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Particle distribution from spectral Mie-scattering: kernel representation and singular-value spectrum¹

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Abstract

This paper deals with the Mie scattering kernels for multi-spectral data. The kernels may be represented in form of power series. Furthermore, the singular-value spectrum and the degree of ill-posedness in dependence on the refractive index of the particles are numerically approximated. A special hybrid regularization technique allows us to determine via inversion the particle distributions of different types.

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1 Introduction

Aerosol particles affect life on earth in several ways. They play an important role in the climate system; the effect of aerosol particles on the global climate system is one of the major uncertainties of present climate predictions. They have a major role in atmospheric chemistry and hence affect the concentrations

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of other potentially harmful budget, in particular in the UV-B part of the spectrum. Moreover, one reason for the ozone depletion is the chlorine (Cl) in CFCs in the stratosphere. On the other side, polar stratospheric clouds (PSCs) a type of aerosol particles are believed to be active in precursor stages of ozone depletion in the winter-cold stratosphere by catalyzing heterogeneous chemical reactions on their surface and by redistributing HNO_3 through sedimentation [22] and [23]. Such particles can be produced by volcanic eruptions in the stratosphere or by environmental pollution of the air above industrial areas. At ground level, they can be harmful, even toxic, to man, animals, and plants. Because of these adverse effects that aerosol particles can have on human life, it is necessary to achieve an advanced understanding of the processes that generate, redistribute, and remove aerosol particles in the atmosphere. The size distribution of these cloud particles is an important parameter for quantifying those mechanisms, because it relates the total surface to the total mass. This distribution can be determined either by in-situ measurements with optical particle counters or by remote sensing with lidar (light detection and ranging) equipment [5], [6] and [20]. From the optical data of lidar measurements it is necessary to derive the microphysical parameters of the particles, e.g. the knowledge of the particle distribution is necessary to model processes involving ozone chemistry [18]. The exact theory for scattering by a spherical particle which radius is comparable to the emitted wavelength of the lidar set up is solveable from Maxwell's equations and Mie theory [19], [16] and [7]. Similar considerations can be found in [2]. They deal with multi-angular properties by using full angular range for the data and show that the kernel function may be expanded in spherical harmonics. This treatment is applicable to laboratory-based experiments where a sample or detector may be rotated through all angles. We consider real-life-based remote sensing experiments by using multi-spectral properties and deal with the Mie scattering kernel. In [4] they deal with multi-spectral turbidity measurements but the work is limited to the anomalous scattering approximation.

1.1 Mathematical model

The mathematical model of such a lidar measurement process consists of a system of two Fredholm integral equations of the first kind for the backscatter and extinction coefficients β^{Aer} and α^{Aer}

$$\beta^{Aer}(\lambda, z) = \int_{r_0}^{r_1} K_{\pi}(r, \lambda; m) n(r, z) dr = \int_{r_0}^{r_1} \pi r^2 Q_{\pi}(r, \lambda; m) n(r, z) dr, \quad (1)$$

$$\alpha^{Aer}(\lambda, z) = \int_{r_0}^{r_1} K_{ext}(r, \lambda; m) n(r, z) dr = \int_{r_0}^{r_1} \pi r^2 Q_{ext}(r, \lambda; m) n(r, z) dr, \quad (2)$$

where r is the particle radius, m the complex refractive index, r_0 and r_1 represent suitable lower and upper limits of realistic radii, λ is the wavelength, λ_0 the smallest and λ_1 the largest wavelength, z is the height, n the particle size distribution, K_π the backscatter and K_{ext} the extinction kernel. The kernel function reflects shape, size, and material composition of the particles. The following formulas hold for extinction and backscatter coefficients [7]

$$Q_\pi = \frac{1}{k^2 r^2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n - b_n) \right|^2, Q_{ext} = \frac{2}{k^2 r^2} \sum_{n=1}^{\infty} (2n+1) \operatorname{Re}(a_n + b_n), \quad (3)$$

where k is the wave number defined by $k = 2\pi/\lambda$ and a_n and b_n are the coefficients which we get from the boundary conditions for the tangential components of the waves. Now equations (1) and (2) are formulated into a more specific and more solid form

$$\Gamma^{Aer}(\lambda, z) = \int_{r_0}^{r_1} K_{\pi/ext}^v(r, \lambda; m) v(r, z) dr = \int_{r_0}^{r_1} \frac{3}{4r} Q_{\pi/ext}(r, \lambda; m) v(r, z) dr, \quad (4)$$

where the $v(r, z)$ term is the volume concentration distribution, finally, we are looking for. Γ^{Aer} stands for β^{Aer} and/or α^{Aer} , respectively, depending on the measurement data. The determination of the particle volume distribution v from a small number of backscatter and extinction measurements is a nonlinear inverse ill-posed problem since the refractive index m in the kernels $K_{\pi/ext}^v$ is an unknown, too.

1.2 Ill-posed problem and regularization

The equations (1), (2) and (4) are ill-posed on all three counts (existence, uniqueness, and stability), where stability means a solution that changes only slightly with a slight change in the problem [10]. We consider an operator of the form $Tx = y$ where $T : H_1 \rightarrow H_2$ is a compact, linear (but not necessarily self-adjoint) operator from a Hilbert space H_1 into a Hilbert space H_2 . For a bounded linear operator T a solution $x \in H_1$ of the equation $Tx = y$ exists if and only if y belongs to $R(T)$, the range of T . Since T is linear, $R(T)$ is a subspace of H_2 , which in generally does not exhaust H_2 . We may enlarge the class of functions y for which a type of generalized solution exists to a dense subspace of function in H_2 . This accomplished by introducing the idea of a least squares solution. A function $x \in H_1$ is called a least squares solution if

$$\|Tx - y\| = \inf\{\|Tu - y\| : u \in H_1\}. \quad (5)$$

The set of all least squares solutions is closed and convex. Therefore, there is a unique least squares solution of smallest norm which we call generalized solution. The mapping T^\dagger that associates with a given $y \in D(T^\dagger) = R(T) + R(T)^\perp$ the unique least squares solution having smallest norm, $T^\dagger y$, is called the Moore-Penrose generalized inverse of T . In our scheme T^\dagger is then the mechanism which provides a unique solution for any $y \in D(T^\dagger)$. In this sense, T^\dagger settles the issues of existence and uniqueness for generalized solutions. The generalized Pseudoinverse operator $T^\dagger : D(T^\dagger) \rightarrow H_1$ is a closed densely defined linear operator which is bounded if and only if $R(T)$ is closed. Since both lidar integral operators are compact, each of them can have closed range if and only if $R(T)$ is a finite dimensional subspace of H_2 . This is not the case just under the given lidar integral kernels. Therefore, $R(T)$ is not closed so T^\dagger is unbounded, i.e. T^\dagger is discontinuous. Very small changes in the right hand side $y(\lambda)$ can be accounted for by large changes in the solution $x(r)$. That the instability is fundamental, and not just a consequence of some special form of the kernels, follows from the Riemann-Lebesgue lemma.

If we wish to obtain a well-posed problem we need a so called regularization. In general regularizations are families of operators

$$T_\gamma : H_2 \rightarrow H_1 \quad \text{with} \quad \lim_{\gamma \rightarrow 0} T_\gamma y = T^\dagger y \quad \text{for all } y \in D(T^\dagger) \quad , \quad (6)$$

i.e. the convergence is pointwise on $D(T^\dagger)$ [17]. The parameter γ is the so-called regularization parameter. In the case of noisy data y^δ with $\|y - y^\delta\| \leq \delta$ we determine as solution

$$x_\gamma^\delta = T_\gamma y^\delta . \quad (7)$$

However, the total error consists of two parts , i.e. two summands,

$$x_\gamma^\delta - x = T_\gamma(y^\delta - y) + (T_\gamma - T^\dagger)y . \quad (8)$$

The first part is the data error and the second part the approximation error or regularization error. In generally if $\gamma \rightarrow 0$ the approximation error tends to zero (with respect to the H_1 -norm) while the data error tends to infinity. Therefore, the total error can never be zero and we are in a dilemma. We have to look for an "optimal" regularization parameter γ which minimizes the total error.

The paper is organized as follows. In section 2 we prove some convergence results and the representation in power series. In section 3 we determine an approximation for the singular value expansion and for the degree of ill-posedness. Finally, in section 4 we show by example some inversion results via a specially developed hybrid regularization technique.

2 A representation of the lidar-operators kernels

In this section the lidar-operators kernels are represented in the form of power series with respect to the product $x = r\nu$ of the particle radius and wave number, respectively; it may be useful by practical calculations. Apart from anything else this representation points to the connection with the classical moments problem.

We start with some notations. Put $\nu = 2\pi/\lambda$, $\nu_i = 2\pi/\lambda_{1-i}$ ($i = 0, 1$), $\mu = m$, $H_1 = L_2(r_0, r_1)$, $H_2 = L_2(\nu_0, \nu_1)$. Let us consider two families of integral operators $T_\pi^{(\mu)} : H_1 \rightarrow H_2$ and $T_{ext}^{(\mu)} : H_1 \rightarrow H_2$ which kernels are

$$K_\pi^{(\mu)}(r, \nu) = \pi\nu^{-2} \left| \sum_{n=1}^{\infty} (2n+1)(-1)^n (a_n - b_n) \right|^2,$$

$$K_{ext}^{(\mu)}(r, \nu) = 2\pi\nu^{-2} \sum_{n=1}^{\infty} (2n+1) Re(a_n + b_n) = 2\pi\nu^{-2} Re \sum_{n=1}^{\infty} (2n+1)(a_n + b_n), \quad (9)$$

respectively. The restrictions on the real variables r, ν and on a complex parameter μ may be written in the form

$$0 < x < \infty, \quad \mu^2 \notin M_\varepsilon \quad (10)$$

where $x := r\nu$, $\varepsilon > 0$ is an arbitrary small fixed number and M_ε denotes the domain $\{\mu : |\frac{\mu-1}{\mu}| < 2 - \varepsilon\}$ on the complex plain. (For the lidar operators problem the values of parameter μ lie in fact strictly inside M_ε).

The functions a_n , b_n ($n \geq 1$) in (9) are expressed by Sommerfeld spherical harmonics ψ and ζ (see (7.2.6; 44-46), [3]) as follows

$$a_n = a_n(x; \mu) = \frac{\mu\psi_n(\mu x)\psi'_n(x) - \psi_n(x)\psi'_n(\mu x)}{\mu\psi_n(\mu x)\zeta'_n(x) - \zeta_n(x)\psi'_n(\mu x)},$$

$$b_n = b_n(x; \mu) = \frac{\psi_n(\mu x)\psi'_n(x) - \mu\psi_n(x)\psi'_n(\mu x)}{\psi_n(\mu x)\zeta'_n(x) - \mu\zeta_n(x)\psi'_n(\mu x)}, \quad (11)$$

where the denominators in (11) do not turn into zero in the definition domain of the kernels.

Lemma 2.1. The functions a_n and b_n may be written as follows

$$a_n = \frac{u_n^{(a)}}{u_n^{(a)} + i v_n^{(a)}}; \quad b_n = \frac{u_n^{(b)}}{u_n^{(b)} + i v_n^{(b)}}, \quad n \geq 1, \quad (12)$$

where

$$\begin{aligned}
u_n^{(a)} &= (\mu^2 - 1)(n+1)J_{n+1/2}(x)J_{n+1/2}(\mu x) - \mu^2 x J_{n+3/2}(x)J_{n+1/2}(\mu x) + \\
&\quad + \mu x J_{n+1/2}(x)J_{n+3/2}(\mu x), \\
v_n^{(a)} &= (\mu^2 - 1)(n+1)Y_{n+1/2}(x)J_{n+1/2}(\mu x) - \mu^2 x Y_{n+3/2}(x)J_{n+1/2}(\mu x) + \\
&\quad + \mu x Y_{n+1/2}(x)J_{n+3/2}(\mu x), \\
u_n^{(b)} &= \mu J_{n+1/2}(x)J_{n+3/2}(\mu x) - J_{n+1/2}(\mu x)J_{n+3/2}(x), \\
v_n^{(b)} &= \mu Y_{n+1/2}(x)J_{n+3/2}(\mu x) - J_{n+1/2}(\mu x)Y_{n+3/2}(x), \quad n \geq 1.
\end{aligned} \tag{13}$$

Here J and Y denote the Bessel functions of the 1st and the 2nd kind, respectively; $v_n^{(a)}$, $v_n^{(b)}$ and the denominators in (12) do not turn into zero in the definition domain of the kernels. Put $\mu_0 = 1 = 1 + 0 \cdot i \in M_\varepsilon$. It should be noted that in virtue of formula (7.11;35), [3], and by definition (13) we have for each $n \geq 1$ and $x > 0$

$$u_n^{(a)}(x; \mu_0) = u_n^{(b)}(x; \mu_0) = 0 ; v_n^{(a)}(x; \mu_0) = \frac{2}{\pi} ; v_n^{(b)}(x; \mu_0) = \frac{2}{\pi \cdot x}. \tag{14}$$

Theorem 2.2. Both of the series in (9) are uniformly (=absolutely) convergent in every bounded subdomain of the definition domain (10).

Proof. Let the inequalities $x < \rho < \infty, |\mu| < M < \infty$ be valued. For $n \geq 1$ one can write

$$|a_n| = \frac{\left| \frac{u_n^{(a)}}{v_n^{(a)}} \right|}{\left| \frac{u_n^{(a)}}{v_n^{(a)}} + i \right|} \leq \frac{\left| \frac{u_n^{(a)}}{v_n^{(a)}} \right|}{\left| \left| \frac{u_n^{(a)}}{v_n^{(a)}} \right| - 1 \right|}; \quad |b_n| = \frac{\left| \frac{u_n^{(b)}}{v_n^{(b)}} \right|}{\left| \frac{u_n^{(b)}}{v_n^{(b)}} + i \right|} \leq \frac{\left| \frac{u_n^{(b)}}{v_n^{(b)}} \right|}{\left| \left| \frac{u_n^{(b)}}{v_n^{(b)}} \right| - 1 \right|}. \tag{15}$$

For the sake, the awkwardness to avoid, put

$$\begin{aligned}
J_{n+1/2}(x) &= \alpha_1; & J_{n+3/2}(x) &= \alpha_2; \\
J_{n+1/2}(\mu x) &= \beta_1; & J_{n+3/2}(\mu x) &= \beta_2; \\
Y_{n+1/2}(x) &= \gamma_1; & Y_{n+3/2}(x) &= \gamma_2; \\
\frac{(\mu^2-1)(n+1)}{\mu^2 x} + \frac{\beta_2}{\mu \beta_1} &= A(\mu, x, n) = A,
\end{aligned}$$

so that by (12)

$$\begin{aligned}
\frac{u_n^{(a)}}{v_n^{(a)}} &= \frac{(\mu^2-1)(n+1)\alpha_1\beta_1 - \mu^2 x \alpha_2 \beta_1 + \mu x \alpha_1 \beta_2}{(\mu^2-1)(n+1)\gamma_1\beta_1 - \mu^2 x \gamma_2 \beta_1 + \mu x \gamma_1 \beta_2} = \frac{\alpha_2}{\gamma_2} \cdot \frac{\frac{\alpha_1}{\gamma_1} A - 1}{\frac{\alpha_2}{\gamma_2} A - 1}; \\
\frac{u_n^{(b)}}{v_n^{(b)}} &= \frac{\mu \alpha_1 \beta_2 - \alpha_2 \beta_1}{\mu \gamma_1 \beta_2 - \gamma_2 \beta_1}.
\end{aligned}$$

Now we have

$$\left| \frac{u_n^{(a)}}{v_n^{(a)}} \right| \leq \left| \frac{\alpha_2}{\gamma_2} \right| \times \frac{\left| \frac{\alpha_1}{\alpha_2} \right| \cdot |A| + 1}{\left| \frac{\gamma_1}{\gamma_2} \right| \cdot |A| - 1}; \quad \left| \frac{u_n^{(b)}}{v_n^{(b)}} \right| \leq \left| \frac{\alpha_2}{\gamma_2} \right| \times \frac{|\mu| \left| \frac{\alpha_1}{\alpha_2} \right| \cdot \left| \frac{\beta_2}{\beta_1} \right| + 1}{\left| \mu \right| \left| \frac{\gamma_1}{\gamma_2} \right| \cdot \left| \frac{\beta_2}{\beta_1} \right| - 1}. \quad (16)$$

For each half-integer Bessel function $J_{\kappa+1/2}(z)$ the relations (3.62)-(3.67), [21], imply the following two-side estimations

$$\left| \frac{(z/2)^{\kappa+1/2}}{\Gamma(\kappa+3/2)} \right| \times O_{1,\kappa} \leq \left| J_{\kappa+1/2}(z) \right| \leq \left| \frac{(z/2)^{\kappa+1/2}}{\Gamma(\kappa+3/2)} \right| \times O_{2,\kappa},$$

where κ are integer and $O_{1,\kappa}, O_{2,\kappa}$ are two real sequences such that $\lim_{|\kappa| \rightarrow \infty} O_{1,\kappa} = 1-, \lim_{|\kappa| \rightarrow \infty} O_{2,\kappa} = 1+.$

Put $\Theta_{1,n} = \min[O(\alpha_1), O(\beta_1), O(\gamma_1)], \Theta_{2,n} = \max[O(\alpha_1), O(\beta_1), O(\gamma_1)],$

then

$$\lim_{|\kappa| \rightarrow \infty} \Theta_{1,\kappa} = 1-, \lim_{|\kappa| \rightarrow \infty} \Theta_{2,\kappa} = 1+, \quad (17)$$

and the following two-side estimation are obtained

$$\begin{aligned} \left| \frac{(x/2)^{n+1/2}}{\Gamma(n+3/2)} \right| \cdot \Theta_{1,n} &\leq \left| \alpha_1 \right| \leq \left| \frac{(x/2)^{n+1/2}}{\Gamma(n+3/2)} \right| \cdot \Theta_{2,n}; \\ \left| \frac{(x/2)^{n+3/2}}{\Gamma(n+5/2)} \right| \cdot \Theta_{1,n} &\leq \left| \alpha_2 \right| \leq \left| \frac{(x/2)^{n+3/2}}{\Gamma(n+5/2)} \right| \cdot \Theta_{2,n}; \\ \left| \frac{(\mu x/2)^{n+1/2}}{\Gamma(n+3/2)} \right| \cdot \Theta_{1,n} &\leq \left| \beta_1 \right| \leq \left| \frac{(\mu x/2)^{n+1/2}}{\Gamma(n+3/2)} \right| \cdot \Theta_{2,n}; \\ \left| \frac{(\mu x/2)^{n+3/2}}{\Gamma(n+5/2)} \right| \cdot \Theta_{1,n} &\leq \left| \alpha_2 \right| \leq \left| \frac{(\mu x/2)^{n+3/2}}{\Gamma(n+5/2)} \right| \cdot \Theta_{2,n}; \\ \left| \frac{(x/2)^{-n-1/2}}{\Gamma(-n+1/2)} \right| \cdot \Theta_{1,n} &\leq \left| \gamma_1 \right| \leq \left| \frac{(x/2)^{-n-1/2}}{\Gamma(-n+1/2)} \right| \cdot \Theta_{2,n}; \\ \left| \frac{(x/2)^{-n-3/2}}{\Gamma(-n-1/2)} \right| \cdot \Theta_{1,n} &\leq \left| \gamma_2 \right| \leq \left| \frac{(x/2)^{-n-3/2}}{\Gamma(-n-1/2)} \right| \cdot \Theta_{2,n}. \end{aligned} \quad (18)$$

Thus we have for the terms of (16)

$$\begin{aligned} \frac{x^{2n+3}}{(2n+3)[(2n+1)!!]^2} \cdot \frac{\Theta_{1,n}}{\Theta_{2,n}} &\leq \left| \frac{\alpha_2}{\gamma_2} \right| \leq \frac{x^{2n+3}}{(2n+3)[(2n+1)!!]^2} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}}; \\ \frac{|\mu|x}{2n+3} \cdot \frac{\Theta_{1,n}}{\Theta_{2,n}} &\leq \left| \frac{\beta_2}{\beta_1} \right| \leq \frac{|\mu|x}{2n+3} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}}; \\ \frac{x}{2n+1} \cdot \frac{\Theta_{1,n}}{\Theta_{2,n}} &\leq \left| \frac{\gamma_1}{\gamma_2} \right| \leq \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}}; \\ \frac{2n+3}{x} \cdot \frac{\Theta_{1,n}}{\Theta_{2,n}} &\leq \left| \frac{\alpha_1}{\alpha_2} \right| \leq \frac{2n+3}{x} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}}. \end{aligned} \quad (19)$$

By virtue of (19) and (10) we have

$$1 - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} |A| \geq 1 - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} + \frac{\beta_2}{\mu \beta_1} \right| \geq 1 - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left(\left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} \right| + \left| \frac{\beta_2}{\mu \beta_1} \right| \right) = 1 - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} \right| - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{\beta_2}{\mu \beta_1} \right| = 1 - \frac{n+1}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{(\mu^2-1)}{\mu^2} \right| - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{\beta_2}{\mu \beta_1} \right| \geq 1 - \frac{n+1}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{(\mu^2-1)}{\mu^2} \right| - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \cdot \frac{x}{2n+3} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} = 1 - \frac{n+1}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{\mu^2-1}{\mu^2} \right| - \frac{x^2}{(2n+1)(2n+3)} \cdot \left(\frac{\Theta_{2,n}}{\Theta_{1,n}} \right)^2$$

Due to the the relations (17) one can choose $N_{\varepsilon,1}$ so that

$$\frac{n+1}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \left| \frac{\mu^2-1}{\mu^2} \right| \leq (1/2 + \varepsilon/4)(2 - \varepsilon) = 1 - \varepsilon^2/4; \quad \frac{x^2}{(2n+1)(2n+3)} \cdot \left(\frac{\Theta_{2,n}}{\Theta_{1,n}} \right)^2 \leq \frac{\rho^2}{(2n+1)(2n+3)} \cdot \left(\frac{\Theta_{1,n}}{\Theta_{2,n}} \right)^2 \leq \varepsilon^2/8; \quad n \geq N_{\varepsilon,1}.$$

This estimation and preceding inequality imply

$$1 - \frac{x}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} |A| = 1 - \left| \frac{\gamma_1}{\gamma_2} \right| |A| = \left| \left| \frac{\gamma_1}{\gamma_2} \right| A - 1 \right| \geq \varepsilon^2/8 > 0. \quad (20)$$

Now using (19) one can estimate $|A|$ as follows

$$|A| \leq \left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} \right| + \left| \frac{\beta_2}{\mu \beta_1} \right| \leq \left| \frac{\mu^2-1}{\mu^2} \right| \cdot \frac{(n+1)}{x} + \frac{x}{2n+3} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \leq C_{\varepsilon,1} \frac{(n+1)}{x} + \frac{x}{2n+3} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}},$$

where by reason of (10) the constant $C_{\varepsilon,1} = 2 - \varepsilon$ depends only on the fixed ε, ρ and M but does not depend on $\mu \in M_\varepsilon$, n and $x > 0$.

From (19) we have

$$\left| \frac{\alpha_1}{\alpha_2} \right| \cdot |A| \leq C_{\varepsilon,1} \frac{(2n+3)(n+1)}{x^2} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} + \frac{\Theta_{2,n}^2}{\Theta_{1,n}^2},$$

and by virtue of (17)

$$\left| \frac{\alpha_1}{\alpha_2} \right| \cdot |A| \leq C_{\varepsilon,2} \frac{(2n+3)(n+1)}{x^2},$$

where $C_{\varepsilon,2} > 2$ and does not depend on x, n and μ .

Now we have

$$\left| \frac{\alpha_1}{\alpha_2} \right| \left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} + \frac{\beta_2}{\mu \beta_1} \right| + 1 = \left| \frac{\alpha_1}{\alpha_2} \right| \cdot |A| + 1 < C_{\varepsilon,3} \frac{(2n+3)(n+1)}{x^2} \\ \left| \frac{\gamma_1}{\gamma_2} \right| \cdot \left| \frac{(\mu^2-1)(n+1)}{\mu^2 x} + \frac{\beta_2}{\mu \beta_1} \right| = \left| \frac{\gamma_1}{\gamma_2} \right| \cdot |A| < \frac{n+1}{2n+1} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} + \frac{x^2}{(2n+1)(2n+3)} \cdot \frac{\Theta_{2,n}^2}{\Theta_{1,n}^2}.$$

Using this estimations we obtain from (16), (19), (20) and preceding inequalities that for large n the following inequalities are true

$$\left| \frac{u_n^{(a)}}{v_n^{(a)}} \right| \leq \left| \frac{\alpha_2}{\gamma_2} \right| \times \frac{\left| \frac{\alpha_1}{\alpha_2} \right| \cdot |A| + 1}{\left| \frac{\gamma_1}{\gamma_2} \right| \cdot |A| - 1} \leq 8 \frac{x^{2n+3}}{(2n+3)[(2n+1)!!]^2} \cdot C_{\varepsilon,3} \frac{(2n+3)(n+1)}{x^2 \varepsilon^2} \leq C_{\varepsilon,4} \frac{\rho^{2n+1}}{(2n+1)[(2n-1)!!]^2}.$$

Therefore $\lim_{n \rightarrow \infty} \left| \frac{u_n^{(a)}}{v_n^{(a)}} \right| = 0$. Now the series $\sum_{n=1}^{\infty} (2n+1)a_n$ absolutely converges since in view of (15) we have: $|a_n| \leq \left| \frac{u_n^{(a)}}{v_n^{(a)}} \right| (1 - \left| \frac{u_n^{(a)}}{v_n^{(a)}} \right|)$ while the series

$\sum_{n=1}^{\infty} \frac{\rho^{2n+1}}{(2n+1)[(2n-1)!!]^2}$ is convergent.

Now we obtain from (19)

$$|\mu| \cdot \left| \frac{\alpha_1}{\alpha_2} \right| \left| \frac{\beta_2}{\beta_1} \right| \leq |\mu^2| \cdot \frac{\Theta_{2,n}^2}{\Theta_{1,n}^2} < M^2 C_{\varepsilon,5}.$$

On the other hand from (19) we have as well

$$|\mu| \cdot \left| \frac{\gamma_1}{\gamma_2} \right| \left| \frac{\beta_2}{\beta_1} \right| \leq \frac{|\mu^2| \cdot x^2}{(2n+1)(2n+3)} \cdot \frac{\Theta_{2,n}^2}{\Theta_{1,n}^2} \leq C_{\varepsilon,6} \frac{M^2 \cdot g^2}{(2n+1)(2n+3)},$$

and the term in right hand side converges to zero so that from (16) and (19) follows

$$\left| \frac{u_n^{(b)}}{v_n^{(b)}} \right| < \frac{\rho^{2n+3}}{(2n+3)[(2n+1)!!]^2} \cdot \frac{\Theta_{2,n}}{\Theta_{1,n}} \cdot \frac{C_{\varepsilon,7}+1}{1 - \frac{|M^2| \rho^2}{(2n+1)(2n+3)} \cdot \frac{\Theta_{2,n}^2}{\Theta_{1,n}^2}} < \frac{C_{\varepsilon,8}}{(2n+3)[(2n+1)!!]},$$

so we have $\left| \frac{u_n^{(b)}}{v_n^{(b)}} \right|$ converges to zero as $n \rightarrow \infty$. From the preceding estimation

and from the second of the inequalities (15) obtain $|b_n| \leq \left| \frac{u_n^{(b)}}{v_n^{(b)}} \right| (1 - \left| \frac{u_n^{(b)}}{v_n^{(b)}} \right|)$

thus the series $\sum_{n=1}^{\infty} (2n+1)b_n$ absolutely converges. \triangle

Theorem 2.3. Both of the functions $\nu^2 K_{ext}^{(\mu)}$ and $\nu^2 K_{\pi}^{(\mu)}$ may be represented in the form of power series with respect to $x = r\nu$ namely $\sum_{j=0}^{\infty} \gamma_j^{(ext)} x^j$ and $\sum_{j=0}^{\infty} \gamma_j^{(\pi)} x^j$, respectively, where γ 's are some real coefficients.

Proof. By Theorem 2.2 the Taylor expansions in powers of $\mu - \mu_0$ for both of the functions in Theorem 2.3 converge uniformly in the domain (10). Our goal is to prove that each of them has coefficients which are entire functions, i.e. power series, with respect to x . Note that if it is the case then both of the sums on the right hand side of (9) may be obviously represented in the form $\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \gamma_{i,j}^{(ext)} x^j \right) (\mu - \mu_0)^i$ and $\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} \gamma_{i,j}^{(\pi)} x^j \right) (\mu - \mu_0)^i$, respectively, with some complex γ 's. Then $Re(\sum \dots)$ and $|\sum \dots|^2$ have to be represented in the form $\sum_{j=0}^{\infty} \gamma_j^{(ext)} x^j$ and $\sum_{j=0}^{\infty} \gamma_j^{(\pi)} x^j$, respectively, with some real γ 's so Theorem 2.3 follows.

For an arbitrary function $h = h(x; \mu)$ we put $h^{(-k)} = 0$ for each $k \geq 1$; $h^{(0)} = h$, $h^{(1)} = h'$ denotes the derivative of h with respect to μ ; $h^{(k)}$ denotes k -th derivative, $k \geq 1$. Fix a positive integer n . In what follows it will be clear from the context, what is meant by $u(x; \mu)$: $u_n^{(a)}(x; \mu)$ or $u_n^{(b)}(x; \mu)$, and the same for $v(x; \mu)$. Define the analytical functions $g = g(x; \mu)$, $f = f(x; \mu)$ and the sequence of analytical functions $F_k = F_k(x; \mu)$, $k \geq 1$, as follows

$$g = \frac{u(x; \mu)}{v(x; \mu)}, \quad f = \frac{g}{g+i}; \quad F_1 = g, \quad F_{k+1} = F_k'(g+i) - kg'F_k; \quad k \geq 1,$$

hence by (14) $g(x; \mu_0) = F_1(x; \mu_0) = 0$ for every $x > 0$. By induction the following two formulas are valid for each $k \geq 1$

$$1^\diamond. \quad f^{(k)} = \frac{F_{k+1}}{(g+i)^{k+1}}.$$

2° . $F_{k+1} = F'_k(g+i) + \sum_{m=1}^{k-1} (-1)^m (k-m+1)_m (g')^m F'_{k-m}(g+i) + (-1)^k k! (g')^k F_1$,
where $(k-m+1)_m$ denotes Pochhammer symbol

$$(k-m+1)_m = (k-m+1)(k-m+2) \cdots (k-2)(k-1)k, \quad m \geq 1.$$

Because $F_1(x; \mu_0) = g(x; \mu_0) = 0$ we have

$$F_{k+1}(x; \mu_0) = i \left(F'_k(x; \mu_0) + \sum_{m=1}^{k-1} (-1)^m (k-m+1)_m (g'(x; \mu_0))^m F'_{k-m}(x; \mu_0) \right).$$

Substituting in 1° obtain

$$f^{(k)}(x; \mu_0) = (-i)^k \left(F'_k(x; \mu_0) + \sum_{m=1}^{k-1} (-1)^m (k-m+1)_m (g'(x; \mu_0))^m F'_{k-m}(x; \mu_0) \right) \quad (21)$$

For an analytical function h the Taylor expansion at the point μ_0 has the form $\sum_{k=0}^{\infty} \frac{1}{k!} h^{(k)}(x; \mu_0) (\mu - \mu_0)^k$ so we have to show that $f^{(k)}(x; \mu_0), k \geq 1$, are the entire functions with respect to x . By induction one can easily derive that any term of the recurrent sequence $\{F_j(x; \mu_0)\}_{j=1}^{\infty}$ is represented in the form of sum of products (with some complex coefficients) of derivatives of various degrees of the function g which are calculated at the point μ_0 . Thus (21) shows that only entirety of the functions $g^{(k)}(x; \mu_0), k \geq 1$, is to be proved.

By the Leibniz formula k -th derivatives of the function $g = \frac{u}{v}$ may be written as follows

$$g^{(k)} = u(1/v)^{(k)} + \binom{k}{1} u'(1/v)^{(k-1)} + \binom{k}{2} u''(1/v)^{(k-2)} + \cdots + u^{(k)}(1/v).$$

Therefore it is sufficient to show that the functions $u^{(k)}$ and $(1/v)^{(k)}$ are entire for every $k \geq 1$. Here the properties (14) are very important. Using the relation

$$Y_{n+1/2} = (-1)^{n-1} J_{-n-1/2},$$

(see [3], (7.11;5)) one can rewrite (13) as follows

$$\begin{aligned} u_n^{(a)}(x; \mu) &= (\mu^2 - 1)(n+1)J_{n+1/2}(x)J_{n+1/2}(\mu x) - \mu^2 x J_{n+3/2}(x)J_{n+1/2}(\mu x) + \\ &\quad + \mu x J_{n+1/2}(x)J_{n+3/2}(\mu x), \\ v_n^{(a)}(x; \mu) &= (-1)^{n-1} \left((\mu^2 - 1)(n+1)J_{-n-1/2}(x)J_{n+1/2}(\mu x) + \right. \\ &\quad \left. + \mu^2 x J_{-n-3/2}(x)J_{n+1/2}(\mu x) + \mu x J_{-n-1/2}(x)J_{n+3/2}(\mu x) \right), \end{aligned}$$

$$\begin{aligned}
u_n^{(b)}(x; \mu) &= \mu J_{n+1/2}(x) J_{n+3/2}(\mu x) - J_{n+1/2}(\mu x) J_{n+3/2}(x), \\
v_n^{(b)}(x; \mu) &= (-1)^{n-1} \left(\mu J_{-n-1/2}(x) J_{n+3/2}(\mu x) + J_{n+1/2}(\mu x) J_{-n-3/2}(x) \right).
\end{aligned} \tag{22}$$

For the sake of convenience by next calculating we rename the factors not containing μ as following

$$(n+1)J_{n+1/2}(x) = C_1; \quad xJ_{n+3/2}(x) = C_2; \quad xJ_{n+1/2}(x) = C_3.$$

so that

$$u_n^{(a)}(x; \mu) = (\mu^2 - 1)C_1 J_{n+1/2}(\mu x) - \mu^2 C_2 J_{n+1/2}(\mu x) + \mu C_3 J_{n+3/2}(\mu x) = (C_1 \mu^2 - C_1 - C_2 \mu^2) J_{n+1/2}(\mu x) + \mu C_3 J_{n+3/2}(\mu x).$$

Denoting $C_1 - C_2 = C_3$ obtain

$$u_n^{(a)}(x; \mu) = (C_0 \mu^2 - C_1) J_{n+1/2}(\mu x) + \mu C_3 J_{n+3/2}(\mu x).$$

Let us calculate the k -th derivative of the function $u_n^{(a)}(x; \mu)$, $k \geq 1$, at the point μ_0 . By induction consequently obtain

$$\begin{aligned}
(u_n^{(a)}(x; \mu))^{(k)}(\mu_0) &= A_1^{(k)} \cdot (C_1 - C_2) [J_{n+1/2}(\mu x)]^{(k-2)}(\mu_0) + \\
&+ A_2^{(k)} \cdot (C_1 - C_2) [J_{n+1/2}(\mu x)]^{(k-1)}(\mu_0) - A_3^{(k)} \cdot C_2 [J_{n+1/2}(\mu x)]^{(k)}(\mu_0) + \\
&+ A_4^{(k)} \cdot C_3 [J_{n+3/2}(\mu x)]^{(k-1)}(\mu_0) + A_5^{(k)} C_3 [J_{n+3/2}(\mu x)]^{(k)}(\mu_0),
\end{aligned}$$

where $A_j^{(k)}$ ($j = 1, 2, 3, 4, 5$) are some real coefficients. Replacing C_1, C_2, C_3 one can it rewrite as

$$\begin{aligned}
(u_n^{(a)}(x; \mu))^{(k)}(\mu_0) &= A_1^{(k)} (n+1) J_{n+1/2}(x) [J_{n+1/2}(\mu x)]^{(k-2)}(\mu_0) \\
&- A_1^{(k)} x J_{n+3/2}(x) [J_{n+1/2}(\mu x)]^{(k-2)}(\mu_0) + \\
&+ A_2^{(k)} (n+1) J_{n+1/2}(x) [J_{n+1/2}(\mu x)]^{(k-1)}(\mu_0) \\
&- A_2^{(k)} x J_{n+3/2}(x) [J_{n+1/2}(\mu x)]^{(k-1)}(\mu_0) - \\
&- A_3^{(k)} x J_{n+3/2}(x) [J_{n+1/2}(\mu x)]^{(k)}(\mu_0) + A_4^{(k)} x J_{n+1/2}(x) [J_{n+3/2}(\mu x)]^{(k-1)}(\mu_0) + \\
&+ A_5^{(k)} x J_{n+1/2}(x) [J_{n+3/2}(\mu x)]^{(k)}(\mu_0)
\end{aligned} \tag{23}$$

One can similarly obtain the following formulas

$$\begin{aligned}
(v_n^{(a)}(x; \mu))^{(k)}(\mu_0) &= (-1)^{n-1} \left(B_1^{(k)} (n+1) J_{-n-1/2}(x) [J_{n+1/2}(\mu x)]^{(k-2)}(\mu_0) + \right. \\
&+ B_1^{(k)} x J_{-n-3/2}(x) [J_{n+1/2}(\mu x)]^{(k-2)}(\mu_0) + \\
&+ B_2^{(k)} (n+1) J_{-n-1/2}(x) [J_{n+1/2}(\mu x)]^{(k-1)}(\mu_0) + \\
&+ B_2^{(k)} x J_{-n-3/2}(x) [J_{n+1/2}(\mu x)]^{(k-1)}(\mu_0) + \\
&+ B_3^{(k)} x J_{-n-3/2}(x) [J_{n+1/2}(\mu x)]^{(k)}(\mu_0) + B_4^{(k)} x J_{-n-1/2}(x) [J_{n+3/2}(\mu x)]^{(k-1)}(\mu_0) + \\
&\left. + B_5^{(k)} x J_{-n-1/2}(x) [J_{n+3/2}(\mu x)]^{(k)}(\mu_0) \right),
\end{aligned} \tag{24}$$

$$(u_n^{(b)}(x; \mu))^{(k)}(\mu_0) = kJ_{n+1/2}(x)[J_{n+3/2}(\mu x)]^{(k-1)}(\mu_0) + \\ + J_{n+1/2}(x)[J_{n+3/2}(\mu x)]^{(k)}(\mu_0) - J_{n+3/2}(x)[J_{n+1/2}(\mu x)]^{(k)}(\mu_0), \quad (25)$$

$$(v_n^{(b)}(x; \mu))^{(k)}(\mu_0) = (-1)^{n-1} \left(kJ_{-n-1/2}(x)[J_{n+3/2}(\mu x)]^{(k-1)}(\mu_0) + \right. \\ \left. + J_{-n-1/2}(x)[J_{n+3/2}(\mu x)]^{(k)}(\mu_0) + J_{-n-3/2}(x)[J_{n+1/2}(\mu x)]^{(k)}(\mu_0) \right). \quad (26)$$

Our next goal is to prove that the minimal power order of the variable x in the terms of the power series for $[J_{n+1/2}(\mu x)]^{(k)}(\mu_0)$, $k \geq 1$, n is integer, is equal to $n + 1/2$ therefore it do not depend on k .

Using the recurrence formula

$$J'_\tau(z) = 1/2 \cdot (J_{\tau-1}(z) - J_{\tau+1}(z))$$

(where τ, z are two arbitrary complex numbers, see [3], (7.2.8;57)) one can obtain by induction the following one

$$(J_\tau(\mu x))_\mu^{(k)} = (x/2)^k \left(J_{\tau-k}(\mu x) - K_1 J_{\tau-(k-2)}(x) + K_2 J_{\tau-(k-4)}(\mu x) - \dots + \right. \\ \left. (-1)^{k-1} K_{k-1} J_{\tau+(k-2)}(\mu x) + (-1)^k J_{\tau+k}(\mu x) \right).$$

where K_j are integers, $j = 1, \dots, k-1$. It follows immediately from the standard representation of Bessel function in the form of power series (see e.g. [3], (7.2.1;2) that for the case of half-integer $\tau = n + 1/2$ (n is integer) and for $\mu = \mu_0$ the term above which contains the minimal power of x is equal to $(x/2)^k J_{\tau-k}(x)$. Moreover this power may be calculated as following

$$(x/2)^k J_{n+1/2-k}(x) = (x/2)^k (x/2)^{n+1/2-k} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^m}{m! \Gamma(n+1/2-k+m+1)} = (x/2)^{n+1/2} \cdot \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^m}{m! \Gamma(n+1/2-k+m+1)}.$$

Therefore our assertion is true.

Returning to the (23)-(26) one can easy see that every summand of the right hand side contains x with the minimal power $2n + 1$, 0 , $2n + 2$ and -1 respectively. Therefore it is shown that $(u_n^{(a)}(x; \mu))^{(k)}(\mu_0)$ and $(u_n^{(b)}(x; \mu))^{(k)}(\mu_0)$ are entire with respect to x with minimal powers of x which are equal to $2n + 1$ and $2n + 2$, respectively.

Now we are going to prove that $(1/v_n^{(a)}(x; \mu))^{(k)}(\mu_0)$ and $(1/v_n^{(b)}(x; \mu))^{(k)}(\mu_0)$ are entire with respect to x as well, and calculate the minimal powers of x among the terms of the corresponding sums.

As the reasons for $(1/v_n^{(a)}(x; \mu))^{(k)}(\mu_0)$ and for $(1/v_n^{(b)}(x; \mu))^{(k)}(\mu_0)$ are the same we use in (27) for both of the functions the notation $(1/v(x; \mu))^{(k)}$. By induction it is easy to show that the following general representation is true

$$(1/v)^{(k)} = \frac{\sum A_{i_0 \dots i_k}(v)^{i_0} (v^{(1)})^{i_1} \dots (v^{(k)})^{i_k}}{(v)^{k+1}} \quad (27)$$

where A 's denote some real coefficients while the integer exponents i_j 's satisfy the conditions $0 \leq i_0, i_1, \dots, i_k; \sum_{j=0}^k i_j = k$.

Note that the maximal power of v in the numerator of (27) is equal to $k - 1$. On the other hand it was showed above that the minimal power of x in $(v_n^{(b)}(x; \mu))^{(k)}(\mu_0)$ is equal to -1 while by (20) $v_n^{(b)}(x; \mu_0) = 2/\pi x$. So the minimal power of x in $(1/v_n^{(b)}(x; \mu))^{(k)}(\mu_0)$ is equal to 1 . Analogously the minimal power of x in $(v_n^{(a)}(x; \mu))^{(k)}(\mu_0)$ is equal to 0 while $(v_n)^{(a)}(x; \mu_0) = 2/\pi$. So the minimal power of x in $(1/v_n^{(a)}(x; \mu))^{(k)}(\mu_0)$ is equal to 0 .

Thus in the Leibniz formula the functions $(g^{(a)})^{(k)}$ and $(g^{(b)})^{(k)}$ are entire with respect to x with minimal powers $2n + 1$ and $2n + 3$ respectively. Now formulas (12) and (21) show that the functions $a_n^{(k)}(x; \mu_0)$ and $b_n^{(k)}(x; \mu_0)$ are represented as power series with respect to x which begin with x^{2n+1} and x^{2n+3} respectively. Therefore in the series of formula (9) the minimal powers of x occur provided $n = 1$, namely 3 and 6 , respectively. \triangle

3 Degree of ill-posedness

The operators $T^*T : H_1 \rightarrow H_1$ and $TT^* : H_2 \rightarrow H_2$, see section 1.2, are compact self-adjoint linear operators where T^* is the adjoint operator of T . The nonzero eigenvalues of T^*T or of TT^* (they have the same eigenvalues) can be enumerated as $\lambda_1 \geq \lambda_2 \geq \dots$. If we designate by v_1, v_2, \dots , an associated sequence of orthonormal eigenfunctions of T^*T , then $\{v_1, v_2, \dots\}$ is complete in the range $\overline{R(T^*T)} = N(T)^\perp$ (orthogonal compliment of the null space of T). Let $\mu_j = \sqrt{\lambda_j}$ then $Tv_j = \mu_j u_j$ and $T^*u_j = \mu_j v_j$. Moreover, $TT^*u_j = \mu_j T v_j = \mu_j^2 u_j = \lambda_j u_j$ and it is easy to see that the orthonormal eigenfunctions $\{u_j\}$ of TT^* form a complete orthogonal set for $\overline{R(TT^*)} = N(T^*)^\perp$. The system $\{v_j, u_j; \mu_j\}$ is called a singular system for T and the numbers μ_j are called singular values of T . Every square-integrable kernel of a linear integral operator has a singular value expansion (SVE) which is a mean convergent expansion of the form

$$K(r, \lambda) = \sum_{i=1}^{\infty} \mu_i u_i(r) \overline{v_i(\lambda)}, \quad r \in I_r = [r_0, r_1], \lambda \in I_\lambda = [\lambda_0, \lambda_1], \quad (28)$$

where $\{u_i, v_i\}$ are the left and right singular functions of the kernel (see [11]). The behaviour of the singular values and functions is strongly connected with the properties of the kernel. Roughly speaking, the smoother the kernel the faster the singular values μ_i decay to zero where smoothness is measured by the number of continuous partial derivatives of the kernel, see [9] and [8]. It holds under some assumptions $\mu_i = o(i^{-p-3/2})$, $i \rightarrow \infty$, where $p - 1$ is the number of continuous partial derivatives with respect to the first variable. This number p of the given lidar kernels, if one exists, is hardly to derive. We

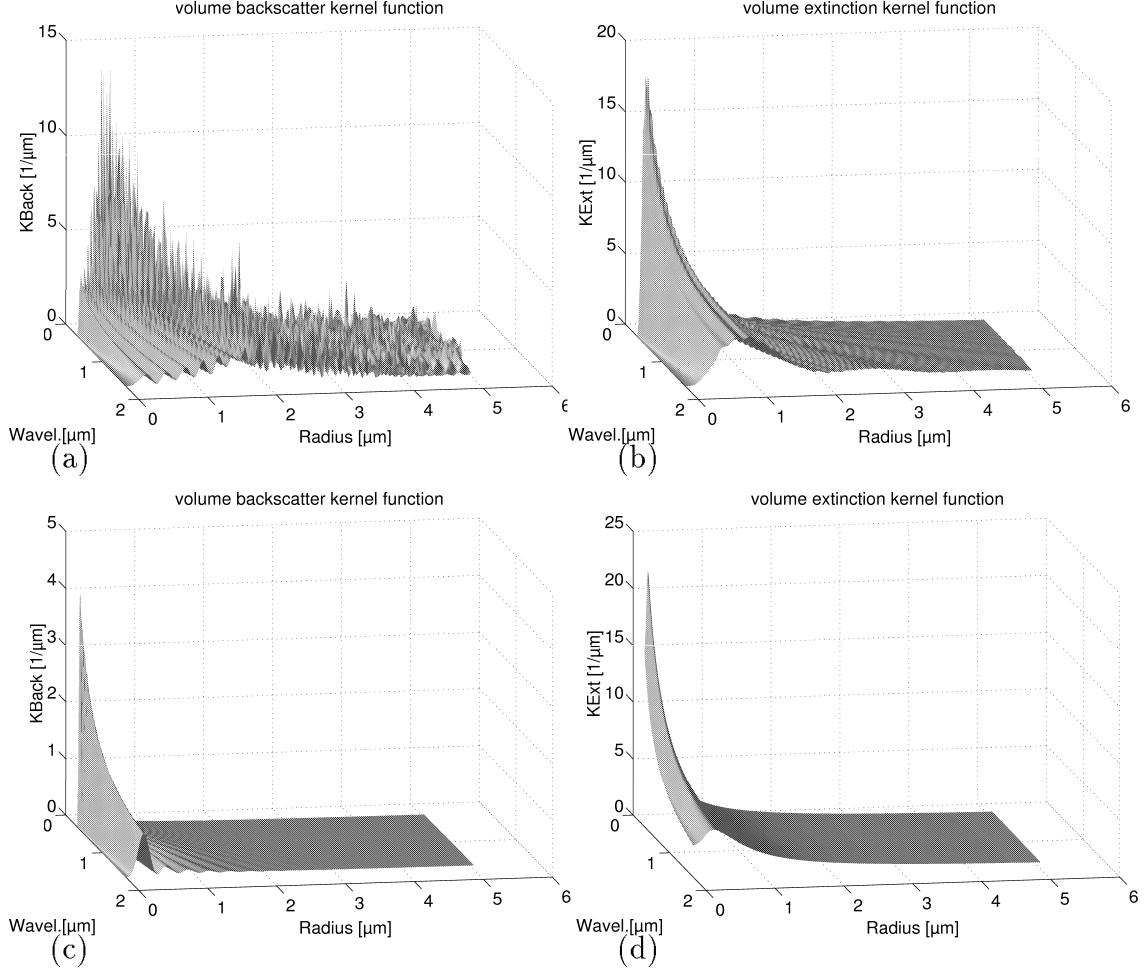


Fig. 1. Volume backscatter and extinction kernel function $K_{\pi/ext}^v$ for different refractive indices: first case without absorption, i.e. $m_1 = 1.5 + 0.0i$ (a) and (b); second case with strong absorption, i.e. $m_2 = 1.5 + 0.5i$ (c) and (d), see equation (4).

decide to deal with a numerical approximation. The smaller the μ_i , the more oscillations in the singular functions u_i and v_i , see Fig. 3(a),(b).

The inversion of the Mie backscatter kernel is potentially interesting because the kernel itself possesses a high degree of oscillation if the absorption is weak, i.e. the imaginary part of the refractive index is small, see Fig. 1(a), which suggest that the classic instability due to the smoothing out of fast oscillatory components in the solution space may not occur. However, in practice the oscillation of the kernel is so fine that the particle distribution would need to be computed on an extremely fine quadrature grid. This itself produces attendant problems with noise in the data. On the other side if the absorption is strong, i.e. the imaginary part of the refractive index is large, see Fig. 1(b), the kernel is smooth. In contrast to the backscatter kernel the Mie extinction kernel is very smooth in both absorption cases, see Fig. 1(c),(d).

The SVE is a powerful analysis tool, but unfortunately it is only known analytically in a limited number of cases. Hence approximations to the SVE

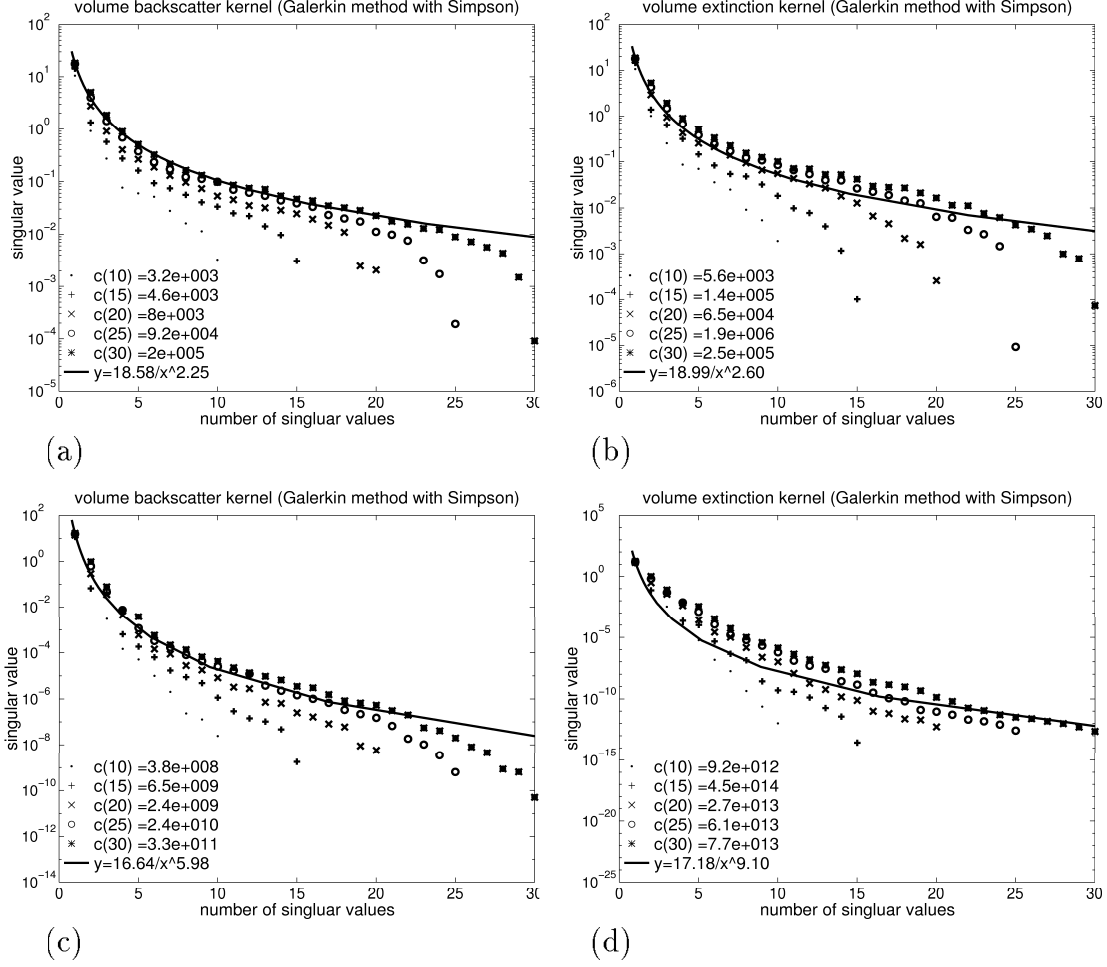


Fig. 2. An approximation to the singular values and to the degree of ill-posedness of the volume backscatter lidar operator K_{π}^v in dependence on the absorption (a) without and (c) with strong absorption as well as of the volume extinction lidar operator K_{ext}^v (b) and (d); with the refractive indices $m_1 = 1.5 + 0.0i$ (a) and (b) as well as $m_2 = 1.5 + 0.5i$ (c) and (d); the $c(i)$ are the condition numbers of the resulting coefficient matrices, see equation (29), in dependence on the discretization dimension n .

can always be computed numerically when (1) and (2) or (4), respectively, are discretized by means of the Galerkin method followed by computation of the singular value decomposition (SVD) of the such obtained matrix. Choose orthonormal basis ϕ_1, \dots, ϕ_n and ψ_1, \dots, ψ_n in the spaces $L_2(I_r)$ and $L_2(I_{\lambda})$, respectively, and define a matrix $A = (a_{i,j})$ as follows

$$a_{ij} = (\psi_i, T\phi_j) \quad i, j = 1, \dots, n \quad (29)$$

where (\cdot, \cdot) denotes the scalar product in the space $L_2(I_{\lambda})$. Then the SVD of A immediately gives approximations to the SVE of the kernel. Let $A \in R^{n,n}$

be a square matrix. Then the SVD of A is a decomposition of the form

$$A = UDV^T = \sum_{i=1}^n u_i \sigma_i v_i^T, \quad (30)$$

where $U = (u_1, \dots, u_n), V = (v_1, \dots, v_n) \in R^{n^2}$ are matrices with orthonormal columns and where the diagonal matrix $D = \text{diag}(\sigma_1, \dots, \sigma_n)$ has nonnegative diagonal elements appearing in nonincreasing order such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. The numbers σ_i are the singular values of A while the vectors u_i and v_i are the left and right singular vectors of A , respectively. In connection with discrete ill-posed problems, two characteristic features of the SVD are very often found. First, the singular values decay gradually to zero with no particular gap in the spectra. An increase of the dimensions of A will increase the number of small singular values, see Fig. 2, as in our lidar application. Second, the left and right singular vectors tend to have more sign changes in their elements as the index i increases, i.e. σ_i decreases. Both features are consequences of the fact that the SVD of A is closely related to the SVE of the underlying kernel, see [11], [1] and [24].

The singular values μ_j of T are then approximated by the algebraic singular values σ_j of A . In detail, the n singular values $\sigma_i^{(n)}$ of A are approximations to the n singular values of the kernel. Moreover, if we introduce the functions

$$\bar{u}_j(\lambda) = \sum_{i=1}^n u_{ij} \psi(\lambda), j = 1, \dots, n, \quad (31)$$

$$\bar{v}_j(r) = \sum_{i=1}^n v_{ij} \phi(r), j = 1, \dots, n, \quad (32)$$

where u_{ij} and v_{ij} are the elements of U and V , then these functions are approximations to n left and right singular functions of the kernel. We compute the double integrals in (29) by Simpsons numerical quadrature scheme, so that we can expect that the quadrature errors do not exceed the approximation errors caused by the basis functions. Due to [12] and [13] the singular values $\sigma_i^{(n)}$ (where n is the number of basis functions) are increasingly better approximations to the true singular values μ_i , in other words it holds

$$\sum_{i=1}^n (\mu_i - \sigma_i^{(n)})^2 \leq \Delta_n^2 \quad (33)$$

and

$$\sigma_i^{(n)} \leq \sigma_i^{(n+1)} \leq \mu_i, 0 \leq \mu_i - \sigma_i^{(n)} \leq \sqrt{\|T\|^2 - \|A\|_F^2} =: \Delta_n, i = 1, \dots, n. \quad (34)$$

The true singular values μ_i of T are bounded by the computed singular values $\sigma_i^{(n)}$ as follows: $\sigma_i^{(n)} \leq \mu_i \leq ((\sigma_i^{(n)})^2 + \Delta_n^2)^{1/2}$. If $\Delta_n \rightarrow 0$ for $n \rightarrow \infty$, then the approximate singular values $\sigma_i^{(n)}$ converge uniformly in n to the true singular values μ_i , see Fig. 2(a)-(d).

Moreover, for the singular functions hold

$$\max\{\|u_i - \bar{u}_i\|_2, \|v_i - \bar{v}_i\|_2\} \leq \left(\frac{2\Delta_n}{\mu_i - \mu_{i+1}}\right)^{1/2}. \quad (35)$$

that means, the corresponding approximate singular functions converge in the mean to the true singular functions, see Fig. 2(a),(b). Notice the square root in (35) which means that the singular value estimates $\sigma_i^{(n)}$ are usually much more accurate than the approximate singular functions.

The intervalls I_r and I_λ are each divided into n subintervalls $\{I_r^{(i)}\}$ and $\{I_\lambda^{(i)}\}$ of the same lengths h_r and h_λ , respectively, and the basis functions are then given by

$$\phi_i(r) = \begin{cases} h_r^{-1/2} & : r \in I_r^{(i)} \\ 0 & : \text{else} \end{cases}, \psi_i(\lambda) = \begin{cases} h_\lambda^{-1/2} & : \lambda \in I_\lambda^{(i)}, i = 1, \dots, n \\ 0 & : \text{else} \end{cases} \quad (36)$$

Let $r_0 = 0.001\mu\text{m}$, $r_1 = 5\mu\text{m}$, $\lambda_0 = 300\text{nm}$ and $\lambda_1 = 1100\text{nm}$, which are real-life domains in the lidar field, then the formulas give second-order approximations to the singular values μ_i see Fig. 2.

All singular values of A , which arise in such a discrete ill-posed problem

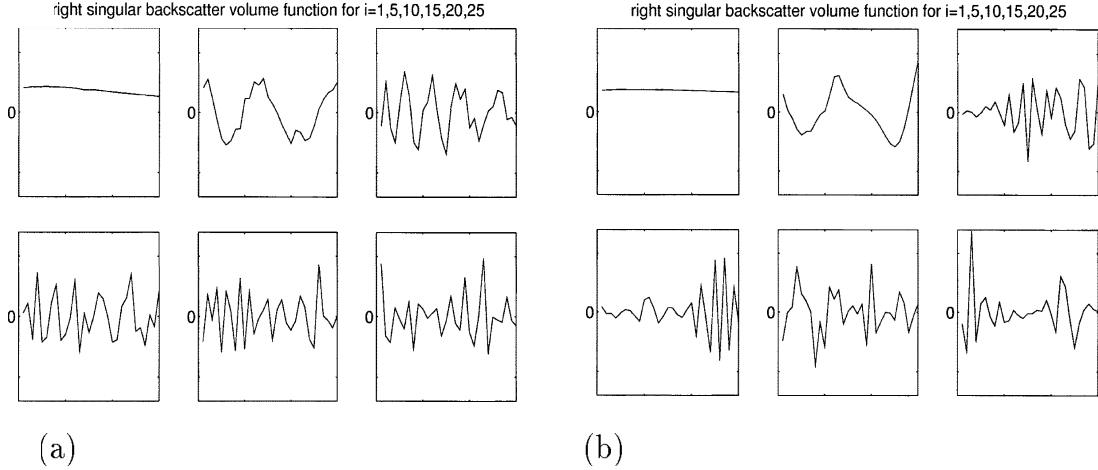


Fig. 3. A qualitative approximation to six right singular functions of the volume backscatter lidar operator v_i for $i = 1, 5, 10, 15, 20, 25$; (a) $m_1 = 1.5 + 0.0i$ and (b) $m_2 = 1.5 + 0.5i$. We see that the higher the index, the more high-frequency components are present in v_i .

from the sampling of a Fredholm integral equation of the first kind, decay on

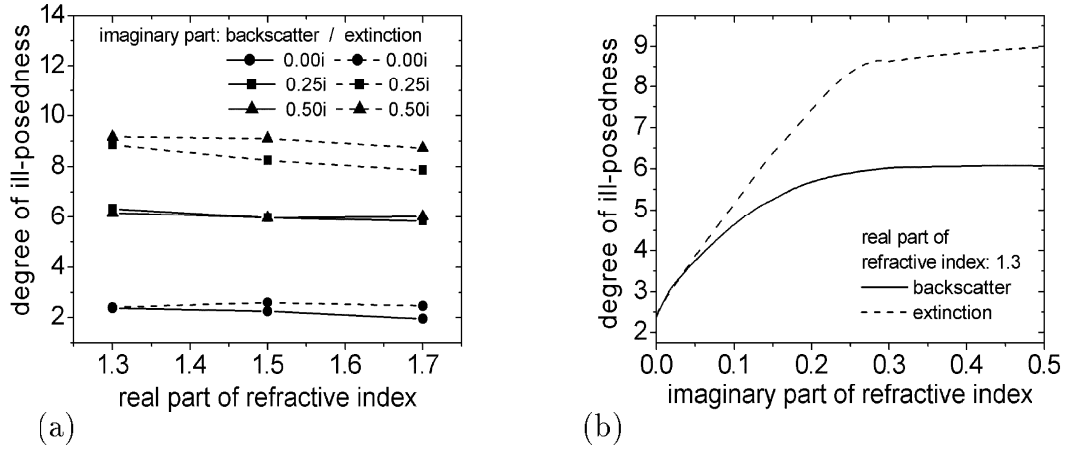


Fig. 4. The degree of ill-posedness of the volume lidar operators in dependence on the real and imaginary part of the refractive index. We see on the one hand no remarkable influence of the real part and on the other hand the important influence of the imaginary part, i.e. the absorption influence.

the average to zero. There is no practical gap in the singular value spectrum, typically, the singular values follow a harmonic progression $\sigma_i \simeq i^{-\alpha}$ or a geometric progression $\sigma_i \simeq e^{-\alpha i}$, where α is a positive real constant. The decay rate of the singular values μ_i is so fundamental for the behaviour of ill-posed problems that it makes sense to use this decay rate to characterize the degree of ill-posedness of the problem. Hofmann [14] and [15] gives the following definition: if there exist a positive real number α such that the singular values satisfy $\mu_i = O(i^{-\alpha})$, then α is called the degree of ill-posedness. The problem is characterized as *mildly* or *moderately ill-posed* if $\alpha \leq 1$ or $\alpha > 1$, respectively. On the other hand, if $\mu_i = O(e^{-\alpha i})$, i.e. the singular values decay very rapidly, then the problem is termed *severely ill-posed*.

For our lidar operators (4) we determined the condition numbers and by a numerical weighted nonlinear least squares method fit an approximation to the degree of ill-posedness, see Fig. 2(a)-(d). In general, one observes that the lidar operators are moderately ill-posed, since α is between 2.25 and 9.10. In detail, one can see that the degree of the extinction operator is higher as the degree of the backscatter one, see Fig. 2(a),(b) or Fig. 2(c),(d), respectively. Moreover, if the absorption of the particles is strong then the degree grows rapidly, see Fig. 2(c),(d). Realizing the logarithmic scale in Fig. 2(d) one can see that the singular values are almost located on a straight line. Therefore, this extinction operator with strong absorption is nearly severely ill-posed. As one expects the condition numbers grow with n and they show the same behaviour as the degree. The matrices are always highly ill-conditioned, and its numerical null space is spanned by vectors with many sign changes. Fig. 3(a),(b) shows qualitatively six approximations $v_i, i = 1, 5, 10, 15, 20, 25$, of the right singular functions and the typical behaviour that the higher the index i the higher the spectral components in v_i . Due to the higher degree of ill-

posedness it seems that the behaviour of the oscillations is more unstructured in the strong absorption case Fig. 3(b).

With respect to the evaluation of lidar measurements it is necessary to know, how the degree of ill-posedness depends on the real and imaginary part of the refractive index of the particles. This dependence is shown in Fig. 4(a),(b). On the one hand there is no influence of the real part in the real-life domain between 1.3 and 1.7, see Fig. 4(a). On the other hand the imaginary part in the domain between 0 and 0.5 has a very important influence on the degree. The degree grows rapidly between 0 and 0.25 which is a very realistic domain in our atmosphere. The growth rate can be compared with a root function in the underlying domain.

4 Conclusions

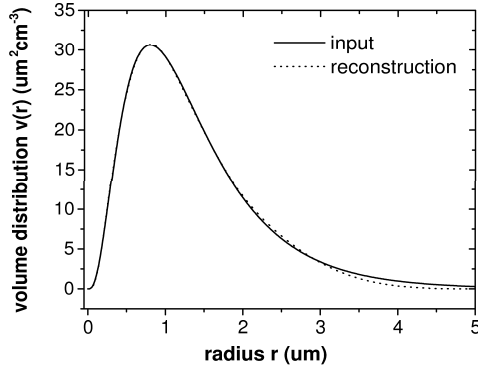
Based on this knowledge we had to develop a special hybrid regularization technique in the sense of (6) and (8) which is described in detail in [6] and [5]. We briefly give some numerical results here. Using for ψ_i the delta distribution the Galerkin method changes to a limit case the collocation method, which does not need so much computer time as the first one. We deal with B-spline functions for the ϕ_i , see equations (29) and (36). The dimension n and the order of the B-spline functions are taken as regularization parameters. The resulting linear equation systems from M backscatter $\beta^{Aer}(\lambda_i)$ and N extinction $\alpha^{Aer}(\lambda_j)$ measurements is solved by truncated singular value decomposition. This hybrid regularization technique with three regularization parameters works for different distributions with different modes, see Fig. 5(a),(b),(d). Moreover, this technique promises good results with additional unknown refractive index, see Fig. 5(c),(d).

Acknowledgements

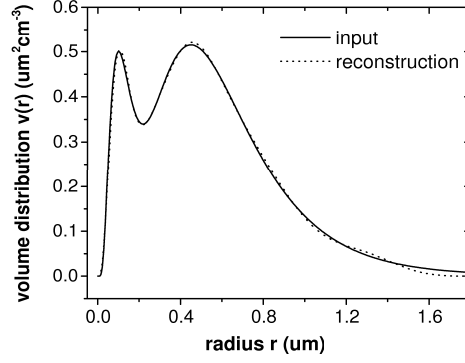
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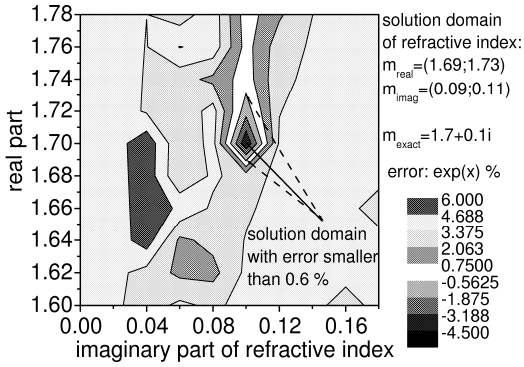
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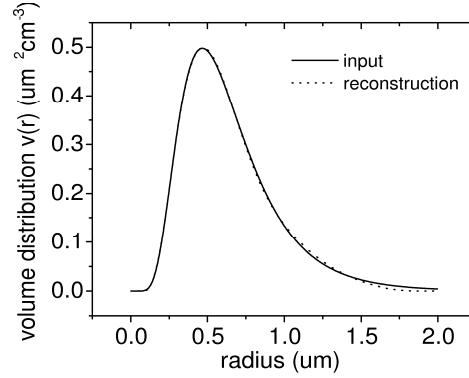
(a)



(b)



(c)



(d)

Fig. 5. Regularized inversion results, i.e. the determination of the particle distributions of different types from six backscatter $\beta^{Aer}(\lambda_i)$ and two extinction $\alpha^{Aer}(\lambda_j)$ coefficients: (a) Gamma distribution, (b) Bimodal-log-normal distribution, (d) Monomodal-log-normal distribution and (c) distribution (d) with additional unknown refractive index m in the kernel functions $K_{\pi/ext}^v(r, \lambda, m)$.

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