M,E)}$ by $|\cdot|_{L^\infty(K,E|_K)}$. Obviously (6.4) holds since $\gamma(S^1\times[0,T))\subset K$. Then similarly as in the proof of Theorem 6.6 one extends the solution to $S^1\times[0,T+\epsilon]$ (for $\epsilon>0$ sufficient small) and produces a contradiction to the definition of T. $\square$

Conclusion and outlook. We see that the energy estimates (Corollary 4.6) are crucial to make the "long time existence proof" work. If $k=\dim(\Sigma)=1$, the maximum principle can be applied to obtain good a priori estimates for the energy densities. Even in the case $k>1$, the maximum principle is not applicable and the proof breaks down. The greater $k>1$ is, the worse the nonlinearities become. Perhaps in $\dim(\Sigma)=2$, where the nonlinearities are "only" of quadratic order in du, i.e. $|Z((du)^{\underline{k}})|\le C|du|^k$ ($C>0$ a constant) for a bounded k-force Z, existence of weak long time solution can be shown. It would be an interesting task to prove the existence of long time solutions in this case especially regarding the relevance of this question in String theory. Also an open question is the third item of program presented in Chapter 4: Does one always find a convergent subsequence of a long time solution to the IVP (6.11) when (M,G) is compact Riemannian manifold of nonpositive curvature, $K^M\le 0$?

Appendix A

Notation and definitions

(a) Geometric notation.

Let (Σ, g) and (M, G) be Riemannian manifolds and (E, ∇^E, h) be a Riemannian vector bundle over M. For simplicity we denote the metrics g, G, h and all the induced metrics and connections on the various tensor bundles by $\langle \cdot, \cdot \rangle$ and ∇, respectively. If Σ is oriented, then we denote by vol_g the canonical volume form on Σ (similarly for M). If Σ is not orientable, then in expressions

$$\int_\Sigma f \, d\mathrm{vol}_g,$$

where $f : \Sigma \to \mathbb{R}$ is an integrable function, the symbol $d\mathrm{vol}_g$ is to mean the Riemannian measure which can be defined for any Riemannian manifold. The notion of integrability of functions etc., in the sense of Lesbeque, is defined with respect to this measure. A set $A \subset \Sigma$ is called measurable if it belongs to the Borel algebra of Σ. A map $f : \Sigma \to M$ is called measurable if the preimage (under f) of any measurable subset of M is measurable. The characteristic function of a measurable set $A \subset \Sigma$ is denoted by χ_A and its volume by $V(A) = \int_\Sigma \chi_A \, d\mathrm{vol}_g$.

Definition A.1. A *Riemannian vector bundle* over M is a triple (E, ∇^E, h) consisting of a real vector bundle E and a connection ∇^E on E that is compatible with the metric h of E, i.e. $\nabla^E_X h = 0$ for all $X \in T_x M$, for all $x \in M$.

For a vector valued k-form $\omega \in \Gamma(\Lambda^k T^*M \otimes E)$ the covariant derivative is defined as usual by (the index E is suppressed in the following formulas):

$$(\nabla_X \omega)(X_1, \ldots, X_k) = \nabla_X(\omega(X_1, \ldots, X_k)) - \sum_j \omega(X_1, \ldots, \nabla_X X_j, \ldots, X_k), \tag{A.1}$$

where $X_1, \ldots, X_k \in \Gamma(TM)$. The induced curvature is defined by

$$R(X_1, X_2)\omega = \{\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]}\}\omega. \tag{A.2}$$

The *exterior differential* $d : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^{k+1} T^*M \otimes E)$ is analogously defined by formula (2.10). Similarly one defines the *co-differential* $\delta : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^{k-1} T^*M \otimes E)$ by

$$\delta\omega(X_1, \ldots, X_{k-1}) = -(\nabla_{e_i}\omega)(e_i, X_1, \ldots, X_{k-1}), \tag{A.3}$$

where $\{e_i\}$ is a local orthonormal frame field. The *Hodge-Laplace operator* $\Delta : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^k T^*M \otimes E)$ then is given by

$$\Delta = -\{d\delta + \delta d\}, \tag{A.4}$$

and the *rough Laplacian* $\bar{\Delta} : \Gamma(\Lambda^k T^*M \otimes E) \to \Gamma(\Lambda^k T^*M \otimes E)$ by

$$\bar{\Delta}\omega = \{\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i}e_i}\}\omega \tag{A.5}$$

Furthermore on $\Gamma(\Lambda^k T^*M \otimes E)$ we use the following convention for the induced metric. Let $\{e_i\}$ be an orthonormal frame near $x \in M$, then for $\alpha, \beta \in \Gamma(\Lambda^k T^*M \otimes E)$ we define

$$\langle \alpha, \beta \rangle_\wedge := \sum_{i_1 < \cdots < i_k} \langle \alpha(e_{i_1}, \ldots, e_{i_k}), \beta(e_{i_1}, \ldots, e_{i_k}) \rangle.$$

We distinguish this metric from that naturally induced metric for non totally skew-symmetric k-linear vector valued tensor fields $\alpha, \beta \in \Gamma(\bigotimes^k T^*M \otimes E)$ which is given by

$$\langle \alpha, \beta \rangle = \langle \alpha(e_{i_1}, \ldots, e_{i_k}), \beta(e_{i_1}, \ldots, e_{i_k}) \rangle.$$

Note that this two definitions are related by a factor $1/k!$, namely for $\alpha, \beta \in \Gamma(\Lambda^k T^*M \otimes E) \subset \Gamma(\bigotimes^k T^*M \otimes E)$, we have

$$\langle \alpha, \beta \rangle_\wedge = \frac{1}{k!} \langle \alpha, \beta \rangle.$$

In this paper we supress the subscript $\wedge$ with the convention that $\langle \alpha, \beta \rangle$ is to mean $\langle \alpha, \beta \rangle_\wedge$ if $\alpha, \beta \in \Gamma(\Lambda^k T^*M \otimes E)$. In particular, for the volume element we have $|\mathrm{vol}_G| = \langle \mathrm{vol}_G, \mathrm{vol}_G \rangle^{1/2} = 1$ due to this convention.

We recall some notions from Linear Algebra. Let (V, g) and (W, h) be a Euclidean vector spaces. There is a canonical isomorphism $V \to V^*$, $\xi \mapsto g(\xi, \cdot)$. We denote it by $\xi^\flat$ for $\xi \in V$ and its inverse by $\omega^\sharp$ for $\omega \in V^*$. One can extend these musical isomorphisms from k-vectors $\Lambda^k V$ to k-forms $\Lambda^k V^*$. On decomposable k-vectors it is defined by $(\xi_1 \wedge \ldots \wedge \xi_k)^\flat := \xi_1^\flat \wedge \ldots \wedge \xi_k^\flat$ and extended by linearity. In the same way one defines $(\omega_1 \wedge \ldots \wedge \omega_k)^\sharp := \omega_1^\sharp \wedge \ldots \wedge \omega_k^\sharp$ on decomposable k-forms. It should always be clear from the context to which metric $\sharp$ and $\flat$ are referring so that we are not picking up the reference to the metric in our notation. Here we use the convention

$$(\omega_1 \wedge \ldots \wedge \omega_k)(\xi_1, \ldots, \xi_k) = \sum_{\sigma \in S_k} (-1)^\sigma \omega_1(\xi_{\sigma(1)}) \cdots \omega_k(\xi_{\sigma(k)}),$$

where S_k denotes the permutation group of order k, i.e. σ runs over all k-permutations. The sign $(-1)^\sigma$ of the permutation equals $+1$ if the permutation σ is even and -1 if it is odd. More general, for linear maps $A_1, \ldots, A_k$ from V to W we define a skew-symmetric k-linear map $A_1 \tilde{\wedge} \ldots \tilde{\wedge} A_k : V^k \to \Lambda^k W$ by

$$(A_1 \tilde{\wedge} \ldots \tilde{\wedge} A_k)(\xi_1, \ldots, \xi_k) = \sum_{\sigma \in S_k} (-1)^\sigma A_1(\xi_{\sigma(1)}) \wedge \ldots \wedge A_k(\xi_{\sigma(k)}).$$

By the universal property of the exterior product this induces a linear map $A_1\tilde{\wedge}\dots\tilde{\wedge}A_k : \Lambda^k V \to \Lambda^k W$, denoted by the same symbol, such that on decomposable k-vectors, we have

$$(A_1\tilde{\wedge}\dots\tilde{\wedge}A_k)(\xi_1 \wedge \dots \wedge \xi_k) = \sum_{\sigma\in S_k} (-1)^\sigma A_1(\xi_{\sigma(1)}) \wedge \dots \wedge A_k(\xi_{\sigma(k)}).$$

For a single linear map $A : V \to W$ we define a linear map $A^{\underline{k}} : \Lambda^k V \to \Lambda^k W$ by

$$A^{\underline{k}} := \frac{1}{k!} A^k, \tag{A.6}$$

where A^k denotes the k-fold $\tilde{\wedge}$-product of A with itself,

$$A^k = \underbrace{A\tilde{\wedge}\dots\tilde{\wedge}A}_{\text{k-times}}.$$

Note that for $\xi_1, \dots, \xi_k \in V$ with $|\xi_i| \le 1$ $(i = 1, \dots, k)$, we have $|A^{\underline{k}}(\xi_1 \wedge \dots \wedge \xi_k)| \le |A|^k$. Here $|A| = \left[\sum_i \langle A(e_i), A(e_i)\rangle\right]^{1/2}$ for any orthonormal basis $\{e_i\}$ of V. Let $X = \mathrm{Hom}(V, W)$ denote the vector space of endomorphisms from V to W and set $\tilde{\Lambda}^k X = \mathrm{Hom}(\Lambda^k V, \Lambda^k W)$. There is a natural product $\tilde{\Lambda}^k X \otimes \tilde{\Lambda}^l X \to \tilde{\Lambda}^{k+l} X$ given by

$$(A_1\tilde{\wedge}\dots\tilde{\wedge}A_k) \otimes (A_{k+1}\tilde{\wedge}\dots\tilde{\wedge}A_{k+l}) \mapsto A_1\tilde{\wedge}\dots\tilde{\wedge}A_{k+l}$$

which is associative and symmetric. Note that in general $\tilde{\Lambda}^k X \neq \Lambda^k X$, e.g. for $A \in X$ we have $A\tilde{\wedge}A \neq 0$ in $\tilde{\Lambda}^k X$, but $A \wedge A = 0$ in $\Lambda^k X$. Let $A_1, \dots, A_k : E \to F$ be bundle homomorphisms, (E, ∇^E) and (F, ∇^F) be bundles with connection over a Riemannian manifold (M, G) and $\eta, \xi_1, \dots, \xi_k \in \Gamma(TM)$. Then we define a connection $\tilde{\nabla}$ on $\tilde{\Lambda}^k X$ (here $X = \mathrm{Hom}(E, F)$) by

$$\begin{aligned}&\big(\tilde{\nabla}_\eta(A_1\tilde{\wedge}\dots\tilde{\wedge}A_k)\big)(\xi_1 \wedge \dots \wedge \xi_k) := \\ &\quad \nabla_\eta(A_1\tilde{\wedge}\dots\tilde{\wedge}A_k)(\xi_1 \wedge \dots \wedge \xi_k) - (A_1\tilde{\wedge}\dots\tilde{\wedge}A_k)\big(\nabla_\eta(\xi_1 \wedge \dots \wedge \xi_k)\big).\end{aligned}$$

For convenience we have denoted the natural induced connections on $\Lambda^k E$ and $\Lambda^k F$, respectively, simply by ∇. It follows immediately that the Leibniz rule is satisfied, i.e.

$$\tilde{\nabla}_\eta(A_1\tilde{\wedge}\dots\tilde{\wedge}A_k) = \tilde{\nabla}_\eta A_1\tilde{\wedge}A_2\tilde{\wedge}\dots\tilde{\wedge}A_k + \dots + A_1\tilde{\wedge}A_2\tilde{\wedge}\dots\tilde{\wedge}A_{k-1}\tilde{\wedge}\tilde{\nabla}_\eta A_k.$$

(b) Function spaces.

Let (Σ, g) and (M, G) be compact Riemannian manifolds and (E, ∇^E, h) be a Riemannian vector bundle over M. Then we define the following spaces.

$C^0(\Sigma, M) = \{u : \Sigma \to M \mid u \text{ is continuous}\}$
$C^k(\Sigma, M) = \{u : \Sigma \to M \mid u \text{ is } k\text{-times continuous differentiable }\}$ for $k \ge 1$
$C^\infty(\Sigma, M) = \{u : \Sigma \to M \mid u \text{ is smooth }\} = \bigcap_{k=0}^\infty C^k(\Sigma, M)$
$\Gamma(M, E) = C^\infty(M, E) = \{s : M \to E \text{ is a smooth section in } E\}$

If the reference to the base space is clear, we just write $\Gamma(E) = \Gamma(M, E)$.

For $M = \mathbb{R}$ and $0 \le k \le \infty$ we set $C^k(\Sigma) = C^k(\Sigma, \mathbb{R})$.

Let $|\cdot|$ denote the norm induced by the h of E. Then for $1 \leq p \leq \infty$ the L^p-spaces are defined as follows.

$L^p(M, E) = \{s : M \to E \mid s$ is s measurable section and $|s|_{L^p(M,E)} < \infty\}$ for $p \neq \infty$

Here for $p = \infty$, we put $|s|_{L^\infty(M,E)} = \inf\{r \in \mathbb{R} \mid |u| \leq r$ holds a.e.$\}$ and for $1 \leq p < \infty$

$$|s|_{L^p(M,E)} = \left(\int\limits_M |s|^p \, d\mathrm{vol}_G \right)^{1/p}.$$

If $E = M \times \mathbb{R}^q$ is the trivial bundle with canonical metric and trivial connection over M, for $1 \leq p \leq \infty$ we set $L^p(M, \mathbb{R}^q) = L^p(M, M \times \mathbb{R}^q)$ and especially for $q = 1$ we write $L^p(M) = L^p(M, \mathbb{R})$. By $L^p(\Sigma, M)$ we mean the space $\{u \in L^p(\Sigma, \mathbb{R}^q) \mid u(\Sigma) \subset M\}$, where $M \subset \mathbb{R}^q$ is regarded as an isometrically imbedded submanifold in some Euclidean space $\mathbb{R}^q$.

Let $0 < \alpha < 1$ be a positive real number, k be a nonnegative integer, and $U \subset \mathbb{R}^n$ be an open subset in $\mathbb{R}^n$. For $u \in C^k(U)$ we define

$$|u|_{C^{k+\alpha}(U)} = \sum_{|\beta| \leq k} \sup_U |D^\beta u| + \sum_{|\beta| = k} \langle D^\beta u \rangle_U^{(\alpha)}.$$

Here

$$\langle u \rangle_U^{(\alpha)} = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

and $D^\beta u$ is given by

$$D^\beta u = D_1^{\beta_1} D_2^{\beta_2} \cdots D_n^{\beta_n} u, \qquad D_i = \frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n,$$

where $\beta = (\beta_1, \ldots, \beta_n)$ denotes a multi-index consisting of n nonnegative integers β_i's and $|\beta| = \beta_1 \cdots \beta_n$ denotes its length. Then for $k \geq 1$ the Hölder spaces are given by

$C^{k+\alpha}(U) = \{u : U \to \mathbb{R} \mid u$ is k-times continuous differentiable and $|u|_{C^{k+\alpha}(U)} < \infty\}$,

and for $k = 0$ we set $C^{0+\alpha}(U) = C^\alpha(U) = \{u \in C^0(U) \mid |u|_{C^{0+\alpha}(U)} < \infty\}$.
On a Riemannian manifold M one defines the Hölder spaces $C^{k+\alpha}(M)$ as follows: Let $d(x, y)$ be the Riemannian distance function on M and let $\mathrm{injrad}(x)$ denote the injectivity radius for a point $x \in M$. For a function $u \in C^k(M)$, we set

$$\langle D^k u \rangle_M^{(\alpha)} = \sup_{\substack{y \in B(x, \mathrm{injrad}(x)) - \{x\} \\ x \in M}} \frac{|(\nabla^k u)_x - \widetilde{(\nabla^k u)_y}|}{d(x, y)^\alpha},$$

where $\widetilde{(\nabla^k u)_y}$ denotes the parallel translated tensor along the unique minimal geodesic joining x and y. On M we define a Hölder norm by

$$|u|_{C^{k+\alpha}(M)} = \sum_{i=0}^{k} \sup_M |D^i u| + \langle D^k u \rangle_M^{(\alpha)}$$

and the space $C^{k+\alpha}(M)$ as above. Here, $|D^i u| = \langle \nabla^i u, \nabla^i u \rangle^{1/2}$. If U is a convex neighborhood one defines for $u \in C^k(U)$

$$|u|_{C^{k+\alpha}(U)} = \sum_{i=0}^{k} \sup_U |D^i u| + \langle D^k u \rangle_U^{(\alpha)},$$

where

$$\langle D^k u \rangle_U^{(\alpha)} = \sup_{\substack{x,y \in U \\ x \neq y}} \frac{|(\nabla^k u)_x - \widetilde{(\nabla^k u)_y}|}{d(x,y)^\alpha}.$$

The local Hölder space $C^{k+\alpha}(U)$ is defined as above. If in addition M is compact, we see that a function $u \in C^k(M)$ belongs to $C^{k+\alpha}(M)$ iff the restriction $u|_U$ belongs to $C^{k+\alpha}(U)$ for any convex neighborhood $U \subset M$. For a vector valued function $u : M \to \mathbb{R}^q$, we say that u belongs to $C^{k+\alpha}(M, \mathbb{R}^q)$ if all its components u^i belong to $C^{k+\alpha}(M)$. Finally, by $C^{k+\alpha}(\Sigma, M)$ we mean the space $\{u \in C^{k+\alpha}(\Sigma, \mathbb{R}^q) \mid u(\Sigma) \subset M\}$, where $M \subset \mathbb{R}^q$ is regarded as an isometrically imbedded submanifold in some Euclidean space $\mathbb{R}^q$.

Appendix B

Analytical toolbox

(a) Differentiability of solutions

Let $U \subset \mathbb{R}^n$ be a bounded and connected open set, and let P be a linear elliptic partial differential operator given by

$$P = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x^i} + d(x).$$

Theorem B.1. (1) *Given $0 < \alpha < 1$, assume that $a^{ij}, b^i, d, f \in C^{\alpha}(U)$. Then $u \in C^{2+\alpha}(U)$ holds if $u \in C^2(U)$ satisfies the linear partial differential equation $Pu(x) = f(x)$.* (2) *Furthermore, if $a^{ij}, b^i, d, f \in C^{k+\alpha}(U)$ for a given $k \geq 1$, then a solution u to* (1) *is $C^{k+2+\alpha}$. In particular, if $a^{ij}, b^i, d, f \in C^{\infty}(U)$, then $u \in C^{\infty}(U)$.*

There is an analogous result for linear parabolic partial differential equations. Given $T > 0$, set $Q = U \times (0, T)$. For a function $u : Q \to \mathbb{R}$, we set

$$\langle u \rangle_x^{(\alpha)} = \sup_{\substack{(x,t),(x',t) \in Q \\ x \neq x'}} \frac{|u(x,t) - u(x',t)|}{|x - x'|^{\alpha}},$$

$$\langle u \rangle_t^{(\alpha)} = \sup_{\substack{(x,t),(x,t') \in Q \\ t \neq t'}} \frac{|u(x,t) - u(x,t')|}{|t - t'|^{\alpha}}.$$

The norms $|u|_Q^{(\alpha,\alpha/2)}$ and $|u|_Q^{(2+\alpha,1+\alpha/2)}$ are defined as (5.4) in Chapter 5 and the Hölder spaces $C^{\alpha,\alpha/2}(Q)$, $C^{2+\alpha,1+\alpha/2}(Q)$ with respect to these norms are as given in Appendix A(b). We then have the following.

Theorem B.2. (1) *Given $0 < \alpha < 1$, assume that $a^{ij}, b^i, d \in C^{\alpha}(U)$ and $f \in C^{\alpha,\alpha/2}(Q)$. Then $u \in C^{2+\alpha,1+\alpha/2}(Q)$ holds, if $u \in C^{2,1}(Q)$ satisfies the following linear parabolic partial differential equation*

$$\left(P - \frac{\partial}{\partial t}\right) u(x,t) = f(x,t).$$

(2) *Let p, q be nonnegative integers. Given β, κ with $|\beta| \leq p$, $|\beta| + 2\kappa \leq p$, $\kappa \leq q$, assume that $D_x^{\beta} a^{ij}, D_x^{\beta} b^i, D_x^{\beta} d \in C^{\alpha}(U)$ and $D_x^{\beta} D_t^{\kappa} f \in C^{\alpha,\alpha/2}(Q)$. Then a solution u to* (1) *satisfy $D_x^{\beta} D_t^{\kappa} u \in C^{\alpha,\alpha/2}(Q)$ for any β, κ with $|\beta| + 2\kappa \leq p+2$, $\kappa \leq q+1$. In particular, $a^{ij}, b^i, d \in C^{\infty}(U)$ and $f \in C^{\infty}(Q)$ imply that $u \in C^{\infty}(Q)$.*

Concerning the above mentioned results see [11], [10] and [18].

(b) Schauder estimates.

Given $r > 0$, set $B(0,r) = \{x \in \mathbb{R}^n \mid |x| < r\}$. For an $0 < \alpha < 1$, assume that

$$a^{ij}, b^i, d \in C^\alpha(B(0,r)),\ 1 \le i, j \le m,$$

and that P is uniformly elliptic, i.e. that

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a^{ij}(x)\xi^i\xi^j \le \Lambda|\xi|^2$$

holds for some constants $0 < \lambda \le \Lambda < \infty$ and for any $x \in B(0,r)$ and $\xi \in \mathbb{R}^n$. Then, concerning the linear elliptic partial differential operator

$$P = \sum_{i,j=1}^n a^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n b^i(x)\frac{\partial}{\partial x^i} + d(x)$$

and the linear parabolic partial differential operator

$$L = P - \frac{\partial}{\partial t}$$

the following holds.

Theorem B.3. (1) *If $f \in C^\alpha(B(0,r))$ and $u \in C^2(B(0,r))$ satisfy*

$$Pu(x) = f(x)$$

then $u \in C^{2+\alpha}(B(0,r))$ and

$$\begin{aligned}
|u|_{C^{1+\alpha}(B(0,r/2))} &\le C\big(|f|_{L^\infty(B(0,r))} + |u|_{L^\infty(B(0,r))}\big),\\
|u|_{C^{2+\alpha}(B(0,r/2))} &\le C\big(|f|_{C^\alpha(B(0,r))} + |u|_{L^\infty(B(0,r))}\big)
\end{aligned}$$

hold. Here, C is a constant only determined by $n, \alpha, \Lambda/\lambda, |a^{ij}|_{C^\alpha(B(0,r))}$, $|b^i|_{C^\alpha(B(0,r))}, |d|_{C^\alpha(B(0,r))}$.

(2) *Let $0 \le t \le T$. If $f(\cdot,t) \in C^\alpha(B(0,r))$ and $u(\cdot,t) \in C^2(B(0,r))$ satisfy*

$$Lu(x,t) = f(x,t),$$

then $u(\cdot,t) \in C^{2+\alpha}(B(0,r))$ and

$$\begin{aligned}
|u(\cdot,t)|_{C^\alpha(B(0,r/2))} &\le C\Big(\sup_{t\in[0,T]}|f(\cdot,t)|_{L^\infty(B(0,r))} + \sup_{t\in[0,T]}|u(\cdot,t)|_{L^\infty(B(0,r))}\Big)\\
|u(\cdot,t)|_{C^{2+\alpha}(B(0,r/2))} &+ \Big|\frac{\partial u}{\partial t}(\cdot,t)\Big|_{C^\alpha(B(0,r))}\\
&\le C\Big(\sup_{t\in[0,T]}|f(\cdot,t)|_{C^\alpha(B(0,r))} + \sup_{t\in[0,T]}|u(\cdot,t)|_{L^\infty(B(0,r))}\Big)
\end{aligned}$$

hold. Here, C is a constant only determined by $n, \alpha, \Lambda/\lambda, |a^{ij}|_{C^\alpha(B(0,r))}$, $|b^i|_{C^\alpha(B(0,r))}, |d|_{C^\alpha(B(0,r))}$.

Concerning the above mentioned results see [11] and [25].

Remark B.4. The Schauder estimates are used in Section 6; due to compactness the local estimates presented here carry over to the entire manifold Σ in Proposition 6.3. Namely, let $r > 0$ be a positive number and $x_1, \ldots x_N \in \Sigma$ be a finite number of points such that $B(x_i, r)$ $(i = 1, \ldots N)$ is a convex neighborhood of x_i and such that $\Sigma = \bigcup_{i=1}^N B(x_i, r)$. Here, $B(x, r) = \{y \in \Sigma \mid d(x, y) < r\}$ denotes the open ball with center $x \in \Sigma$ and radius r with respect to the Riemannian distance function $d(x, y)$ on Σ. Set $\mathcal{U} = \{B(x_i, r)\}$. For a function $u \in C^k(\Sigma)$, we define

$$|u|_{C^{k+\alpha}(\mathcal{U})} = \max_i \{|u|_{C^{k+\alpha}(B(x_i,r))}\}.$$

For a function $u : \Sigma \to \mathbb{R}^q$ with components $u^j \in C^k(\Sigma)$ $(j = 1, \ldots, q)$, we set

$$|u|_{C^{k+\alpha}(\mathcal{U})} = \max_j \{|u^j|_{C^{k+\alpha}(\mathcal{U})}\}.$$

Recalling the definition of the Hölder norm (see Appendix A(b) above), one easily verifies that there exist constants $C_1 = C_1(\mathcal{U}), C_2 = C_2(\mathcal{U}) > 0$, only depending on $\mathcal{U}$, such that

$$C_1 |u|_{C^{k+\alpha}(\mathcal{U})} \leq |u|_{C^{k+\alpha}(\Sigma)} \leq C_2 |u|_{C^{k+\alpha}(\mathcal{U})}$$

holds for all $u \in C^{k+\alpha}(\Sigma)$. Put another way, $|\cdot|_{C^{k+\alpha}(\Sigma)}$ and $|\cdot|_{C^{k+\alpha}(\mathcal{U})}$ are equivalent norms on the space $C^{k+\alpha}(\Sigma)$ for a compact Riemannian manifold Σ. So, we see that the constant $C = C(\Sigma, M, Z, f, \alpha, T)$ in Proposition 6.3 actually depends on the covering $\mathcal{U} = \mathcal{U}(\Sigma)$ so that, to be exact, one should write $C = C(\mathcal{U}(\Sigma), M, Z, f, \alpha, T)$. However, we ignore this subtlety and just mention that in the proof of Proposition 6.3 one fixes some covering and does the estimates with respect to it.

(c) Miscellaneous.

Lemma B.5 (Mean Value Theorem)**.** *Let (M, g) be a Riemannian manifold and $u \in C^1(M)$ be a C^1 differentiable $\mathbb{R}$-valued function on M. Denote by $d(x, y)$ the Riemannian distance function on M. Then for any compact convex subset $K \subset M$*

$$|u(x) - u(y)| \leq \sup_K |du|\, d(x, y)$$

holds for all $x, y \in K$. Here we denote all norms which are induced by g by $|\cdot|$.

Proof. Let $x, y \in K$ be arbitrary points. Choose the unique minimal geodesic $\gamma : [0, 1] \to M$ joining x and y in K. Applying the Mean Value Theorem to $u \circ \gamma : [0, 1] \to \mathbb{R}$, we see

$$|u(x) - u(y)| = |\int_0^1 (u \circ \gamma)'(s) ds| \leq \sup_K |du| \int_0^1 |\gamma'(s)| ds = \sup_K |du|\, d(x, y).$$

□

Theorem B.6 (Divergence Theorem)**.** *Let (M, g) be a compact Riemannian manifold and $X \in \Gamma(TM)$ be a vector field on M. Then*

$$\int_M \operatorname{div}(X)\, d\mathrm{vol}_g = 0,$$

where dvol_g *denotes the Riemannian measure of* M*. In particular, if* $u : M \to \mathbb{R}$ *is a smooth map, then we have*

$$\int_M \Delta u \, \mathrm{dvol}_g = 0.$$

Here $\mathrm{div}(X) = \mathrm{trace}\, \nabla X$ *and* $\Delta u = \mathrm{div}(\mathrm{grad}\, u)$*. A proof can be found in* [21].

Lemma B.7 (Gronwall's Lemma)**.** *Let* $h(t)$ *be a nonnegative function, continuous on* $[0, T]$ *and differentiable on* $(0, T)$*, which satisfies for all* $t \in (0, T)$ *the differential inequality*

$$h'(t) \le a(t)h(t) + b(t), \tag{B.1}$$

where $a(t)$ *and* $b(t)$ *are nonnegative, continuous functions on* $[0, T]$*. Then*

$$h(t) \le e^{\int_0^t a(s)ds} \Big\{ h(0) + \int_0^t b(s)ds \Big\} \tag{B.2}$$

for all $0 \le t \le T$*. In particular, if in addition,* $b \equiv 0$ *on* $[0, T]$ *and* $h(0) = 0$ *hold, then* $h \equiv 0$ *on* $[0, T]$*.*

Proof. Put $A(t) = \int_0^t a(\tau)d\tau$ and $B(t) = \int_0^t b(\tau)d\tau$. From (B.1) we see

$$\frac{d}{ds}\Big\{ h(s)e^{-A(s)} \Big\} = e^{-A(s)}\{h'(s) - a(s)h(s)\} \le e^{-A(s)}b(s)$$

for all $s \in (0, T)$. Let $\delta, \epsilon > 0$ be small positive numbers. Then for any $t \in [\epsilon, T - \delta]$ we have

$$h(t)e^{-A(t)} \le h(\epsilon)e^{-A(\epsilon)} + \int_\epsilon^t e^{-A(s)}b(s)ds \le h(\epsilon)e^{-A(\epsilon)} + \int_\epsilon^t b(s)ds.$$

Thus, (B.2) follows by letting $\delta, \epsilon \to 0$. □

Definition B.8. Let X, Y be Banach spaces and $U \subset X$ be an open subset of X. A map $f : U \to Y$ is called (*Fréchet*) *differentiable* at $p \in U$ if there exists a bounded linear operator $T : X \to Y$ such that

$$\lim_{h \to 0} \frac{|f(p + h) - f(p) - T(h)|}{|h|} = 0.$$

The unique operator T satisfying this condition is called the (*Fréchet*) *derivative* of f at p and is denoted by df_p, f_{*p}, $f'(p)$.

Theorem B.9 (Inverse Function Theorem)**.** *Let* X, Y *be Banach spaces. Let* $U \subset X$ *be an open set and* $f : U \to Y$ *be a map. Assume that its derivative* f' *exists and is continuous in* U*. Furthermore assume that at* $p \in U$*,* $df_p : X \to Y$ *is a linear isomorphism of Banach spaces, i.e.* $df_p : X \to Y$ *is a bounded linear operator, and bijective, with bounded inverse* $(df_p)^{-1} : Y \to X$*. Then there exists an open neighborhood* $W \subset U$ *of* p *such that* $f(W)$ *is an open subset in* Y *and* $f : W \to f(W)$ *is a homeomorphism.*

For a proof of this theorem, see [17].

Appendix C

Basics of gerbes

This chapter is not to be understood as an introduction to gerbes at full length. We just review some basic facts about Deligne cohomology $H^\bullet(M, \mathcal{D}^\bullet)$ of a manifold M, namely we only present here the local description of a gerbe and its holonomy in terms of an open good cover. Roughly speaking, the Deligne complex is a truncated Čech–de Rham complex. The cohomology of the total complex then is called the Deligne cohomology. Similarly to Čech cohomology, one defines this in terms of an open good cover and then passes to the direct limit with respect to refinements of open good covers. These means that a k-Deligne class $\mathcal{G} \in H^k(M, \mathcal{D}^k)$ is a certain cohomology class not depending on any choices of open good covers. However, for a concrete description of Deligne cohomology one can choose an open good cover and express it in terms of local data, i.e. we choose a Čech representative for Deligne cohomology. For a more detailed discussion of this topic see [3], [5], [13].

Definition C.1 (Čech representative of a gerbe)**.** Let M be a smooth manifold and $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be an open good cover of M, that is, every finite intersection of elements on $\mathcal{U}$ is contractible. A $(k-1)$-*gerbe* (or a k-*Deligne class*) $\mathcal{G} = [g, A^1, \ldots, A^k]$ over M can be represented by a $(k+1)$-tupel of families of forms of ascending degree that satisfy certain cocycle conditions. Namely, g is a family of smooth $U(1)$-valued maps $g_{\alpha_0,\ldots,\alpha_k} : U_{\alpha_0} \cap \ldots \cap U_{\alpha_k} \to U(1)$ and A^r $(1 \le r \le k)$ is a family of smooth $i\mathbb{R}$-valued differential forms $A^r_{\alpha_0,\ldots,\alpha_{k-r}}$ of degree r which are locally defined on the $(k+1-r)$-fold overlaps $U_{\alpha_0} \cap \ldots \cap U_{\alpha_{k-r}}$, satisfying the following cocycle conditions:

$$
\begin{aligned}
(\delta g)_{\alpha_0,\ldots,\alpha_{k+1}} &= 1 \quad \text{on} \quad U_{\alpha_0} \cap \ldots \cap U_{\alpha_{k+1}}, \\
(\delta A^1)_{\alpha_0,\ldots,\alpha_k} &= (-1)^{k-1} g^{-1}_{\alpha_0,\ldots,\alpha_k} d g_{\alpha_0,\ldots,\alpha_k} \quad \text{on} \quad U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}, \\
(\delta A^2)_{\alpha_0,\ldots,\alpha_{k-1}} &= (-1)^{k-2} d A^1_{\alpha_0,\ldots,\alpha_{k-1}} \quad \text{on} \quad U_{\alpha_0} \cap \ldots \cap U_{\alpha_{k-1}}, \\
&\vdots \\
A^k_\beta - A^k_\alpha &= d A^{k-1}_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta,
\end{aligned}
$$

where $i \in \mathbb{C}$ denotes the imaginary unit, δ the Čech differential and d the exterior differential on forms. Adopting the physicist's convention, we will repeatedly write the top degree term of a k-Deligne class as $B := -iA^k$. Physicists call it the *B-field* or *Kalb-Ramond field*.

Definition C.2. The *curvature* of a gerbe $\mathcal{G} = [g, A^1, \ldots, A^{k-1}, iB]$ over M is the globally defined $(k+1)$-form $\Omega = dB_\alpha$ which only depends on the gerbe $\mathcal{G}$. In fact, from the last one of the above cocycle conditions we see that $dB_\alpha = dB_\beta$ on $U_\alpha \cap U_\beta$, i.e. the family of locally defined $(k+1)$-forms dB_α patch together to a global well-defined $(k+1)$-form Ω. For the fact that Ω does not dependent on the choice of a Čech representative, see Remark C.8 below. A gerbe is called *flat* if its curvature Ω vanishes. Moreover $[\Omega]/2\pi \in H^{k+1}(M, \mathbb{R})$ is the image of an integral class and one refers to it, being the characteristic class of the gerbe, as *Diximier-Duady* class $c(\mathcal{G})$ of $\mathcal{G}$.

Remark C.3. In physics the curvature of a gerbe is denoted by H and called the *H-field.* Unfortunately this letter is reserved for the mean curvature of Riemannian hypersurfaces. Therefore, we prefer the notation Ω for the curvature of a gerbe.

Definition C.4. Let Σ and M be smooth manifolds and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \mathcal{A}}$ be an open good cover of M. Let $\mathcal{G} = [g, A^1, \ldots, A^{k-1}, iB]$ be a gerbe and assume that Σ is compact, oriented and without boundary. For a smooth map $\varphi : \Sigma \to M$ the *holonomy of $\mathcal{G}$ along φ* is given by the formula

$$\begin{aligned} \mathrm{hol}_{\mathcal{G}}(\varphi) = \prod_{\sigma^k} \exp i \int_{\sigma^k} \varphi^* B_{\rho(\sigma^k)} \cdot \prod_{r=1}^{k-1} \prod_{\sigma^{k-r} \subset \cdots \subset \sigma^k} & \exp \int_{\sigma^{k-r}} \varphi^* A^{k-r}_{\rho(\sigma^k) \ldots \rho(\sigma^{k-r})} \\ \cdot \prod_{\sigma^0 \subset \cdots \subset \sigma^k} & g_{\rho(\sigma^k) \ldots \rho(\sigma^0)}(\varphi(\sigma^0)). \end{aligned} \tag{C.1}$$

For example the last product is over all $(k+1)$-tupels $(\sigma^0, \ldots, \sigma^k)$ with $\sigma^0 \subset \ldots \subset \sigma^k$ etc. Here we have assumed that all the $\sigma^r \in \tau$ are the r-dimensional faces of an adequate chosen triangulation τ of Σ and $\rho : \tau \to \mathcal{A}$ is an index map picking up a finite number of members of the induced cover $\{\varphi^{-1}(U_\alpha)\}_{\alpha \in \mathcal{A}}$ such that $\sigma^0 \subset \varphi^{-1}(U_{\rho(\sigma^0)}), \ldots, \sigma^k \in \varphi^{-1}(U_{\rho(\sigma^k)})$ for all $\sigma^0, \ldots, \sigma^k \in \tau$ (this is possible because Σ is compact). Often we will abbreviate $\mathrm{hol} = \mathrm{hol}_{\mathcal{G}}$ provided that misunderstandings are excluded. In the special case $M = \Sigma$ and $\varphi = \mathrm{id}$ we call $\mathrm{hol}_{\mathcal{G}}(M) := \mathrm{hol}_{\mathcal{G}}(\mathrm{id})$ the *holonomy of $\mathcal{G}$ over M*. Note that $\mathrm{hol}_{\mathcal{G}}(M)$ is only defined for compact, oriented M without boundary.

Example C.5. If $k = 0$, then a Deligne class is a smooth map $f : M \to U(1)$ and the one-form associated to the class is $f^*\theta$, where θ denotes the Maurer-Cartan form on $U(1)$. The holonomy of the smooth map is just the product over the evaluations of f at all points p, namely, $\prod_{p \in M} f(p)$. Due to compactness this product is finite.

Example C.6. If $k = 1$, then a Deligne class can be represented by an isomorphism class of complex line bundles with connection. The holonomy is the classical holonomy of a connection.

Example C.7. If $k = 2$, then a Deligne class can be represented by a stable isomorphism class of bundle gerbes with connection and curving. The holonomy is the holonomy of a connection and curving. See [5], [16] for more details.

Definition C.8 (Gauge transformations)**.** By a *gauge transformation* we mean replacing a Čech representative of Deligne class $\mathcal{G} = [g, A^1, \ldots, A^k]$ by another Čech representative, with respect to a fixed open good cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$, namely

$$(g, A^1, \ldots, A^k) \to (g, A^1, \ldots, A^k) + D(t, T^1, \ldots, T^{k-1}),$$

where D is the differential of the Deligne complex. Explicitly, this reads

$$\begin{aligned}
g &\to g\,\delta(t), \\
A^1 &\to A^1 + \delta(T^1) + (-1)^{k-1}\,t^{-1}dt, \\
A^2 &\to A^2 + \delta(T^2) + (-1)^{k-2}dT^1, \\
&\;\vdots \\
A^k &\to A^k + dT^{k-1},
\end{aligned} \tag{C.2}$$

where h is a family of smooth $U(1)$-valued maps $h_{\alpha_0,\ldots,\alpha_{k-1}} : U_{\alpha_0} \cap \ldots \cap U_{\alpha_{k-1}} \to U(1)$ and T^r $(1 \le r \le k-1)$ is a family of smooth $i\mathbb{R}$-valued differential forms $T^r_{\alpha_0,\ldots,\alpha_{k-1-r}}$ of degree r which are locally defined on the $(k-r)$-fold overlaps $U_{\alpha_0} \cap \ldots \cap U_{\alpha_{k-1-r}}$. Also we see that the curvature Ω is unaffected by a gauge transformation, namely we have $d(-i(A^k + dT^{k-1})) = d(-iA^k) = \Omega$. By the way this shows that in fact Ω does not depend on the choice of a Čech representative, but only on the gerbe $\mathcal{G} = [g, A^1, \ldots, A^k]$.

Example C.9. Let $U(1) \hookrightarrow P \to M$ be a principal bundle over M with structure group $U(1)$ and connection one-form ω on P. Assume that $\mathcal{U} = \{U_\alpha\}_{\alpha\in\mathcal{A}}$ is a open good cover such that P can be trivialized over every U_α with trivialization $s_\alpha : U_\alpha \to P$. If $U_\alpha \cap U_\beta \neq \emptyset$, s_α and s_β differ on the overlap by a transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$, namely $s_\beta = s_\alpha\, g_{\alpha\beta}$. Letting the cocycle $\{g'_{\alpha\beta}\}$ come from another trivialization $\{s'_\alpha\}$, there must exist a family of functions $\{t_\alpha : U_\alpha \to U(1)\}$ such that on U_α we have $s'_\alpha = s_\alpha\, t_\alpha$. Then on the overlap $U_\alpha \cap U_\beta$, $s_\beta\, t_\beta = s'_\beta = s'_\alpha\, g'_{\alpha\beta} = s_\alpha\, t_\alpha\, g'_{\alpha\beta} = s_\beta\, g^{-1}_{\alpha\beta}\, t_\alpha\, g'_{\alpha\beta}$ must hold. This implies

$$g'_{\alpha\beta} = g_{\alpha\beta}\, t_\beta\, t_\alpha^{-1} \quad \text{on } U_\alpha \cap U_\beta. \tag{C.3}$$

If we put $A_\alpha := i(s^*_\alpha\, \omega)$ and $A'_\alpha := i(s'_\alpha)^*\omega$, then from the transformation formula for connection one-forms (see [2], [14] for example) we see that on U_α

$$A'_\alpha = A_\alpha + t_\alpha^{-1} dt_\alpha \tag{C.4}$$

must hold. Obviously (C.3) and (C.4) together form a gauge transformation in the sense of (C.2). Expressing the t_α's as $t_\alpha = \exp(i\varphi_\alpha)$ with real valued functions $\{\varphi_\alpha : U_\alpha \to \mathbb{R}\}$ (this is possible since the U_α's are contractible) we can rewrite (C.4) as

$$(s'_\alpha)^*\omega = s^*_\alpha\, \omega + d\varphi_\alpha.$$

This is what in electrodynamics is called a gauge transformation of a "vector potential" $s^*_\alpha\, \omega$.

Bibliography

[1] A. Bahri, I. A. Taimanov: *Periodic Orbits in Magnetic Fields and Ricci Curvature of Lagrangian Systems*, Transactions of the American Mathmatical Society, Vol. 350, No. 7, pp. 2697–2717; 1998

[2] D. Bleecker: *Gauge Theory and Variational Principles*, Addison-Wesley Publishing Company; 1981

[3] J.-L. Brylinski: *Loop Spaces, Charakteristic Classes and Geometric Quantization*, Progress in Mathmatics, Birkhäuser; 1993

[4] K. Burns, G. P. Paternain: *Anosov magnetic flows, critical values and topological entropy*, Institute of Physics Publishing, Nonlinearity 15, pp. 281–314; 2002

[5] A. Carey, S. Johnson, M. Murray: *Holonomy on D-branes*, J. Geom. Phys. 52, No. 2, pp. 186–216; 2004

[6] G. Contreras, L. Macarini, G. P. Paternain: *Periodic Orbits for Exact Magnetic Flow on Surfaces*, IMRN International Mathematics Research Notes, No. 8; 2004

[7] J. Eells, Jr. J. H.Sampson: *Harmonic Mappings of Riemannian Manifolds*, American Journal of Mathmatics, Vol. 86, No. 1, pp. 109–160; 1964

[8] J. Eells, A. Ratto: *Harmonic Maps and Minimal Immersions with Symmetries*, Annals of Mathematics Studies, No. 130, Princeton University Press; 1993

[9] L. C. Evans: *Partial Differential Equations*, AMS, Graduate Studies in Mathematics, Vol. 19; 1998

[10] A. Friedman: *Partial differential equations of parabolic type*, Prentice Hall; 1964

[11] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer; 1997

[12] A. Haraux: *Nonlinear Evolution Equations*, Lecture Notes in Mathematics, No. 841, Springer-Verlag; 1981

[13] N. Hitchin: *Lectures on special Lagrangian submanifolds*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), pp. 151–182, AMS/IP Stud. Adv. Math., 23, Amer. Math. Soc., Providence, RI; 2001

[14] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry*, Vol. 1, Interscience Tracts in Pure and Applied Mathematics, Interscience Publishers, John Wiley and Sons, Inc.; 1963

[15] O. A. Ladyženskaya, V. A. Solonnikov, and N. N. Ural'ceva: *Linear and Quasilinear Equations of Parabolic Type*, American Math. Society; 1968

[16] M. K. Murray: *Bundle gerbes*, J. Lond. Math. Soc. 54 (2), pp. 403–416; 1996

[17] S. Lang: *Introduction to Differentiable Manifolds*, Interscience; 1962

[18] G. M. Lieberman: *Second order parabolic differential equations*, World Scientific; 1996

[19] S. Nishikawa: *Variational Problems in Geometry*, AMS, Translations of Math. Monographs, Vol. 205; 2002

[20] M. Renardy, R. Rogers: *An Introduction to Partial Differential Equations*, Springer, Text in Applied Mathematics, Vol. 13; 1993

[21] T. Sakai: *Riemannian Geometry*, AMS, Translations of Math. Monographs, Vol. 149; 1995

[22] I. A. Taimanov: *Closed extremals on two-dimensional manifolds*, Uspekhi Mat. Nauk 47:2, pp. 143–185; 1992

[23] M. E. Taylor: *Partial Differential Equations*, Vol. III, Springer; 1996

[24] H. Urakawa: *Calculus of Variations and Harmonic Maps*, AMS, Translations of Math. Monographs, Vol. 132; 1993

[25] C. Wang, Z. Wu, J. Yin: *Elliptic and parabolic equations*, World Scientific; 2006

[26] Y. Xin: *Geometry of Harmonic maps*, Birkhäuser; 1996