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Simultaneous activity and attenuation reconstruction in emission tomography

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Simultaneous Activity and Attenuation Reconstruction in Emission Tomography

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Abstract. In Single Photon Emission Computed Tomography (SPECT) one is interested in reconstructing the activity distribution f of some radiopharmaceutical. The data gathered suffer from attenuation due to the tissue density μ. Each imaged slice incorporates noisy sample values of the nonlinear attenuated Radon transform

$$A(f,\mu)(\omega,s) = \int_{-\infty}^{\infty} f(s\omega^{\perp} + t\omega) \exp(-\int_{t}^{\infty} \mu(s\omega^{\perp} + \tau\omega)\, d\tau)\, dt \ . \qquad (1)$$

Traditional theory for SPECT reconstruction treats μ as a known parameter. In practical applications, however, μ is not known, but either crudely estimated, determined in costly additional measurements or plainly neglected. We demonstrate that an approximation of both f and μ from SPECT data alone is feasible, leading to quantitatively more accurate SPECT images. The result is based on nonlinear Tikhonov regularization techniques for parameter estimation problems in differential equations combined with Gauss-Newton-CG minimization.

Keywords

SPECT, tomography, attenuated Radon transform, nonlinear inverse problem, Tikhonov regularization, nonlinear optimization.

Mathematical subjects classification

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1. Introduction

Some of the most challenging problems at the crossroads of mathematics, engineering and medicine arise from the development of tomographic methods for medical diagnosis, especially those involving nonlinear models with very noisy data. A prominent example is the use of single photon emission computed tomography (SPECT) to examine massive body parts in nuclear medical diagnosis. When SPECT is used to visualize the metabolism of a bodily organ, a disease can often be diagnosed much earlier than is possible with conventional tomographic devices (e.g. CT, Ultrasound) that detect anatomical changes only.

SPECT (similar to PET, positron emission tomography) functions by reconstructing the activity distribution of a γ-radiopharmaceutical. Unfortunately,

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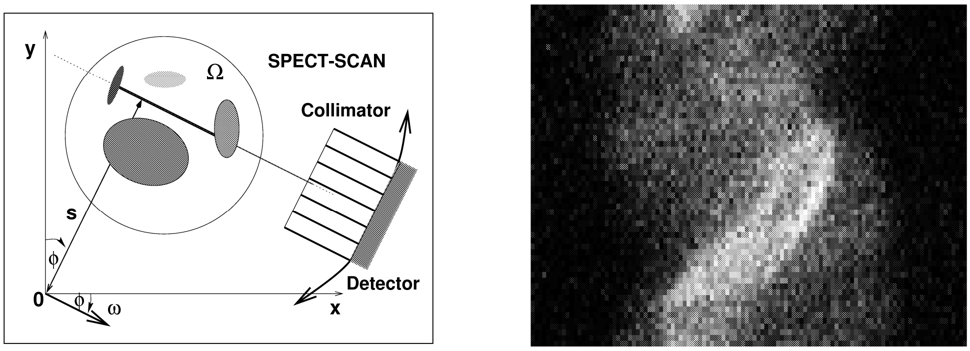


Figure 1. Scaning geometry (a) and clinical sinogram (b)

γ-photons are absorbed and scattered in dense media. This absorption may cause strong artifacts in the reconstruction algorithms which are currently used and it is therefore desirable to find a clinically usable method for attenuation correction in SPECT imaging, preferably one using no additional measurements besides the standard emission data (cf. Figure 1(b)).

A mathematical model of the imaging process in SPECT is the attenuated Radon transform (ATRT). For SPECT scans the count rates of the scintillation cameras in a certain energy range are approximately given by line integrals over the activity f and the attenuation μ in the cross-section under consideration.

Along the line ℓ in direction of the unit vector $\omega = (\cos(\varphi), \sin(\varphi)) \in S^1$ at distance $s \in [-\varrho, \varrho]$ to the origin (see Figure 1(a)) the count rate is approximately proportional to $A(f, \mu)(\omega, s)$ as given by Equation (1). Mainly due to scatter effects, integration over cones rather than lines, movement of the radiopharmaca during the measurements and especially the stochastic nature of the radioactive decay the measured data (called sinograms after cross-section wise rearrangement, as in Figure 1(b))

$$y^\delta(\omega_k, s_l) \simeq y^0(\omega_k, s_l) := A(f, \mu)(\omega_k, s_l) \tag{2}$$

are rather noisy sample values of the ATRT (cf. Figure 1). More precise models are known, but may not be treated numerically with current computers (cf. [Dic97]). The ATRT is also a mathematical model for a number of other inverse problems. It plays a prominent role, for example, in the analysis of plasma physics experiments, optics of semi-transparent media, non-destructive testing (e.g. for the inspection of containers full of nuclear waste), and astronomical measurements.

In SPECT the task is to find quantitatively accurate estimates for the density distribution f of the radiopharmaceutical. Obviously knowledge is required of the attenuation map μ. However, μ is ignored in the clinical standard backprojection reconstruction algorithms. This leads to pronounced ghosts in the reconstructions, especially when used in thoracic or abdominal diagnosis. More sophisticated algorithms try to estimate μ from additional transmission measurements, possibly leading to increased measurement times, more expensive equipment, higher data noise and radiation doses.

F. Natterer showed in [Nat83, Nat93] that at least some information on the attenuation μ may be recovered from sampled SPECT data alone. Recently several authors investigated the problem of identifying μ from the ATRT data alone [CGLT79, Bro95, NWCG95, You95, KMJ$^+$96]. To our knowledge all these attempts have had but limited success.

This paper investigates a *Tikhonov-IntraSPECT* ansatz, i.e. whether a good approximation of both the activity and attenuation maps f and μ can be obtained using only the SPECT data and a nonlinear version of the Tikhonov regularization method developed based on the work of H. Engl *et al* on parameter identification in partial differential equations.

In the second section we summarize some general results on nonlinear Tikhonov regularization. Section 3 will introduce a factorization of the ATRT and some mapping properties of the constituting operators. Based on this factorization our main theoretical results are given in Section 4. The following section gives a brief introduction and some results on a numerical method to minimize the Tikhonov functional efficiently. The remaining Section 6 is an appendix that contains proofs of some more technical results used in Section 3.

2. Nonlinear Tikhonov regularization

Our research is based on the famous main theorem of Engl, Kunisch and Neubauer in [EKN89]. Let us fix some notation. We consider a continuous, nonlinear operator $A : \mathcal{D}(A) \subset X \to Y$ between Banach spaces X, Y. The aim is to find an estimate for a signal x which is mapped by A to perfect data $y = y^0$

$$A(x) = y \tag{3}$$

given some approximate data y^δ with

$$\|y^\delta - y\|_Y \leq \delta \ . \tag{4}$$

In the SPECT problem we have to deal with 2 components of $x = (f, \mu)$.

An element $x^\dagger \in \mathcal{D}(A)$ satisfying

$$\begin{aligned} \|A(x^\dagger) - y\|_Y &= \min_{x \in \mathcal{D}(A)} \|A(x) - y\|_Y = \operatorname{dist}(\mathcal{R}(A), y) \quad \text{and} \\ \|x^\dagger - x_*\|_X &= \min_{x \in \mathcal{D}(A)} \{\|x - x_*\|_X \mid \|A(x) - y\|_Y = \operatorname{dist}(\mathcal{R}(A), y)\} \end{aligned}$$

is called x_**-minimum-norm-least-square-solution*, if furthermore $A(x^\dagger) = y$ then $x^\dagger$ is a x_**-minimum-norm-solution.* The basic idea is to investigate whether an (approximate) minimizer of the Tikhonov functional

$$T(x) := \|A(x) - y^\delta\|_Y^2 + \alpha \|x - x_*\|_X^2$$

is a good estimate for the solution of the inverse problem ((3),(4)). In the Tikhonov functional x_* is called a start value (or initial guess) and $\alpha > 0$ the regularization parameter. Let $\eta \geq 0$. By $x_\alpha^{\delta,\eta}$ we denote an element of $\mathcal{D}(A)$ satisfying

$$T(x_\alpha^{\delta,\eta}) \leq \inf_{x \in \mathcal{D}(A)} T(x) + \eta \ .$$

In [EKN89] conditions were established under which nonlinear Tikhonov regularization, i.e. approximating the solution $x^\dagger$ by such $x_\alpha^{\delta,\eta}$, is order-optimal. We summarize and slightly generalize them in the following definition and theorem.

Definition: 2.1 The nonlinear inverse problem ((3),(4)) of approximating the preimage next to $x_* \in X$ of (the perfect data) y under the operator A satisfies *condition T1-T7* (condition T for short) if:

T1 $A : \mathcal{D}(A) \subset X \to Y$ has a convex domain $\mathcal{D}(A)$ in a Hilbert space X.

T2 Y is a Banach space.

T3 A x_*-minimum norm solution $x^\dagger$ with $A(x^\dagger) = y$ exists.

T4 The operator is Fréchet differentiable in some neighborhood $B_\rho(x^\dagger)$, $\rho > 0$.

T5 The Fréchet derivative A' is Lipschitz continuous at $x^\dagger$ (with constant L)

$$\|A'(z) - A'(x^\dagger)\|_{L(X,Y)} \leq L \|z - x^\dagger\|_X \text{ for all } z \in B_\rho(x^\dagger) \ .$$

T6 The operator A is weakly sequentially closed, i.e. if $x_n \rightharpoonup x \in X$ and $A(x_n) \rightharpoonup y \in Y$ then $x \in \mathcal{D}(A)$ with $A(x) = y$.

T7 The solution $x^\dagger$ fulfills a first order source condition with respect to the initial guess x_*, i.e. there exists $w \in Y'$ with

$$x^\dagger - x_* = A'(x^\dagger)^* w \in \mathcal{R}(A'(x^\dagger)^*) \text{ and } L\|w\| < 1 .$$

Here $A'(x^\dagger)^*$ denotes the Banach space dual of $A'(x^\dagger)$ defined by

$$\langle\, y \,|\, A'(x^\dagger)x \,\rangle_{Y' \times Y} = \langle\, A'(x^\dagger)^* y \,|\, x \,\rangle_{Y' \times Y} .$$

□

Theorem: 2.2 *Let the nonlinear ill-posed problem ((3),(4)) of approximating the preimage $x^\dagger$ next to $x_* \in X$ of $y \in Y$ under the operator $A : \mathcal{D}(A) \subset X \to Y$ given y^δ satisfy condition T. Further, let $(y^\delta)_\delta$ be a sequence of improved data with $\|y^\delta - y\| \leq \delta$ and data error bound $\delta \searrow 0$. If a regularization parameter $\alpha \sim \delta$ and a tolerable minimization error $\eta = \mathcal{O}(\delta^2)$ are chosen and if $\rho > 2\|x^\dagger - x_*\| + \frac{\delta}{\sqrt{\alpha}} + \sqrt{\frac{\eta}{\alpha}}$ $(= 2\|x^\dagger - x_*\| + \mathcal{O}(\sqrt{\delta})$) in T4,T5, then for some $C_e, C_r < \infty$ the order optimal error estimate*

$$\|x_\alpha^{\delta,\eta} - x^\dagger\|_X \leq C_e\sqrt{\delta}$$

holds and the residual fulfills

$$\|A(x_\alpha^{\delta,\eta}) - y\|_Y \leq C_r\delta .$$

Proof: For Y a Hilbert space the theorem was first proved in [EKN89, thm. 2.4]. A version for $\eta = 0$ can also be found in [EHN96, thm. 10.4]. A proof dealing also with higher order source conditions is given in [EHN96, thm. 10.7]. To include the case where Y is a Banach space (a minor generalization) for first order source condition one simply has to replace the adjoint of the derivative in the topology generated by the scalar-product in a Hilbert space Y, by the dual of $A'(x^\dagger)$ with respect to the duality of $\langle \diamond \,|\, \diamond \rangle_{Y' \times Y}$ and use the latter instead of the scalar-product $\langle \diamond \,|\, \diamond \rangle_Y$ in the proof of [EKN89, thm. 2.4]. The Hilbert space structure in X is crucial to the known proofs, since they all depend on a polarization identity in X. If $\eta > 0$ is used, the weak closedness condition (T6) may be dropped, it only ensures existence of a minimizer of T.

A close examination of the proof gives the condition on ρ, which is only required to be sufficiently large in the original work. It further follows, that the minimal constant C_e in the error estimate is obtained for $\eta = 0$ and $\alpha = \frac{\delta}{\|w\|}$ and given by $C_e^* = 2\sqrt{\frac{\|w\|}{1 - L\|w\|}}$. For this α the constant $C_r^* = 3$ is obtained. ■

Remark: 2.3 In applications Y will most likely be a Hilbert space, where the norm has a simple Fréchet derivative. Minimizing T for Y say an L^1 function space, for which condition T4, T5 may be more easily satisfied (cf. Theorem 4.10), is more demanding, because e.g. $\|\diamond\|_1$ is not Fréchet differentiable. It is possible, but somewhat technical, to proof weaker results similar to the above in case A' is merely Hölder continuous. The interested reader is referred to [Dic97].

Engl *et al* also discuss other nonlinear regularization techniques in their monograph [EHN96, chp.10]. Most of these methods require additional conditions that strongly restrict the nonlinearity of A. Since these restrictions could not be established for various modifications of the attenuated Radon transform (cf. [Dic97, Sec. 4.3]), we rely on the basic Tikhonov technique. ⋄

Using Taylor-expansion of the operator A it is possible to demonstrate, that the nonlinear Tikhonov-functional $T(x)$ is strongly convex in the vicinity of an approximate minimizer. This ensures convergence of most standard minimization routines applied to T once they are sufficiently close to the solution.

Theorem: 2.4 *Let A be a Fréchet differentiable operator between Hilbert spaces X, Y with Lipschitz continuous derivative, Lipschitz constant L and convex domain $\mathcal{D}(A)$. Further, let x_0 satisfy $A(x_0) \simeq y^\delta$ with $\|A(x_0) - y^\delta\| \leq C\delta$. Then if $C < \frac{\alpha}{L\delta}$ there exists a ball $B_\rho(x_0)$, $\rho = \rho(\alpha, \delta) > 0$ over which the Tikhonov functional $T = T(\diamond; \alpha, x_*, y^\delta)$ is strongly convex.*

The somewhat technical proof of this theorem can be found in [Dic97] and will be given in a forthcoming paper on the details of our numerical method.

Corollary: 2.5 *Let A be as in the previous theorem, then the minimizer x_α^δ of the Tikhonov functional $T(\diamond; \alpha, x_*, y^\delta)$ with perturbed data is locally unique, provided $\alpha > L\|A(x_\alpha^\delta) - y^\delta\|_Y$ holds.*

3. Decomposition of a modified ATRT-operator

It is easy to proof differentiability of the ATRT operator A considered as a map

$$A : L^\infty(\Omega) \times L^\infty(\Omega) \to L^\infty(S^1 \times [-\varrho, \varrho])$$

for Ω the disk $\{x \in \mathbb{R}^2 | \|x\|_2 \leq \varrho\}$. By the Sobolev lemma $H^s(\Omega) \hookrightarrow \mathcal{C}^j(\Omega)$ is a continuous embedding $\forall s > j + \dim(\Omega)/2$, (in particular $\forall f \in H^s(\Omega),\ s > 1 :$ $\|f\|_\infty \leq C_{s,\infty}\|f\|_{H^s}$). Together with the inclusion $L^\infty(\Omega) \hookrightarrow L^2(\Omega)$ one may deduce Fréchet differentiability of arbitrary order of A as a map

$$A : H_0^s(\Omega) \times H_0^s(\Omega) \to L^2(S^1 \times [-\varrho, \varrho])$$

for $s > 1$. This allows the application of the previous theorem, but only under the assumptions that the parameter functions f, μ are in $H_0^1(\Omega)$. Further, in order to satisfy the source condition $x - x_* = (f - f_*, \mu - \mu_*) \in \mathcal{R}(A'(x^\dagger)^*)$, the solution f (resp. μ) has to be in $f_* + \pi H_0^2(\Omega)$ (resp. $\mu_* + \pi H_0^2(\Omega)$), with the orthogonal projection $\pi : H_0^2(\Omega) \to H_0^1(\Omega)$ as a calculation of $A'(x^\dagger)^*$ with respect to the $H_0^1 \times H_0^1, L^2$ topology will show. These conditions are unrealistic in clinical applications.

The natural assumptions on the regularity of the functions f, μ in the application would be to assume that the functions are positive and continuous but with possible jumps on the boundary of some smooth sets (e.g. representing the boundary of an organ). In mathematical terms the desired domain could be modeled by $f, \mu \in H_{0,+}^s(\Omega),\ s < \frac{1}{2}$ (cf. [Nat86, p92ff]). We therefore set forth to establish the existence of a Lipschitz continuous derivative of A over a larger space, namely to a product space of Sobolev spaces of fractional order smaller than $\frac{1}{2}$.

3.1. Operators related to the ATRT

We proceed by studying a decomposition of the ATRT into simpler operators and their mapping properties. Let for this section f, μ denote functions in $\mathcal{C}_c(\mathbb{R}^2)$ and $g, h \in \mathcal{C}_c(S^1 \times \mathbb{R}^2)$ (i.e. continuous functions with compact support). The following operators are then well defined.

3.1.1. Attenuated Radon transform We define the ATRT operator $A : \mathcal{C}_c(\mathbb{R}^2) \times \mathcal{C}_c(\mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^1)$ and a generalized version $\tilde{A} : \mathcal{C}_c(S^1 \times \mathbb{R}^2) \times \mathcal{C}_c(\mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^1)$ that takes angle dependent functions as first argument. It will turn out, that $\tilde{A}$ is related to the derivative of A.

$$A(f, \mu)(\omega, s) := \int_{\mathbb{R}} f(s\omega^\perp + t\omega) \exp(-\int_t^\infty \mu(s\omega^\perp + \tau\omega)\, d\tau)\, dt$$

$$\tilde{A}(g, \mu)(\omega, s) := \int_{\mathbb{R}} g(\omega, s\omega^\perp + t\omega) \exp(-\int_t^\infty \mu(s\omega^\perp + \tau\omega)\, d\tau)\, dt\ . \qquad (5)$$

3.1.2. Radon transform The Radon hyper-plane integral transform in 2 dimensions $R : \mathcal{C}_c(\mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^1)$ agrees with the X-ray line integral transform up to a change of variables. As for the ATRT a generalized variant $\tilde{R} : \mathcal{C}_c(S^1 \times \mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^1)$ taking angle dependent argument functions is introduced.

$$\begin{aligned} Rf(\omega, s) &:= \int_{\mathbb{R}} f(s\omega^\perp + t\omega)\, dt \\ \tilde{R}g(\omega, s) &:= \int_{\mathbb{R}} g(\omega, s\omega^\perp + t\omega)\, dt \ . \end{aligned}$$

3.1.3. Fan-beam transform The analysis of the fan-beam transform

$$D\mu(\omega, x) := \int_0^\infty \mu(x + \tau\omega)\, d\tau$$

$D : \mathcal{C}_c(\mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^2)$ is important for a thorough understanding of the ATRT.

In medical applications the functions f, μ have their support in the disk $\Omega := B_\varrho(0) = \{x \in \mathbb{R}^2 | \|x\|_2 \le \varrho\}$ for some radius $\varrho \sim 15 - 20cm$. We observe that if μ has its support in the disk Ω then for $x \notin \Omega$ either $D\mu(\omega, x) = D\mu(\omega, x_\omega)$ for a suitable $x_\omega \in \partial\Omega$ or $D\mu(\omega, x) = 0$. Hence, it suffices to consider $D\mu$ as function over the compact set $S^1 \times \Omega$.

Also when considering the fan-beam transform as part of the ATRT we notice that for f with compact support in Ω we only need to know $D\mu(\omega, x)$ for $|x| \le \varrho$. If μ too has its support in that disk then for $x \in \Omega$

$$D\mu(\omega, x) = \int_0^{2\varrho} \mu(x + t\omega)\, dt = \int_{\mathbb{R}} \chi_{[0,2\varrho]}(t)\mu(x + t\omega)\, dt$$

Therefore we define for $\varrho \in (0, \infty]$ operators

$$D_\varrho\mu(\omega, x) := \int_{\mathbb{R}} \chi_{[0,2\varrho]}(t)\mu(x + t\omega)\, dt \ .$$

Later it will be demonstrated, that unlike D the operators D_ϱ are bounded between some L^p or Sobolev spaces over $\mathbb{R}^2$. Let μ be a function with support in the disk $\Omega = B_\varrho(0)$. Then $D_\varrho\mu$ has its support in $S^1 \times 3\Omega$. For simplicity we may write D for either D or D_ϱ or their restriction to $S^1 \times \Omega$.

3.1.4. Exponential Operator The unbounded growth of the exponential operator $E : \mathcal{C}(S^1 \times \mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^2)$

$$E(g)(\omega, x) := \exp\left(-g(\omega, x)\right)$$

for functions with negative peaks causes obstacles in the analysis.

Physically admissible arguments to E in a decomposition of the ATRT do not have such peaks. Therefore, we introduce operators $E_\phi : \mathcal{C}(S^1 \times \mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^2)$

$$E_\phi(g)(\omega, x) := \phi\left(-g(\omega, x)\right) \tag{6}$$

for some $\phi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}_+)$ that satisfies $\phi|_{\mathbb{R}_+} = \exp(-\diamond)$ and has $|\phi|, |\phi'|$ and $|\phi''|$ bounded. It will be shown that (unlike E itself) E_ϕ is Fréchet differentiable if considered between suitable function spaces. Under certain conditions on ϕ and the spaces involved Fréchet derivatives of E_ϕ of arbitrary order may exist.

3.1.5. Multiplication Operator Formally we introduce the bilinear multiplication $M : \mathcal{C}(S^1 \times \mathbb{R}^2) \times \mathcal{C}(S^1 \times \mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^2)$ as

$$M(g, h)(\omega, x) = g(\omega, x) \cdot h(\omega, x) \ .$$

3.1.6. Inclusion The symbol i is used to denote any kind of continuous embedding operator in case it becomes necessary to name it explicitly. For example we use $i_{S^1} : \mathcal{C}(\mathbb{R}^2) \to \mathcal{C}(S^1 \times \mathbb{R}^2)$ and $i_s : H^s \to L^2$, if $s \geq 0$

$$\begin{aligned} i_{S^1}(f)(\omega, x) &:= f(x) \\ i_s(f) &:= f \ . \end{aligned}$$

Here H^s denotes a Sobolev space of fractional order.

Lemma: 3.1 *With the above defined operators we have the factorization*

$$\begin{aligned} A(f, \mu) &= \tilde{R}(M(i(f), E(D\mu))) \\ \tilde{A}(g, \mu) &= \tilde{R}(M(g, E(D\mu))) \end{aligned}$$

of the nonlinear ATRT operator. The nonlinear ingredients are the bilinear Operator M and the composition operator E.

The introduction of $i = i_{S^1} \circ i_s$ and M may seem needless, however, a simple inclusion operator may have a nontrivial adjoint and helpful compactness properties. Considering M as a bilinear operator helps to clarify the differentiability properties of A and the structure of the dual of A'.

For f, μ with support in the disk $\Omega = B_\varrho$ the fan-beam operator D may be replaced by D_ϱ. Similar to D_ϱ we define $R_\varrho, \tilde{R}_\varrho$ for the (generalized) Radon transformation with integration restricted to the interval $[-\varrho, \varrho]$ in place of $\mathbb{R}$. With the restricted operators we define the restricted ATRT

$$\begin{aligned} A_\varrho(f, \mu) &:= \tilde{R}_\varrho(M(i(f), E_\phi(D_\varrho\mu))) \\ \tilde{A}_\varrho(g, \mu) &:= \tilde{R}_\varrho(M(g, E_\phi(D_\varrho\mu))) \ . \end{aligned}$$

The following proposition is then obvious

Proposition: 3.2 *For continuous functions f, μ with support in the disk of radius ϱ and $\mu \geq 0$*

$$A_\varrho(f, \mu) = A(f, \mu) \ .$$

The restricted operators $R_\varrho, \tilde{R}_\varrho, D_\varrho, E_\phi, A_\varrho, \tilde{A}_\varrho$ and the operators M and $i : \mathcal{C}(\Omega) \to \mathcal{C}(S^1 \times \Omega)$ are continuous with respect to the supremum norm. The operators therefore have a continuous extension to the L^∞ spaces over the domain of their arguments ($\mathbb{R}^2$, resp. $S^1 \times \mathbb{R}^2$, $S^1 \times \mathbb{R}$).

We note that the unrestricted R, D etc. are discontinuous and E is not equicontinuous in L^∞ topology.

3.1.7. Some notation We introduce the nonlinear operator

$$F(\mu) := A_\varrho(f, \mu)$$

for some fixed function $f \in L^\infty(\Omega)$. The problem of reconstructing μ given f is described by F if the supports of f and μ lie in $\Omega = B_\varrho$ and $\mu \geq 0$. These assumptions will always be made in the sequel. If not stated otherwise, we restrict our attention to functions with support in the compact disk Ω resp. $S^1 \times \Omega$. Under that assumption we may later drop the subscript ϱ.

For any operator F whose range consists of functions on (subsets of) $S^1 \times \mathbb{R}$ or $S^1 \times \mathbb{R}^2$ we define the directional restrictions by

$$F_\omega(f)(s) := F(f)(\omega, s) \ .$$

3.1.8. Extension to larger domains In order to use Hilbert space theory we have to extend the operators to a Hilbert space setting in a way that the composition A_ϱ becomes Fréchet differentiable. There are some obstacles to overcome, for e.g. the multiplication has no continuous extension to L^2 and the nonlinear composition operator E (resp. E_ϕ) is not always differentiable.

3.2. On mapping properties of certain operators

3.2.1. Continuous inclusions The inclusion $i : L^p(\Omega) \to L^p(\Sigma \times \Omega)$ defined by $if(\sigma,\omega) = f(\omega)$ is continuous if and only if Σ has finite measure, e.g. for all compact Σ (e.g. S^1). We cite (e.g. from [Tri78]) the well known theorems on continuous embeddings of Sobolev spaces.

Theorem: 3.3 *$H^s(\Omega)$ over an open relatively compact domain Ω of dimension n is continuously embedded in $L^q(\Omega)$ whenever $\frac{s}{n} \geq \frac{1}{2} - \frac{1}{q}$, $s \geq 0$. In case $\frac{s}{n}$ is larger than $\frac{1}{2} - \frac{1}{q}$ this embedding is compact. Further the embedding $i : H^s(\Omega) \hookrightarrow H^{s'}(\Omega)$ is compact for any $s > s'$. For the limit case $q = \infty$ a $s > \frac{n}{2}$ is necessary for a continuous embedding (which then is also compact).*

3.2.2. Multiplication Operator With respect to the maximum norm M is a bilinear continuous operator of norm 1. By the Sobolev lemma M may be extended over a relative compact open subset $\Omega \subset \mathbb{R}^2$ to a bilinear continuous operator $M : H^s(\Omega) \times H^s(\Omega) \to L^\infty(\Omega)$, $\forall s > 1$. The operator M further extends to $M : X(\Omega) \times Y(\Omega) \to Z(\Omega)$, $M(x,y)(\omega) = x(\omega) \cdot y(\omega)$ whenever a continuous multiplication in function spaces $(X \times Y, Z)$ over a common domain Ω is defined. By the generalized Hölder inequality (19) M is continuous with norm 1 e.g. for $M : L^p \times L^q \to L^r$ provided $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. This together with the Sobolev embedding into L^p proves that $M : H^s(\Omega) \times H^t(\Omega) \to L^2$ is continuous whenever $s + t \geq \frac{\dim(\Omega)}{2} = 1$ (with exception of the case $s = 1, t = 0$ and vice versa). Since continuous bilinear operators are always Fréchet differentiable, so is M.

3.2.3. Radon transform Due to the well known projection slice theorem (cf. [Nat86]) for the Radon transform R it may be extended as an operator on all tempered distributions $f \in \mathcal{S}'(\Omega)$ whose Fourier transform is representable by a function. That includes all distributions with compact support $f \in \mathcal{E}'$. The question remains whether the continuation is continuous.

For a $g \in L^\infty(S^1 \times \Omega)$ we see L^∞ continuity of $\tilde{R}$ and R from

$$||\tilde{R}g||_\infty \leq \mathrm{diam}(\Omega) \cdot ||g||_\infty = 2\varrho\,||g||_\infty \ . \tag{7}$$

For $1/p + 1/q = 1$; $p \in [1, \infty)$ and $g(\omega, x)$ with support in $S^1 \times \Omega$ we have the following estimates

$$\begin{aligned} |\tilde{R}g(\omega,s)| &= |\int_{|t|<\sqrt{\varrho^2-s^2}} g(\omega, s\omega^\perp + t\omega)\,dt| \\ &\leq (2\sqrt{\varrho^2-s^2})^{1/q} (\int_{|t|<\sqrt{\varrho^2-s^2}} |g|^p(\omega, s\omega^\perp + t\omega)\,dt)^{1/p} \ . \end{aligned}$$

This leads to

$$\int_{-\varrho}^{\varrho} |\tilde{R}g(\omega,s)|^p \frac{ds}{(1-(\frac{s}{\varrho})^2)^{p/2q}} \leq (2\varrho)^{p/q} \int_{-\varrho}^{\varrho} \int_{|t|<\sqrt{\varrho^2-s^2}} |g|^p(\omega, s\omega^\perp + t\omega)\,dt\,ds \ .$$

Integrating this estimate over S^1 proves the following proposition.

Proposition: 3.4 *For the Radon transformations the inequalities*

$$||\tilde{R}_\omega g||_p \leq ||\tilde{R}g(\omega,\diamond)||_{p,\frac{ds}{(1-(s/\varrho)^2)^{(p-1)/2}}} \leq (2\varrho)^{1-1/p} ||g(\omega,\diamond)||_p \tag{8a}$$

$$||\tilde{R}g||_p \leq ||\tilde{R}g||_{p,\frac{d\omega\,ds}{(1-(s/\varrho)^2)^{(p-1)/2}}} \leq (2\pi)^{1/p}(2\varrho)^{1-1/p} ||g||_p \tag{8b}$$

$$||Rf||_p \leq ||Rf||_{p,\frac{d\omega\,ds}{(1-(s/\varrho)^2)^{(p-1)/2}}} \leq (2\pi)^{1/p}(2\varrho)^{1-1/p} ||f||_p \tag{8c}$$

$$||R_\omega f||_p \leq ||R_\omega f||_{p,\frac{ds}{(1-(s/\varrho)^2)^{(p-1)/2}}} \leq (2\varrho)^{1-1/p} ||f||_p \tag{8d}$$

hold for functions f, g over Ω resp. $S^1 \times \Omega$. This shows together with (7) that the operators have continuous extensions $R \in L(L^p(\Omega), L^p(S^1 \times [-\varrho, \varrho])), R_\omega \in L(L^p(\Omega), L^p([-\varrho, \varrho]))$ and $\tilde{R} \in L(L^p(S^1 \times \Omega), L^p(S^1 \times [-\varrho, \varrho]))$ for all $p \in [1, \infty]$. We note that $\tilde{R}_\omega$ does not belong to $L(L^p(S^1 \times \Omega), L^p([-\varrho, \varrho]))$ for $p < \infty$.

3.2.4. Fan-beam transform Over the compact set $\Omega \subset \mathbb{R}^2$ the operator D is continuous with respect to the L^∞ topology

$$\|D(\mu)\|_\infty \leq \mathrm{diam}(\Omega)\, \|\mu\|_\infty = 2\varrho\, \|\mu\|_\infty \; .$$

As for $\tilde{R}$ we estimate $D\mu(\omega, x) = D_\varrho\mu(\omega, x)$ for $\mu \in L^p(\Omega), x = s\omega^\perp + t\omega \in \Omega$

$$\begin{aligned} |D\mu(\omega, s\omega^\perp + t\omega)| &= |\int_0^{2\varrho} \mu(s\omega^\perp + t\omega + \tau\omega)\, d\tau| \\ &\leq (2\varrho)^{1/q} (\int_0^{2\varrho} |\mu|^p (s\omega^\perp + t\omega + \tau\omega)\, d\tau)^{1/p} \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Integrating over x yields

$$\begin{aligned} \int_{3\Omega} |D_\varrho\mu(\omega, x)|^p\, d\omega\, dx &\leq (2\varrho)^{p/q} \int_0^{2\varrho} \int_{3\Omega} |\mu|^p (x - \tau\omega)\, dx\, d\tau \\ &\leq (2\varrho)^{(p+q)/q} \|\mu\|_p^p \; . \end{aligned}$$

This results yields via restriction to Ω and integrating over S^1 the next proposition.

Proposition: 3.5 *The fan-beam transforms D and D_ω satisfy for $\mu \in L^p(\Omega)$*

$$\begin{aligned} \|D_\omega\mu\|_{p,\Omega} &\leq 2\varrho\, \|\mu\|_p \\ \|D\mu\|_{p,\Omega} &\leq 2\varrho(2\pi)^{1/p}\, \|\mu\|_p \; . \end{aligned}$$

Therefore, D and D_ω have for all $p \in [1, \infty]$ continuous extensions to $L(L^p(\Omega), L^p(S^1 \times \Omega))$ resp. $L(L^p(\Omega), L^p(\Omega))$.

This result is, however, still too weak to proof the main theorem under realistic assumptions. A stronger result is the following.

Theorem: 3.6 *The fan beam transform extends to a compact operator $D \in L(H_0^s(\Omega), L^p(S^1 \times \Omega))$ if*

$$s > s_*(p) := \begin{cases} 0 & p \leq 3 \\ 1 - \frac{3}{p} & p \in [3, 6] \\ 1 - \frac{2}{p-2} & p \in [6, \infty] \; . \end{cases} \tag{9}$$

For $s = s_*(p)$, $p \notin \{6, \infty\}$ the extension remains continuous but is not necessarily compact. In particular the smoothing property $D_\varrho \in L(L^2(\Omega), L^3(\Omega))$ is established.

For the proof of this theorem we use two Sobolev estimates on D_ϱ and and interpolation result which we present before the proof. The somewhat technical proofs of these results are given in the appendix.

The Fourier transform of a function over $\mathbb{R}^n$ is defined as

$$\mathcal{F}_n f(\zeta) := \hat{f}(\zeta) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\zeta\cdot x} f(x)\, dx \; .$$

Consider the L^1-function $\theta_\varrho := \chi_{[0,2\varrho]}$ with Fourier transform

$$\hat{\theta}_\varrho = \frac{2\varrho}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}\varrho\xi} \mathrm{sinc}(\varrho\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sin(2\varrho\xi)}{\xi} - i \frac{2\sin^2(\varrho\xi)}{\xi} \right) \; .$$

A simple calculation employing the translation $T_y\mu = \mu(\diamond - y)$ yields

$$\begin{aligned}\mathcal{F}_2(D_\varrho\mu(\omega,\diamond))(\zeta) &= \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}} \theta_\varrho(t)(T_{-t\omega}\mu)(x)\mathrm{e}^{-\mathrm{i}\zeta\cdot x}\,dx\,dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \theta_\varrho(t)\mathrm{e}^{it\omega\cdot\zeta}\,dt\,\hat\mu(\zeta) \\ &= \sqrt{2\pi}\hat\theta_\varrho(-\omega\cdot\zeta)\hat\mu(\zeta)\ .\end{aligned}$$

Thus the formula

$$\mathcal{F}_2(D_\varrho\mu(\omega,\diamond))(\zeta) = \sqrt{2\pi}\hat\theta_\varrho(-\zeta\cdot\omega)\hat\mu(\zeta) = \left(\frac{\sin(2\varrho\zeta\cdot\omega)}{\zeta\cdot\omega} + i\frac{2\sin^2(\varrho\zeta\cdot\omega)}{\zeta\cdot\omega}\right)\hat\mu(\zeta)$$

holds, which yields the Fourier representation

$$D_{\omega,\varrho} = 2\varrho\,\bar{\mathcal{F}}_2\circ M(\mathrm{e}^{-i\zeta\cdot\omega}\mathrm{sinc}(\varrho\zeta\cdot\omega),\diamond)\circ\mathcal{F}_2\ . \tag{10}$$

This representation of $D_{\omega,\varrho}$ allows to prove smoothing properties of D restricted to Ω. § From Equation (10) it is apparent that the directionally restricted fan beam transform $D_{\varrho,\omega}$ is a bounded operator of norm less than 2ϱ between Sobolev spaces of the same order $D_{\varrho,\omega} : H_0^\alpha(\Omega) \mapsto H_0^\alpha(3\Omega)$. We would like to show that D_ϱ has a smoothing property analog to the Radon transform , i.e. $D_{\varrho,\omega} : H_0^\alpha(\Omega) \mapsto H_0^{\alpha+1/2}(3\Omega)$ should be bounded (cf. [Nat86]). Unfortunately, this is not the case. However, if we integrate over $\omega \in S^1$ a similar result holds.

Proposition: 3.7 *The operators D_ϱ are smoothing of order 1/2 between Sobolev spaces over Ω and $S^1\times 3\Omega$, i.e.*

$$D_\varrho : H_0^\alpha(\Omega) \mapsto H_0^{\alpha+1/2}(S^1\times 3\Omega)$$

is bounded. Where $H_0^{\alpha+1/2}(S^1\times 3\Omega)$ is the subspace of $L^2(S^1\times 3\Omega)$ with norm

$$\|f\|_{H^{\alpha+1/2}(S^1\times 3\Omega)} := \|\|f(\omega,\diamond)\|_{H^{\alpha+1/2}(3\Omega)}\|_{L^2(S^1)}\ .$$

For all $\varepsilon > 0$ the operators $D_{\varrho,\omega}$ are not bounded from $H_0^\alpha(\Omega)$ into $H_0^{\alpha+\varepsilon}(3\Omega)$ but $D_{\varrho,\omega}\mu \in H_0^{\alpha+1/2}(3\Omega)$ for almost all ω. We have further

$$D_\varrho : H_0^\alpha(\Omega) \mapsto \mathcal{H}^\alpha(S^1\times 3\Omega)\ ,$$

were the Banach space $\mathcal{H}^\alpha(S^1\times 3\Omega)$ is a subspace of $H^\alpha(S^1\times 3\Omega)$ equipped with the norm

$$\|f\|_{\mathcal{H}^\alpha} := \sup_{\omega\in S^1}\|f(\omega,\diamond)\|_{H^\alpha(3\Omega)}\ .$$

The following interpolation estimate is the second major ingredient to the proof of Theorem 3.6.

Proposition: 3.8 *Let X, Y be some domains, $0 < p_0, p_1, p_2 < \infty$; $p_0, p_1 \le p_2$ and f a function on $X\times Y$ for which we have the estimates*

$$\|f(x,\diamond)\|_{Y,p_1} \le C_1 \quad \forall\ x\in X$$

and

$$\|\,\|f(x,\diamond)\|_{Y,p_2}\|_{X,p_0} \le C_2\ .$$

§ With the help of the Fourier transform of distributions one may calculate

$$\mathcal{F}_2(D\mu(\omega,\diamond))(\eta\omega+\xi\omega^\perp) = (\pi\delta(\eta) + \frac{\mathrm{i}}{\eta})\hat\mu(\eta\omega+\xi\omega^\perp)$$

for the original fan-beam transformation $D\mu(\omega,\diamond)(= D_\infty\mu(\omega,\diamond))$ considered as a function over all of $\mathbb{R}^2$. This is the distributional limit for $\varrho\to\infty$ of the formula for D_ϱ. Consequently, whenever $\int\mu(x)\,dx = 2\pi\hat\mu(0) \ne 0$ the function $D\mu$ does not lie in a classical function space whose Fourier transform would be a function-space as well. This motivates using the restricted fan-beam operator D_ϱ instead of D. A simple restriction like $\chi_\Omega\cdot D$ would have a more complex Fourier representation.

Then $f \in L^p(X \times Y)$ *for* $p = p_1 + p_0 - \frac{p_0 p_1}{p_2} = p_2 - \frac{(p_2-p_0)(p_2-p_1)}{p_2}$ *and the estimate*

$$\|f\|_{X\times Y,p} < C_1^{p_0/p} \cdot C_2^{1-p_0/p}$$

holds. In particular we obtain the estimate

$$\|f\|_p \le \max(\| \, \|f(x,\diamond)\|_{Y,p_1} \|_{X,\infty}, \| \, \|f(x,\diamond)\|_{Y,p_1} \|_{X,p_2}) \ .$$

Proof: (of Theorem 3.6) If μ has its support in Ω, then $D_\varrho \mu$ clearly has its support in $S^1 \times 3\Omega$. It was demonstrated in Proposition 3.7 via Fourier techniques that D_ϱ has smoothing properties, i.e. $D_\varrho \in L(H_0^s(\Omega), H_0^{s+1/2}(S^1 \times 3\Omega))$. Further, we showed that the directional restriction $D_{\varrho,\omega}$ has a continuous extension to $D_{\varrho,\omega} \in L(H^s(\Omega), H_0^s(3\Omega))$. This implies, that D_ϱ has a continuous extension

$$D_\varrho \in L(H^s(\Omega), \mathcal{H}_0^s(S^1 \times 3\Omega))$$

with $\mathcal{H}_0^s(S^1 \times 3\Omega) \subset H_0^s(S^1 \times 3\Omega)$ normed by $\|g\|_{\mathcal{H}^s(S^1\times 3\Omega)}$ defined above. Thus, D_ϱ maps continuously into the intersection space $S := \mathcal{H}_0^s(S^1 \times 3\Omega)) \cap H_0^{s+1/2}(S^1 \times 3\Omega)$ with norm $\|g\|_S = \max(\|g\|_{\mathcal{H}^s(S^1\times 3\Omega)}, \|g\|_{H^{s+1/2}(S^1\times 3\Omega)})$. By the Sobolev embedding theorems $H_0^s(3\Omega)$ is continuously embedded in $L^p(3\Omega)$ whenever $\frac{s}{\dim(\Omega)} \le \frac{1}{2} - \frac{1}{p}$ (i.e. $p \le \frac{2}{1-s}$ for $0 \le s < 1$). Since further the restriction $r : L^p(3\Omega) \to L^p(\Omega)$ has norm 1, some multiple of $\|g\|_{\mathcal{H}^s(S^1\times 3\Omega)}$ dominates the norm $\| \, \|g(\omega,\diamond)\|_{L^{p_1}(\Omega)} \|_{L^\infty(S^1)}$ and a multiple of $\|g\|_{H^{s+1/2}(S^1\times 3\Omega)}$ dominates $\| \, \|g(\omega,\diamond)\|_{L^{p_2}(\Omega)} \|_{L^2(S^1)}$ if $s \ge \max(1 - \frac{2}{p_1}, \frac{1}{2} - \frac{2}{p_2})$, $p_i \ne \infty$. With Proposition 3.8 on interpolation over product spaces we find

$$\|g\|_{L^p(S^1\times\Omega)} \le \max(\| \, \|g\|_{L^{p_1}(\Omega)} \|_{L^\infty(S^1)}, \| \, \|g\|_{L^{p_2}(\Omega)} \|_{L^2(S^1)})$$

if $p = p_1 + p_2 - \frac{p_1 p_2}{2}$, i.e.

$$p \le \begin{cases} 2 + \frac{2}{1-s} - 2\frac{1/2-s}{1-s} = \frac{3}{1-s} & s \in [0, \frac{1}{2}) \\ 2 + \frac{2}{1-s} = \frac{4-2s}{1-s} & s \in (\frac{1}{2}, 1) \\ 2 + \infty = \infty & s > 1 \end{cases}$$

(For $p = 6$, $s > 1/2$ and for $p = \infty$, $s > 1$ are required). If s satisfies (9) choose $\varepsilon > 0$ such that $s - \varepsilon$ also satisfies (9). Then D may be decomposed as $D = D \circ i_{H^s,H^{s-\varepsilon}}$ where $i_{H^s,H^{s-\varepsilon}} : H^s \hookrightarrow H^{s-\varepsilon}$ is a compact embedding of Sobolev spaces. Since compactness is inherited by compositions, D is compact. ■

3.2.5. Exponential Operator Over a bounded domain the exponential operator E maps $L^r \to L^p$ only for $r = \infty$ or when redefined for functions with negative peaks. For the redefined operator we have $E_\phi : L^r(\Omega) \to L^p(\Omega)$, $\forall \ 1 \le p, r \le \infty$ whenever the domain Ω has finite measure. In the appendix continuity and differentiability properties of composition operators are studied. It follows from Proposition 6.4 that $E_\phi : L^r(\Omega) \to L^p(\Omega)$ (defined by (6)) is m times Fréchet differentiable if $r/p > m$ or $r = p = \infty$. Differentiability results are summarized in the next proposition.

Proposition: 3.9 *Let* E *be the extension of the exponential* $g \mapsto \exp(-g)$ *to* $L^\infty(\Omega)$ *Then the operator* E *is Fréchet differentiable at* $g \in L^\infty$ *with derivative*

$$\partial E_\phi(g)\nu = \partial E(g)\nu = -E(g) \cdot \nu = M(-E(g), \nu) \qquad (11)$$

An analog result holds for the redefined exponential composition with ϕ *(cf. (6))* $E_\phi : L^r(\Omega) \to L^p(\Omega)$ *over* Ω *with finite volume. Let* $r > p$, $r, p \in [1, \infty]$ *or* $r = p = \infty$ *. For* $g, \nu \in L^r, g \ge 0$ *the operator* $E_\phi = \phi\circ$ *is Fréchet differentiable at* g *with derivative given by (11).*

Higher derivatives exist of any order for E and E_ϕ on L^∞ and of all orders $m < \frac{r}{p}$ for $E_\phi : L^r \to L^p$ provided $\phi|_{\mathbb{R}_-}$ is sufficiently regular, e.g. has all derivatives up to order $m+2$ bounded. They are given at g (with $g \geq 0$ for E_ϕ) by

$$\partial^m E_\phi(g)(\nu_1, \dots, \nu_m) = \partial^m E(g)(\nu_1, \dots, \nu_m) = (-1)^m E(g) \prod_{i=1}^{m} \nu_i \ . \tag{12}$$

If $m + \varepsilon \leq \frac{r}{p}$, $\varepsilon \in (0,1]$ and the regularity assumption (26) on $\phi^{(m)}$ holds, e.g. if $\phi \in \mathcal{C}^{m+1,1} \subset \mathcal{C}^{m+2}$ then the m^{th} derivative $E_\phi^{(m)}$ of $E_\phi : L^r \to L^p$ is Hölder continuous of order ε. In particular for E'_ϕ to be L-continuous $r \geq 2p$ is required.

Proof: First we consider the case for E and $g, \nu \in L^\infty$. In the Banach algebra L^∞

$$E(g+\nu) - E(g) = E(g)(E(\nu) - 1) = E(g) \sum_{j \geq 1}^{\infty} \frac{(-\nu)^j}{j!}$$

converges. A simple estimate using $\|\nu^j\|_\infty = \|\nu\|_\infty^j$ proves for $\|\nu\|_\infty \leq 1$

$$\|E(g+\nu) - E(g) - (-E(g)\nu)\|_\infty \leq \|E(g)\|_\infty (\mathrm{e} - 2) \|\nu\|_\infty^2$$

This estimate yields the theorem for $m = 1$ and L^∞. Similarly

$$\|E(g+\nu) \prod_i \nu_i - E(g) \prod_i \nu_i - (-E(g)\nu \prod_i \nu_i)\|_\infty = \mathcal{O}(\|\nu\|_\infty^2) \prod_i \|\nu_i\|$$

holds, which proves the result for higher derivatives over L^∞. For the operators E_ϕ the above prove does not work since $(L^p, \cdot)$, $p \in [1, \infty)$ is not a Banach algebra. The proof of differentiability of E_ϕ is somewhat technical, and given after Proposition 6.4 in the appendix. It shows that $E_\phi : L^r(\Omega) \to L^p(\Omega)$ is Fréchet differentiable when Ω is finite and $r > p$ and ϕ satisfies the regularity assumptions of that proposition. This is the case if $m+1$ derivatives of ϕ are bounded over $\mathbb{R}$ and $\phi^{(m+1)}$ is Lipschitz continuous. The derivatives of the composition operator $\phi\circ$ are given by $(\nu_1, \dots, \nu_m) \mapsto \phi^{(m)} \cdot \prod_i \nu_i$. At a non-negative function g the derivatives are therefore also given by (12). An application of Proposition 6.2 to $\phi^{(m)}$ which has a bounded derivative by our assumption, yields the Hölder or Lipschitz continuity of the highest derivative. ■

3.2.6. Attenuated Radon transform Clearly for $f \in L^p(\Omega)$, $f \geq 0, \mu \geq 0$ and μ measurable $p \in [1, \infty]$ the estimate $\|A(f,\mu)\|_p \leq \|Rf\|_p$ holds. For arbitrary $f \in L^p(\Omega)$ using a simple variation of the proof demonstrating the continuity of $\tilde{R}$ or R shows that $\|A(f,\mu)\|_p \leq (2\varrho)^{1-1/p} \|f\|_p$. Accordingly, the ATRT may be extended to arbitrary measurable $\mu \geq 0$ and $f \in L^1(\mathbb{R}^2)$. This includes the case f in $L^p(\Omega)$ for $\Omega = B_\varrho(0)$, $p \in [1, \infty]$. For the redefined exponential operator E_ϕ, we have the estimate $\|E_\phi(g)\|_\infty \leq \|\phi\|_\infty \in [1, \infty)$. Thus, for the redefined $A = A_\varrho = \tilde{R}(M(i, E_\phi \circ D_\varrho)$ we infer from (8*b*) by a similar proof for $(f, \mu) \in L^p(\Omega) \times L^0(\mathbb{R}^2) \subset L^1(\mathbb{R}^2) \times L^0(\mathbb{R}^2)$ (L^0: all measurable functions) the estimate

$$\|A(f,\mu)\|_p \leq (2\varrho)^{1-1/p} \|\phi\|_\infty \cdot \|f\|_p \ . \tag{13}$$

Hence the ATRT may be extended to $L^1(\mathbb{R}^2) \times L^0(\mathbb{R}^2)$ and is for fixed μ a continuous linear operator in $L(L^p(\Omega), L^p(S^1 \times [-\varrho, \varrho]))$ (for $p = 1$ even $\Omega = \mathbb{R}^2$ ($\varrho = \infty$) is permissible). It is, however, not obvious that A is continuous and Fréchet differentiable with respect to μ on this large domain. Actually higher regularity of (f, μ) is required in the following section to prove differentiability of A as nonlinear operator of two argument functions.

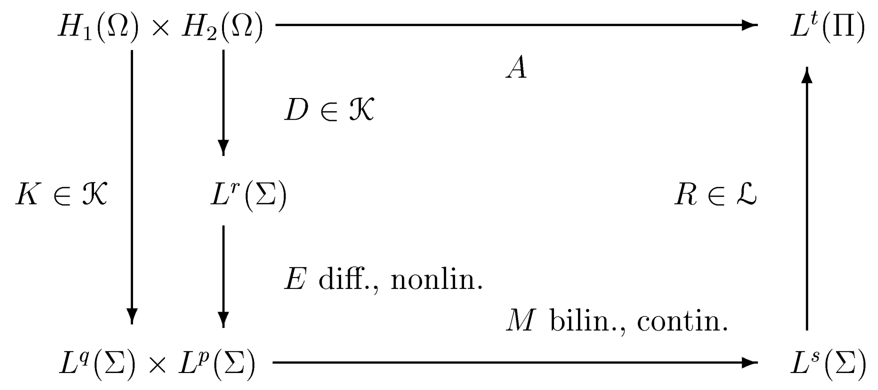


Figure 2. Structure of the ATRT operator

4. Tikhonov-IntraSPECT

4.1. The ATRT in Hilbert space

We aim to apply the regularization theory of [EKN89] to the attenuated Radon transformation A or a redefined version thereof under realistic conditions. To approach this aim we start with two abstract theorems that yields part of the condition T in Definition 2.1 for any operator of a similar structure as the ATRT defined over suitable spaces. In the next theorem we consider a map A that, like the ATRT, has the structure shown in Figure 4.1.

Theorem: 4.1 *Let Ω, Σ, Π be some domains, and $H_1(\Omega), H_2(\Omega)$ be Hilbert spaces continuously embedded by i_1 resp. i_2 into $L^2(\Omega)$. Consider an operator A mapping $X := H_1(\Omega) \times H_2(\Omega)$ into $Y^t := L^t(\Pi)$ for $t \in [1, \infty]$. Let $K : H_1(\Omega) \to L^q(\Sigma)$ and $D : H_2(\Omega) \to L^r(\Sigma)$ be compact linear operators, $E : L^r(\Sigma) \to L^p(\Sigma)$ an n-times Fréchet differentiable operator $M : L^q(\Sigma) \times L^p(\Sigma) \to L^s(\Sigma)$ a continuous, bilinear operator and $R : L^s(\Sigma) \to L^t(\Pi)$ be a continuous linear operator. Assume A admits the decomposition $A := R \circ M \circ (K, E \circ D) : X \to Y^t$ with $A(f, \mu) = R(M(Kf, E(D\mu)))$. Then A is n-times Fréchet differentiable, compact and weakly sequentially closed.*

Proof: Differentiability follows from the chain rule which is valid for Fréchet derivatives. We may decompose $A = \Phi \circ T$ with the compact linear operator $T := (K, D) : X \to Z := L^q(\Sigma) \times L^p(\Sigma)$ and the continuous, nonlinear map $\Phi := R \circ M \circ (Id, E) : Z \to Y^t$. Therefore, A is compact. In order to prove weakly sequentially closedness consider a sequence $x_n \rightharpoonup x$ and $y_n = A(x_n) \rightharpoonup y$. Since $\mathcal{D}(A) = X$ is weakly closed we have $x \in \mathcal{D}(A)$. Assume $y \neq A(x)$. Since T is compact and any weakly convergent sequence is bounded by the theorem of Banach-Steinhaus, there exists a subsequence x'_n such that $z'_n = Tx'_n \to z$ converges in the norm of Z. Assume $z \neq Tx$ then there exists $u \in Z'$ with $\langle z - Tx \,|\, u \rangle \neq 0$ but

$$\langle z - Tx \,|\, u \rangle = \lim \langle Tx'_n - Tx \,|\, u \rangle = \lim \langle T(x'_n - x) \,|\, u \rangle = \lim \langle x'_n - x \,|\, T^* u \rangle = 0 \,.$$

We see thus $z = Tx$. By continuity of Φ $y'_n = \Phi(z'_n) \to \Phi(z) = \Phi(Tx) = A(x)$ in the norm of Y. Thus, y'_n converges weakly to both y and $A(x)$, which by uniqueness of weak limits are therefore equal. ■

Theorem: 4.2 *In the situation of the previous theorem the first Fréchet derivative of A at $x = (f, \mu) \in X$ is given by*

$$A'(f, \mu)(h, \nu) = A(h, \mu) + R(M(Kf, E'(D\mu)D\nu)) \,.$$

The second derivative if existent is given by

$$A''(f,\mu)(h,\nu)^2 = R(M(2Kh, E'(D\mu)D\nu)) + R(M(Kf, E''(D\mu)(D\nu)^2)) \ .$$

If the first Fréchet derivative of E is Hölder continuous of order $\varepsilon \in (0,1]$, then A' is locally Hölder continuous of order ε. The result on the derivative still holds if K, D are merely continuous.

Proof: Noting that M is Fréchet differentiable with derivative $M'(f,g)(h_1,h_2) = M(f,h_2) + M(h_1,g)$ the derivatives may be calculated by the chain rule. We find for the first Fréchet derivative

$$\begin{aligned} A'(f,\mu)(h,\nu) &= R \circ M'(Kf, E \circ D)) \circ (K, E'(D\mu) \circ D)(h,\nu) \\ &= R \circ (M(Kh, E(D\mu)) + M(Kf, E'(D\mu)D\nu)) \\ &= A(h,\mu) + R(M(Kf, E'(D\mu)D\nu)) \ . \end{aligned}$$

The second derivative if existent is calculated from

$$\begin{aligned} A''(f,\mu)((h_1,\nu_1),(h_2,\nu_2)) &= A'(h_1,\mu)(0,\nu_2) \\ &\quad + \ (R(M(Kf, E'(D\mu)D\nu_1)))'(h_2,\nu_2) \\ &= R(M(Kh_1, E'(D\mu)D\nu_2)) \\ &\quad + \ R(M(Kh_2, E'(D\mu)D\nu_1)) \\ &\quad + \ R(M(Kf, E''(D\mu)(D\nu_1, D\nu_2))) \ . \end{aligned}$$

Higher derivatives may be calculated by induction.

Hölder continuity: Let $H_{E'}$ denote the Hölder constant of E'. Then

$$\begin{aligned} \|E(x+u) - E(x)\| &= \|\int_0^1 E'(x+tu)u\,dt\| \\ &\le \|u\| \int_0^1 \|E'(x)\| + \|E'(x+tu) - E'(x)\|\,dt \\ &\le \|u\|(\|E'(x)\| + H_{E'}\|u\|^\varepsilon \int_0^1 t^\varepsilon\,dt) \\ &\le \|E'(x)\|\|u\| + H_{E'}\|u\|^{1+\varepsilon} \ . \end{aligned}$$

Now consider

$$\begin{aligned} &\|A'(f+g,\mu+\eta) - A'(f,\mu)\| \\ &\quad\le \sup_{\|(h,\nu)\|=1} \{ \ \|A'(f+g,\mu+\eta)(h,\nu) - A'(f,\mu+\eta)(h,\nu)\| \\ &\qquad + \ \|A'(f,\mu+\eta)(h,\nu) - A'(f,\mu)(h,\nu)\| \ \} \\ &\quad\le \sup_{\|(h,\nu)\|=1} \{ \ \|R(M(Kg,(E'(D\mu + D\eta))D\nu)\| \\ &\qquad + \ \|R(M(Kh, E(D\mu + D\eta) - E(D\mu)))\| \\ &\qquad + \ \|R(M(Kf,(E'(D(\mu+\eta) - E'(D\mu))D\nu))\| \ \} \\ &\quad\le \|R \circ M\| \sup_{\|(h,\nu)\|=1} \{ \ \|K\|\,\|g\|\,\|E'(D\mu + D\eta)\|\,\|D\|\,\|\nu\| \\ &\qquad + \ \|K\|\,\|h\|(\|E'(D\mu)\| + H_{E'}(\|D\|\,\|\eta\|)^\varepsilon)\|D\|\,\|\eta\| \\ &\qquad + \ \|K\|\,\|f\|H_{E'}(\|D\|\,\|\eta\|)^\varepsilon\|D\|\,\|\nu\| \ \} \\ &\quad\le \|R \circ M\|\,\|K\|\,\|D\| \, (\ (\|E'(D\mu)\| + H_{E'}(\|D\|\,\|\eta\|)^\varepsilon)\|g\| \\ &\qquad + \ (\|E'(D\mu)\| + H_{E'}(\|D\|\,\|\eta\|)^\varepsilon)\|\eta\| + \|f\|H_{E'}(\|D\|\,\|\eta\|)^\varepsilon \ ) \\ &\quad= \|R \circ M\|\,\|K\|\,\|D\| \\ &\qquad \times \ (\|E'(D\mu)\|(\|g\| + \|\eta\|) + (\|g\| + \|\eta\| + \|f\|)H_{E'}(\|D\|\,\|\eta\|)^\varepsilon) \ . \end{aligned}$$

It follows that

$$\|A'(f+g,\mu+\eta) - A'(f,\mu)\| \le C(f,\mu) \cdot \mathcal{O}(\|(g,\eta)\|^\varepsilon), \quad \|(g,\eta)\| \to 0$$

for some $C(f,\mu) < \infty$. Namely for $\|(g,\eta)\| \le \rho \le 1$ and the continuous function $C(f,\mu,\rho) := \|R\circ M\|\,\|K\|\,\|D\|\max(2\rho^{1-\varepsilon}\|E'(D\mu)\|, (\|f\|+2\rho)H_{E'}\|D\|^{\varepsilon})$
the estimate

$$\|A'(f+g,\mu+\eta) - A'(f,\mu)\| \le C(f,\mu,\rho)\|(g,\eta)\|^{\varepsilon}$$

holds. Therefore, A' is locally Hölder continuous. Note that $H_{E'}$ may be replaced by the Hölder constant of E' over the ball $B_{\rho\|D\|}(D\mu)$, i.e. E' need only be locally Hölder continuous.

■

Definition: 4.3 We define over $\Omega = B_{\varrho}(0) \subset \mathbb{R}^2$ the Hilbert space

$$X^{s_1,s_2} := X^{s_1,s_2}_{\zeta_1,\zeta_2}(\Omega) := H_0^{s_1}(\Omega,\zeta_1\,dx) \times H_0^{s_2}(\Omega,\zeta_2\,dx)$$

for some continuous weight functions $0 < c \le \zeta_k(x) \le C < \infty$. The norm on X^{s_1,s_2} is given by

$$\|(f,\mu)\|^2_{X^{s_1,s_2}} := \|\zeta_1(Id-\Delta)^{s_1/2}f\|^2_{L^2} + \|\zeta_2(Id-\Delta)^{s_2/2}\mu\|^2_{L^2}\ .$$

Further, we define the Banach spaces

$$Y^t := L^t(S^1\times[-\varrho,\varrho])$$

for all $t \in [1,\infty]$. □

We note that Y^2 is a Hilbert space. As set $X^{s_1,s_2}_{\zeta_1,\zeta_2}$ agrees with $H_0^{s_1}(\Omega)\times H_0^{s_2}(\Omega) = X^{s_1,s_2}_{1,1}(\Omega)$ without weights. If $\zeta_i \ne 1$ the norm on $X^{s_1,s_2}_{\zeta_1,\zeta_2}$ is, however, different from the norm of $X^{s_1,s_2}_{1,1}$ but equivalent to that norm.

Example: 4.4 *In the situation of Theorems 4.1 and 4.2 let M be the multiplication operation and E a composition operator $\phi\circ$ (cf. Section 6.4) for some suitable differentiable function ϕ satisfying the hypotheses of Proposition 6.4 for Fréchet differentiability of order n. Let further $H_k(\Omega)$ be the weighted Sobolev space $H_0^{s_k}(\Omega,\zeta_k\,dx)$ for some weight functions ζ_k that are bounded above and below ($0 < c \le \zeta_1(x) \le C < \infty$) and $s_k \ge 0$. Then $H_k(\Omega)$ is continuously embedded in $L^2(\Omega)$. Assume that further operators D,K,R are given such that D,E,K,M,R satisfy the conditions of Theorem 4.1 when the composition operator A as in the theorem maps X^{s_1,s_2} to Y^t over some suitable domain Π. Then we have*

$$\begin{aligned}
A(f,\mu) &= R(Kf\cdot\phi(D\mu))\\
A'(f,\mu)(h,\nu) &= R(Kh\cdot\phi(D\mu) + Kf\cdot\phi'(D\mu)\cdot D\nu)\\
A''(f,\mu)(h,\nu)^2 &= R(2Kh\cdot\phi'(D\mu)\cdot D\nu + Kf\cdot\phi''(D\mu)\cdot(D\nu)^2))\ .
\end{aligned}$$

Theorem: 4.5 *The original ATRT operator A has a locally Lipschitz continuous first derivative and is weakly sequentially closed, when considered over X^{s_1,s_2}, $s_1 > \min(0, 1-\frac{2}{t})$, $s_2 > 1$ mapping into Y^t. The redefined operator A_ϱ (with E_ϕ and D_ϱ) has this properties over the larger space X^{s_1,s_2} if* ||

$$s_1 + \frac{4s_2}{3} \ge \frac{7}{3} - \frac{2}{t},\ s_1 \in [0,1),\ s_2 \in [0,\frac{1}{2})\ . \tag{14}$$

The ATRT operator A_ϱ itself is Hölder continuous of order $\varepsilon \in (0,1]$, if only $s_1 \in [0,1), s_2 \in [0,\frac{1}{2})$ satisfy

$$s_1 + \frac{2\varepsilon}{3}s_2 \ge 1 + \frac{2\varepsilon}{3} - \frac{2}{t}\ .$$

If strict inequality holds for some $\varepsilon > 0$ and $s_i > 0$ than A_ϱ is also compact and weakly sequentially closed.

|| If $s_2 \in [\frac{1}{2},1)$ the condition becomes

$$s_1 + \frac{4(s_2-1)}{4-2s_2} \ge 1 - \frac{2}{t},\ s_1 \in [0,1),\ s_2 \in (\frac{1}{2},1)\ .$$

Proof: We are in the situation of the previous example with the generalized Radon transform $\tilde{R}$ in the role of R and $\Sigma = S^1 \times \Omega,\ \Pi = S^1 \times [-\varrho, \varrho]$. The operator D is the restricted fan beam transform D_ϱ and $K := i := i_{S^1} \circ i_{s_1,r} : H^{s_1}(\Omega) \to L^r(\Sigma)$ is a compact inclusion. We note that $H^{s_1}(\Omega)$ is isomorph to $H^{s_1}(\Omega, \zeta\, dx)$ the respective norms are equivalent but not equal for $0 < c \le \zeta \le C < \infty$ and $\zeta \not\equiv 1$. The function ϕ is either $\exp(-\diamond)$ (for the case of the original ATRT) or the ϕ associated with E_ϕ with $\phi|_{\mathbb{R}_+} = \exp(-\diamond)$ (in the A_ϱ case).

For convenience we summarize some results of the previous section in a table

$i:$	$H^{s_1}(\Omega) \to L^q(\Sigma)$	$1/q \ge \frac{1-s_1}{2},\ s_1 \in [0,1)$
$D:$	$H^{s_2}(\Omega) \to L^r(\Sigma)$	$1/r \ge \frac{1-s_2}{3},\ s_2 \in [0, \frac{1}{2})$
		$1/r \ge \frac{1-s_2}{4-2s_2},\ s_2 \in (1/2, 1)$
$E_\phi:$	$L^r(\Sigma) \to L^p(\Sigma)$	n-times F-db, if $r/p > n$,
$E:$	$L^\infty(\Sigma) \to L^q(\Sigma)$	$q \in [1,\infty)$, ∞-F-db
$M:$	$L^q(\Sigma) \times L^p(\Sigma) \to L^t(\Sigma)$	$1/t \ge 1/p + 1/q$
$\tilde{R}:$	$L^t(\Sigma) \to L^t(\Pi)$	$t \in [1, \infty]$

The operators i, D are compact if the respective s_k is larger than required for continuity. If $s_1 > 1$ resp. $s_2 > 1$ than i resp. D map into L^∞. The highest derivative of order $m_* < r/p$ of E_ϕ is Hölder continuous of order $\varepsilon = r/p - m_* \in (0, 1]$. For E to be differentiable $r = \infty$, i.e. $s_2 > 1$ is required.

In the case of the original ATRT A we choose $p, r = \infty,\ q = t$, with $H^{s_1}(\Omega, \zeta_1 dx) \hookrightarrow L^t(\Omega)$ if $s_1 > \min(0, 1 - \frac{2}{t})$. Further, we note that $H^{s_2}(\Omega, \zeta_2 dx) \hookrightarrow L^\infty(\Omega)$ for $s_2 > 1$. In the case of the redefined ATRT A_ϱ we use $r := \frac{3}{1-s_2}$ resp. $r := \frac{4-2s_2}{1-s_2}$ (if $s_2 \in [0, 1/2)$ resp. $s_2 \in (1/2, 1)$), $q := \frac{2}{1-s_1},\ s_1 \in [0, 1)$ and $1/p := 1/q - 1/t$.

The condition in Proposition 6.4 for a ε Hölder continuous n^{th} derivative of E_ϕ requires $r/p \ge n + \varepsilon$ which is satisfied for $\varepsilon, n = 1$ if (14) holds, because e.g. for $s_2 \in [0, 1/2)$ and $\varepsilon_* := \frac{r}{p} = \frac{3}{1-s_2}(\frac{1-s_1}{2} - \frac{1}{t})$ the condition $\varepsilon_* \ge 2$ is equivalent to (14). In order to proof Hölder continuity of order ε of A itself $\varepsilon_* \ge \varepsilon$ is sufficient, which is equivalent to the respective condition in the theorem. The proof of local Hölder continuity of A and $A^{(n)}$ provided E_ϕ resp. $E_\phi^{(n)}$ is Hölder continuous is analog to the one for A' in Theorem 4.2. Concerning A it is somewhat simpler, for $A^{(n)}$ it becomes rather technical. We omit the calculations involved. ■

A note-worthy special case of the above is the following theorem. It states that the redefined ATRT satisfies conditions required by Tikhonov-regularization Theorem 2.2 under more realistic assumptions on the function spaces.

Theorem: 4.6 *The redefined ATRT operator A_ϱ mapping the Hilbert space $X^{s_1,s_2} = H_0^{s_1}(\Omega) \times H_2^{s_1}(\Omega)$ into the Hilbert space $Y^2 = L^2(S^1 \times [-\varrho, \varrho])$ is Hölder continuous for all $s_1 > 0, s_2 \ge 0$. It is compact and weakly sequentially closed for all $s_1 > 0, s_2 > 0$. If $\frac{3}{4}\min(1, s_1) + \min(\frac{1}{2}, s_2) > 1$ the existence of a locally Lipschitz continuous first derivative at every (f, μ) can be guaranteed (e.g. for¶ $s_2 = 1/2,\ s_1 > 2/3$).*

For the original ATRT A these properties only hold under the more restrictive conditions $s_1 \ge 0$ and $s_2 > 1$ (which are in applications unrealistic for μ).

Proposition: 4.7 *The Fréchet derivatives at $(f, \mu) \in X^{s_1,s_2},\ \mu \ge 0$ of the original and the redefined ATRT A, A_ϱ are given if they exist by*

$$\partial^n A(f,\mu)(h_1,\nu_1)\ldots(h_n,\nu_n) = (-1)^n \tilde{A}(\prod_{k=1}^{n} D\nu_k \cdot f - \sum_{j=1}^{n} \prod_{k=1,k\neq j}^{n} D\nu_k \cdot h_j, \mu)\ .$$

Therein $\tilde{A}$ is the generalized ATRT of (5). The analog formula holds for the derivatives of $\tilde{A}$.

¶ A value $s_1 = 2/3$ may not be sufficient, because $s_2 = 1/2$ is a critical value for s_2 .

In particular:

$$\begin{aligned} A'(f,\mu)(h,\nu) &= \tilde{A}(h - D\nu \cdot f, \mu) \\ A''(f,\mu)(h,\nu)^2 &= \tilde{A}(-2D\nu \cdot h + (D\nu)^2 \cdot f, \mu) \\ A'''(f,\mu)(h,\nu)^3 &= \tilde{A}(3(D\nu)^2 \cdot h - (D\nu)^3 \cdot f, \mu) \ . \end{aligned}$$

The derivatives exist for all orders n for $A, A_\varrho : X^{s_1,s_2} \to Y^t, t \in [1,\infty]$, $s_1 \geq \min(0, 1 - \frac{2}{t})$, $s_2 > 1$ and for the redefined ATRT $A_\varrho : X^{s_1,s_2} \to Y^t$ for order $n < u := (\frac{1}{t} - \frac{1}{2} + \frac{s_1}{2})\frac{3}{1-s_2}$, if $s_1 \in [0,1)$ and $s_2 \in [0, \frac{1}{2})$. For $s_2 \in (\frac{1}{2}, 1)$ the factor $\frac{3}{1-s_2}$ has to be replaced by $\frac{4-2s_2}{1-s_2}$. The highest derivative of order $n_* = [u]$ (resp. $n_* = u - 1$, if $u \in \mathbb{N}$) of the ATRT is locally Hölder continuous of order $\varepsilon = u - n_*$.

Proof: With the operators from Theorem 4.5 and Example 4.4 we calculate

$$\begin{aligned} A'(f,\mu)(h,\nu) &= \tilde{R}((h - f \cdot (D\nu)) \exp(-D\mu)) = \tilde{A}(h - fD\nu, \mu) \\ A''(f,\mu)(h,\nu)^2 &= \tilde{R}((f \cdot (D\nu)^2 - 2h \cdot (D\nu)) \exp(-D\mu)) = \tilde{A}(f(D\nu)^2 - 2hD\nu, \mu) \end{aligned}$$

The conditions on the s_i follow as in Theorem 4.5. The results for higher derivatives are proved by a standard inductive argument. ∎

We now proceed to our main technical lemma. It will allow to improve the result of Theorem 4.6.

Lemma: 4.8 *The Fréchet derivative of the redefined ATRT between $X^{s_1,s_2} \to Y^t$ for $s_1 \in [0,1), s_2 \in [0,\frac{1}{2}), t \in [1,\infty]$ exists at f, μ with $f \in L^s$, $s \in [\frac{2}{1-s_1}, \infty]$ if*

$$s_1 > 1 - \frac{2}{t} \ \ \text{and} \ \ s_2 > 1 - 3(\frac{1}{t} - \frac{1}{s}) \ . \tag{15}$$

The derivative is Hölder continuous of order $\varepsilon \in (0,1]$ at f, μ with $f \in L^s$, $s \geq \frac{2}{1-s_1} > t$ if

$$s_1 \geq \frac{5 - 2s_2}{3} - \frac{2}{t} \ \ \text{and} \ \ s_2 \geq 1 - \frac{3}{1+\varepsilon}(\frac{1}{t} - \frac{1}{s}) \ . \tag{16}$$

Proof: Let p, p' be defined via $\frac{1}{p} = \frac{1}{t} - \frac{1}{s} > 0$ and $\frac{1}{p'} = \frac{1}{t} - \frac{1-s_1}{2} > 0$. If $s_2 \geq 1 + \frac{3}{s} - \frac{3}{t}$ and $s_1 > 1 - \frac{2}{t}$ then $A'(f,\mu) \in L(X^{s_1,s_2}, Y^t)$ for $f \in L^s(\Omega)$. Using $\|\tilde{R}\|_{L(L^t,L^t)} = C_{\tilde{R},t,t} < \infty$ this is proved by the following estimates

$$\begin{aligned} \|A'(f,\mu)(h,\nu)\|_t &= \|\tilde{R}(h\phi'(D\mu) + fD\nu\phi(D\mu))\|_{t,S^1\times[-\varrho,\varrho]} \\ &\leq C_{\tilde{R},t,t}\|h\phi'(D\mu) + fD\nu\phi(D\mu)\|_{t,S^1\times\Omega} \\ &\leq C_{\tilde{R},t,t}\|h\phi'(D\mu) + fD\nu\phi(D\mu)\|_{t,S^1\times\Omega} \\ &\leq C_{\tilde{R},t,t}(\|\phi'\|_\infty\|h\|_t + \|\phi\|_\infty\|f\|_s\|D\nu\|_p) \ , \\ \|h\|_t &\leq C_{i,s_1,t}\|h\|_{H^{s_1}} \ \forall \ \frac{s_1}{2} \geq \frac{1}{2} - \frac{1}{t} \\ \|D\nu\|_p &\leq C_{D,s_2,p}\|\nu\|_{H^{s_2}} \ \forall \ \frac{s_2}{3} \geq \frac{1}{3} - \frac{1}{p} \ . \end{aligned}$$

With $\Delta\phi(a,b) := \phi(a+b) - \phi(a)$ we estimate further that

$$\begin{aligned} \clubsuit &:= \|A(f+h,\mu+\nu) - A(f,\mu) - A'(f,\mu)(h,\nu)\|_{t,S^1\times[-\varrho,\varrho]} \\ &\leq \ C_{\tilde{R},t,t}\|f(\Delta\phi(D\mu,\Delta\nu) - \phi'(D\mu)D\nu) + h\Delta\phi(D\mu,D\nu)\|_{t,S^1\times\Omega} \\ &\leq \ C(\|f\|_s\|\Delta\phi(D\mu,D\nu) - \phi'(D\mu)D\nu\|_p + \|h\|_{H^{s_1}}\|\Delta\phi(D\mu,D\nu)\|_{p'}) \ . \end{aligned}$$

The composition $\phi\circ : L^r \to L^p$ is differentiable if $r > p$ by Proposition 6.4, with an remainder estimate (cf. (31)) of order $\varepsilon = \min(2, r/p)$. Since $D\nu \in L^r$ for $\frac{1}{r} = \frac{1-s_2}{3}$, $s_2 \in [0,\frac{1}{2})$ we find $\frac{r}{p} = \frac{3}{1-s_2}(\frac{1}{t} - \frac{1}{s}) > 1$ (by (15)). Further, $\phi\circ : L^r \to L^{p'}$

is Hölder continuous of order $\varepsilon' = \min(1, \frac{r}{p'})$ (cf. (27)) with $\frac{r}{p'} = \frac{3}{1-s_2}(\frac{1}{t} - \frac{1-s_1}{2}) > 0$. It follows that there exists some $R_\phi, H_\phi < \infty$ such that for small $\|\nu\|_{H^{s_2}}$

$$\clubsuit \le C\left(\|f\|_s R_\phi(C_{D,s_2,r}\|\nu\|_{H^{s_2}})^\varepsilon + \|h\|_{H^{s_1}} H_\phi(C_{D,s_2,r}\|\nu\|_{H^{s_2}})^{\varepsilon'}\right)$$
$$= \mathcal{O}(\|(h,\nu)\|_{X^{s_1,s_2}}^{\min(\varepsilon,1+\varepsilon')}) \ .$$

Because $\varepsilon_* = \min(\varepsilon, 1+\varepsilon') = \min(2, \frac{3}{1-s_2}(\frac{1}{t} - \frac{1}{s}), 1 + \frac{3}{1-s_2}(\frac{1}{t} - \frac{1-s_1}{2})) > 1$ if the assumptions (15) are satisfied, we find that A is Fréchet differentiable in the (X^{s_1,s_2}, Y^t) topology at $f \in L^s$.

In order to demonstrate Hölder continuity of $A'(x)$ at $x = (f,\mu)$ with $f \in L^s$ we need to consider

$$\begin{aligned}\spadesuit &:= \|A'(f+g, \mu+\eta)(h,\nu) - A'(f,\mu)(h,\nu)\|_{t,S^1\times[-\varrho,\varrho]} \\ &= \|\tilde{R}((f+g)D\nu\phi'(D\mu + D\nu) + h\phi(D\mu+D\nu) - fD\nu\phi'(D\mu) - h\phi(D\mu))\|_t \\ &\le C_{\tilde{R},t,t}\|h\Delta\phi(D\mu, D\eta) + D\nu(g\phi'(D\mu+D\nu) + f\Delta\phi'(D\mu, D\nu))\|_{t,S^1\times\Omega}\end{aligned}$$

Let p', p'' be defined by $\frac{1}{p'} = \frac{1}{t} - \frac{1-s_1}{2}$ and $\frac{1}{p''} = \frac{1}{t} - \frac{1}{s} - \frac{1-s_2}{3}$ (by (15)). We have $p', p'' > t$ by our assumptions for Hölder continuity and may estimate

$$\begin{aligned}\spadesuit &\le C\|h\|_{H^{s_1}}\|\Delta\phi(D\mu, D\eta)\|_{p'} + C\|g\|_{H^{s_1}}\|\nu\|_{H^{s_2}}\|\phi'\|_\infty \\ &\quad + C\|f\|_s\|\nu\|_{H^{s_2}}\|\Delta\phi'(D\mu, D\eta)\|_{p''} \ .\end{aligned}$$

The estimate for the term containing g holds because $\frac{1}{t} \ge \frac{1-s_1}{2} + \frac{1-s_2}{3}$ is equivalent to the first of the assumptions (16) for Hölder continuity. The maps $\phi\circ, \phi'\circ : L^r \to L^p$ are Hölder continuous of order ε if $r \ge \varepsilon p$. Now since $\|D\eta\|_r \le C_{D,s_2,r}\|\eta\|_{H^{s_2}}$ for $\frac{1}{r} \ge \frac{1-s_2}{3}$ we obtain under assumptions (16) on s_1, s_2, s, t

$$\spadesuit \le C(H_\phi\|h\|_{H^{s_1}}\|\eta\|_{H^{s_2}}^{\varepsilon'} + \|\phi'\|_\infty\|\nu\|_{H^{s_2}}\|g\|_{H^{s_1}} + H_\phi\|\nu\|_{H^{s_2}}\|f\|_s\|\eta\|_{H^{s_2}}^{\varepsilon''})$$

with $\varepsilon' = \min(1, r/p')$ and $\varepsilon'' = \min(1, r/p'')$, where

$$\frac{r}{p'} = \frac{3}{1-s_2}(\frac{1}{t} - \frac{1-s_1}{2}) \text{ and } \frac{r}{p''} = \frac{3}{1-s_2}(\frac{1}{t} - \frac{1}{s}) - 1 \ .$$

This demonstrates $\spadesuit = \|h,\nu\|_{X^{s_1,s_2}}(1+\|f\|_s)\mathcal{O}(\|g,\eta\|_{X^{s_1,s_2}}^{\min(\varepsilon',\varepsilon'')})$ for $\|(g,\eta)\|_{X^{s_1,s_2}} \to 0$, i.e. local Hölder continuity of the derivative of order $\varepsilon_* = \min(\varepsilon', \varepsilon'') = \min(1, \frac{3}{1-s_2}(\frac{s_1}{2} + \frac{1}{t} - \frac{1}{2}), \frac{3}{1-s_2}(\frac{1}{t} - \frac{1}{s}) - 1)$. Now it follows from the first inequality in (16) that $\frac{3}{1-s_2}(\frac{s_1}{2} + \frac{1}{t} - \frac{1}{2}) \ge 1$. The second part of (16) yields the desired estimate. ■

Let us consider two special cases of the previous lemma.

Example: 4.9 *For the redefined ATRT A_ϱ mapping X^{s_1,s_2} into the Hilbert space $Y^2 = L^2(S^1 \times [-\varrho, \varrho])$ the Fréchet derivative exists everywhere if $\frac{3}{2}s_1 + s_2 > 1$, (e.g. $s_1, s_2 > \frac{2}{5}$ or $s_1 = \frac{1}{2}$, $s_2 > \frac{1}{4}$ or $s_2 = \frac{1}{2}$, $s_1 > \frac{1}{3}$). Further, it exists at all elements with bounded $f \in L^\infty$ already for any $s_1 > 0$, $s_2 \ge 0$. If $\frac{3}{2}s_1 + s_2 \ge 1$ the derivative is Hölder continuous at such elements of order $\varepsilon = \min(1, \frac{3}{2(1-s_2)} - 1) \ge \frac{1}{2}$. Lipschitz continuity is reached for $s_2 \ge \frac{1}{4}$.*

In addition the redefined ATRT is everywhere Fréchet differentiable as operator $A : L^2(\Omega) \times L^2(\Omega) \to L^t(S^1 \times [-\varrho, \varrho])$ if $t < \frac{6}{5}$. For $t \in [1, \frac{6}{5})$ the derivative is Lipschitz continuous at all elements with $f \in L^6$ (e.g. bounded f).

Proof: For the specialization to $t = 2$ we note that every $f \in H^{s_1}$ is in L^s, $s = \frac{2}{1-s_2}$. Inserting this in (15) yields the condition $\frac{3}{2}s_1 + s_2 > 1$. The case for $f \in L^\infty$ is obvious. Condition (15) is satisfied for $s_1 = s_2 = 0$ for any $t \in [1, \frac{6}{5})$. For such t (16) holds with $\varepsilon_* = \min(1, \varepsilon(s))$, $\varepsilon(s) = 3(\frac{1}{t} - \frac{1}{s}) - 1) \ge \frac{3}{2} - \frac{3}{s}$, i.e. at $f \in L^6$ the derivative is Lipschitz. For $t = 1$ one obtains $\varepsilon_* = 1$ already for $f \in L^3$. We may finally conclude that the derivative of $A_\varrho : L^2(\Omega) \times L^2(\Omega) \to L^t(\Sigma)$

for $t < \frac{6}{5}$ is Hölder continuous of order $\varepsilon = \frac{3}{t} - \frac{5}{2} > 0$ at least at any element $(f,\mu) \in L^2(\Omega) \times L^2(\Omega)$. ■

Summarizing the last results, we may now state the main theoretical result. We define in view of Lemma 4.8 the following domain of functions

$$D_{s_1,s_2,p,C} := \{(f,\mu) \in X^{s_1,s_2} \| \|f\|_{L^p} \le C\} \ . \tag{17}$$

Theorem: 4.10 *The redefined ATRT satisfies conditions T1-T6 of Definition 2.1 required for optimal order Tikhonov regularization by Theorem 2.2 when considered as operator on* $\mathcal{D}(A) := D_{s_1,s_2,p,C} \sup H_0^{s_1}(\Omega) \times H_0^{s_2}(\Omega)$ *mapping into* $Y^t = L^t(S^1 \times [-\varrho,\varrho])$ *if*

$$s_1 \ge \frac{5-2s_2}{3} - \frac{2}{t} \quad \text{and} \quad s_2 \ge 1 - \frac{3}{2}(\frac{1}{t} - \frac{1}{p}) \ .$$

In particular for the case with the Hilbert space Y^2*, bounded activity* f $(p = \infty, C < \infty)$ *the conditions* $s_2 \in [\frac{1}{4}, \frac{1}{2})$ *and* $s_1 \ge \frac{2}{3}(1 - s_2)$ *(e.g.* $(s_1,s_2) = (\frac{1}{2},\frac{1}{4})$, $(\frac{2}{5},\frac{2}{5})$ *or* $(\frac{1}{3},\frac{1}{2})$*) are sufficient for the existence of a Lipschitz continuous derivative at all* $(f,\mu) \in \mathcal{D}(A)$*. If* $s_2 \in [0, 1/4)$ *the conditions are nearly met with* A' *being only* ε*-Hölder continuous of order* ε *over the domain* $D_{\frac{2}{3}(1-s_2),s_2,\infty,C}$ *with* $\varepsilon = 3/(2-2s_2) - 1 \ge 1/2$ *at least.*

Proof: The results were already proved in Theorem 4.5 and Lemma 4.8. ■

We note again that for the original definition of the ATRT $A : L^2 \times H^s \to Y^2$, $s > 1$ is the setting required to prove differentiability.

Corollary: 4.11 *The operator* $F : H^s := H^s(\Omega, \zeta dx) \to Y^t = L^t(S^1 \times [-\varrho,\varrho])$, $0 < c \le \zeta(x) \le C < \infty$ *defined by* $F(\mu) = A_\varrho(f,\mu)$ *for some* $f \in L^\infty(\Omega)$*, that governs the problem of approximating* μ *given an activity estimate* f *and SPECT data* y^δ*, is Fréchet differentiable in the* $H^s \to L^t$ *topology when* $s \ge 0$*. For* $s > 0$ *the operator is also compact and weakly-sequentially closed. The derivative is given at* $\mu \ge 0$ *by*

$$\begin{aligned} F'(\mu)\nu(\omega,s) &= -\tilde{R}(fD\nu \cdot \mathrm{e}^{-D\mu}) \\ &= -\int_{-\varrho}^{\varrho} \nu(s\omega^\perp + t\omega) D(f\mathrm{e}^{-D_\omega\mu})(-\omega, s\omega^\perp + t\omega)\, dt \ . \end{aligned}$$

It is Hölder continuous at least of order $\varepsilon = \min(1, 3/(2-2s) - 1) \ge 1/2$*. Lipschitz continuity of* F' *can be guaranteed over* $H^{1/4}$*. If higher derivatives exist they are given by*

$$\partial^n F(\mu)\nu = \partial^n A_\varrho(f,\mu)(0,\nu) = (-1)^n \tilde{A}_\varrho(f(D\nu)^n, \mu) \ .$$

For $s > 1$ *derivatives of any order exist. This remains true over* H^s, $s > 1$ *when we replace* A_ϱ *by the original ATRT* A *defined with the proper exponential function instead of the substitute* ϕ.

Proof: We may proof this reusing the proof of Lemma 4.8 with $h, g = 0$. The conditions on s_1 may be dropped. The derivative of $F = A \circ (f, \diamond)$ is $F'(\mu)\nu = A'(f,\mu)(0,\nu)$. From this the remaining results follow easily with Theorem 4.5. Only the alternative representation of the partial derivative of $A_\varrho : X^{s_1,s_2} \to Y$ with respect to μ needs to be proved We show that for sufficiently regular ν and f and $\mu \ge 0$ it may be rewritten as

$$\partial_\mu A(f,\mu)\nu(\omega,s) = -R_\omega(\nu \cdot D_{-\omega}(fE(D_\omega\mu))(s) \ .$$

We have with $\ell(t) = s\omega^\perp + t\omega$

$$\begin{aligned} -\partial_\mu A(f,\mu)\nu(\omega,s) &= -A(f,\mu)'(0,\nu)(\omega,s) = -\tilde{A}(0 - f \cdot D\nu, \mu) \\ &= \int_{\mathbb{R}} f \circ \ell(t) \int_R \theta_\varrho(\tau - t)\nu \circ \ell(\tau)\, d\tau E(D\mu(\omega, \ell(t)))\, dt \end{aligned}$$

$$= \int_{\mathbb{R}} \nu \circ \ell(\tau) \int_{\mathbb{R}} \theta_{\varrho}(\tau - t) f \circ \ell(t) E(D\mu(\omega, \ell(t)))\, dt\, d\tau$$
$$= \int_{-\varrho}^{\varrho} \nu \circ \ell(\tau) \int_{-\varrho}^{\tau} f \circ \ell(t) E(D\mu(\omega, \ell(t)))\, dt\, d\tau$$
$$= R_{\omega}(\nu \cdot D_{-\omega}(fE(D_{\omega}\mu))(s)\ .$$

■

5. Numerical results

The Tikhonov functional minimization has been implemented in form of an adapted Gauss-Newton minimization procedure.

The standard Gauss-Newton method starts from the observation that for linear least-square problems a number of efficient algorithms are known to find a minimizer. The operator A in the Tikhonov functional is replaced by its first order Taylor series

$$A(x) \sim A(x^k) + A'(x^k)(x - x^k)$$

and the resulting quadratic form is minimized

$$T_{\alpha,x_*}(x) \sim \tilde{T}^k(x) := \|A(x^k) + A'(x^k)(x - x^k) - y^{\delta}\|_Y^2 + \alpha\|x - x_*\|_X^2$$

With the vanishing gradient condition

$$\nabla\tilde{T}^k(x) = 2[A'(x^k)^*(A(x^k) + A'(x^k)(x - x^k) - y^{\delta}) + \alpha(x - x_*)] = 0$$

one obtains the update formula

$$x^{k+1} := x^k - (A'(x^k)^* A'(x^k) + \alpha \cdot I)^{-1} \left(A'(x^k)^*(A(x^k) - y^{\delta}) + \alpha(x^k - x_*)\right)\ .$$

This may be rewritten with the gradient

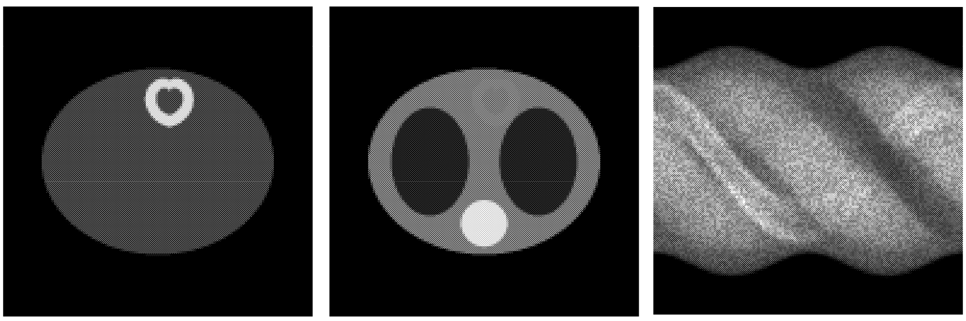
$$g^k = \nabla\tilde{T}^k(x^k) = 2[A'(x^k)^*(A(x^k) - y^{\delta}) + \alpha(x^k - x_*)]$$

as the linear system

$$Q^k d^k = -g^k, \qquad Q^k = \nabla^2\tilde{T}^k = 2(A'(x^k)^* A'(x^k) + \alpha \cdot I)\ .$$

for the optimal Gauss-Newton update $d^k = x^{k+1} - x^k$, with a symmetric positive definite operator $\tilde{Q}^k$. The method equals Newton's method for $\nabla T = 0$ with A'' ignored.

In our implementation the linear problems are solved iteratively by a conjugate gradient type method, that independently searches for updates Δf and $\Delta\mu$.



Activity f Attenuation μ Sinogram $A(f,\mu)$+ noise

Figure 3. Phantom data

In Figure 3 a phantom data study of a myocard diagnosis is shown. We use an activity f constant in the whole body except the heart muscle where it accumulates.

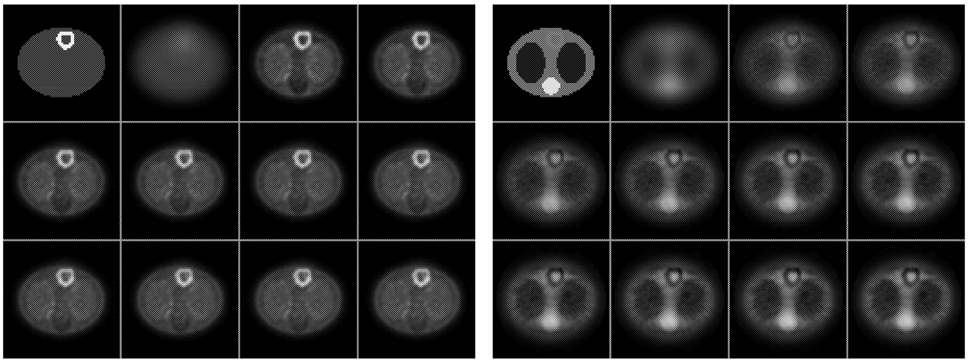


Figure 4. Simultaneous activity and attenuation reconstruction

The nonuniform attenuation map μ is reduced to the heavily attenuating backbone ($0.28cm^{-1}$), muscle tissue ($0.16cm^{-1}$) and the hardly attenuating lungs ($0.04cm^{-1}$).

For this setting SPECT data where simulated through calculating $A(f,\mu)$ and adding 20% multiplicative uniform random noise.

Figure 4 shows the first iterates for f, μ of the reconstruction obtained with our algorithm. $^{+}$

Details of the implementation may be found in [Dic97, Chp. 5] and will be published together with more numerical results in a forthcoming paper.

6. Appendix

The appendix collects some technical proofs.

6.1. Smoothing properties of the restricted fan-beam transform

Proof:(of Proposition 3.7) Using polar-coordinates $r\vartheta$ and the parameterization $\omega = (\cos\varphi, \sin\varphi)$ and further $\tilde{\mu}(r,\vartheta) := \hat{\mu}(r\cos\vartheta, r\sin\vartheta)$ we get from equation (10)

$$\begin{aligned}
\|D\mu\|^2_{H^{\alpha+1/2}(S^1\times\mathbb{R}^2)} &= \oint_{S^1} \|D\mu(\omega,\diamond)\|^2_{H^{\alpha+1/2}(\mathbb{R}^2)}\, d\omega \\
&= \oint_{S^1} \|(1+r^2)^{\alpha/2+1/4}\mathcal{F}_2 D\mu(\omega,\diamond)\|_2^2\, d\omega \\
&= 4\varrho^2 \int_0^{2\pi}\int_0^{\infty}\int_0^{2\pi} (1+r^2)^{\alpha+1/2}|\tilde{\mu}|^2(r,\vartheta)\mathrm{sinc}^2(\varrho r\cos(\vartheta-\varphi))\, d\vartheta\, r dr\, d\varphi \\
&= 4\varrho^2 \int_0^{\infty}\int_0^{2\pi} (1+r^2)^{\alpha+1/2}|\tilde{\mu}|^2(r,\vartheta)\int_0^{2\pi}\mathrm{sinc}^2(\varrho r\cos(\vartheta-\varphi))\, d\varphi\, d\vartheta\, r dr\ .
\end{aligned}$$

We consider (using $\gamma = r\cos\varphi$ and $0 < \varepsilon < r$)

$$\begin{aligned}
\int_0^{2\pi} \mathrm{sinc}^2(r\cos(\vartheta-\varphi))\, d\varphi &= \int_0^{2\pi} \mathrm{sinc}^2(r\cos(\varphi))\, d\varphi \\
\text{(by symmetry)} \qquad &= 4\int_0^{\pi/2} \mathrm{sinc}^2(r\cos(\varphi))\, d\varphi \quad = \\
4\int_0^{r} \mathrm{sinc}^2(\gamma)\frac{d\gamma}{\sqrt{r^2-\gamma^2}} \qquad &= 4\{\int_0^{\varepsilon} + \int_{\varepsilon}^{r}\}\frac{\mathrm{sinc}^2(\gamma)\, d\gamma}{\sqrt{r^2-\gamma^2}}
\end{aligned}$$

$^{+}$ The electronically published version contains some video clips illustrating the progress of the iteration, see http://rubens.math.uni-potsdam.de/ volker/diss/SPECT.htm [redacted](To be changed if paper is accepted)

$$\leq 4\int_0^{\varepsilon}\frac{d\gamma}{\sqrt{r^2-\gamma^2}}+4\int_{\varepsilon}^{r}\frac{d\gamma}{\gamma^2\sqrt{r^2-\gamma^2}}$$
$$=4\mathrm{arcsin}\frac{\varepsilon}{r}+4\frac{\sqrt{r^2-\varepsilon^2}}{r^2\varepsilon};$$

(cf. e.g. [BS84, p.44no164p.45no169] for the integrals). With $\arcsin(x)\leq\frac{\pi}{2}x$ and choosing $\varepsilon=1$ when $\varrho r>1$ and using sinc ≤ 1 elsewhere, we arrive at

$$\int_0^{2\pi}\mathrm{sinc}^2(\varrho r\cos(\vartheta-\varphi))\,d\varphi\leq\begin{cases}(2\pi+4)(\varrho r)^{-1} & \text{if } \varrho r>1\\ 2\pi & \text{if } \varrho r\leq 1\end{cases}$$

and thus for $\|D_\varrho\mu\|^2_{\alpha+1/2}:=\|D_\varrho\mu\|^2_{H^{\alpha+1/2}(S^1\times\mathbb{R}^2)}$

$$\begin{aligned}\|D_\varrho\mu\|^2_{\alpha+1/2}\quad&\leq 8\pi\sqrt{1+\frac{1}{\varrho^2}}\varrho^2\int_0^{1/\varrho}\int_0^{2\pi}(1+r^2)^\alpha|\tilde{\mu}|^2(r,\vartheta))\,d\vartheta\,r\,dr\\&+(8\pi+16)\sqrt{1+\frac{1}{\varrho^2}}\varrho\int_{1/\varrho}^{\infty}\int_0^{2\pi}(1+r^2)^\alpha|\tilde{\mu}|^2(r,\vartheta)\,d\vartheta\,r\,dr\\&\leq C(\varrho)\cdot\|\mu\|^2_{H^\alpha(\Omega)}\end{aligned}$$

For the second half of the theorem we use Fubinis theorem. It proves that since the double integral with measures $d\omega$ and $(1+|\zeta|^2)^{\alpha+1/2}\,d\zeta$ of the positive function $(\hat{D}_\varrho(\mu(\omega,\zeta))^2$ exists, the integral with $(1+|\zeta|^2)^{\alpha+1/2}\,d\zeta$ exists for almost all $\omega\in S^1$. Accordingly, $D_{\varrho,\omega}\mu$ lies in $H^{\alpha+1/2}$ for almost all ω. In order to demonstrate unboundedness we consider the image of a characteristic function of a square under $D_{\varrho,\omega}$. Let w.l.o.g. $\varrho>\sqrt{2}$ and $\sigma=\frac{\pi}{2}\chi_{[-1,1]^2}$. It is well known that $\sigma\in H^s(\Omega)\setminus H^{1/2}(\Omega)$ for all $s<\frac{1}{2}$, and $\hat{\sigma}=\mathrm{sinc}\otimes\mathrm{sinc}$.

We calculate for $D_{\varrho,\omega}\sigma$ with $\omega=(\cos\varphi,\sin\varphi)$

$$\|D_{\omega,\varrho}\sigma\|^2_{H^\alpha(\mathbb{R}^2)}=4\varrho^2\int_0^\infty\int_0^{2\pi}(1+r^2)^\alpha|\tilde{\sigma}|^2(r,\vartheta)\mathrm{sinc}^2(\varrho r\cos(\vartheta-\varphi))\,d\vartheta\,r\,dr\ .$$

The critical cases are $\varphi=0,\pi/2,\pi,3/4\pi$. For them we find* e.g. for $\varphi=0$

$$\begin{aligned}\frac{1}{\varrho^2}\|D_{(1,0),\varrho}\sigma\|^2_{H^\alpha}&\geq 4\int_{\pi/2}^\infty r^{2\alpha+1}\int_0^r\frac{\mathrm{sinc}^2(\gamma)\mathrm{sinc}^2(\sqrt{r^2-\gamma^2})\mathrm{sinc}^2(\varrho\gamma)\,d\gamma}{\sqrt{r^2-\gamma^2}}\,dr\\&\geq 4(\frac{2}{\pi})^2\int_{\pi/2}^\infty r^{2\alpha+1}\int_0^{\pi/2}\mathrm{sinc}^2(\sqrt{r^2-\gamma^2})\mathrm{sinc}^2(\varrho\gamma)\frac{d\gamma}{\sqrt{r^2}}\,dr\\&\geq 4(\frac{2}{\pi})^2(\frac{2\varrho}{\pi})^2\int_{\pi/2}^\infty r^{2\alpha+1-3}\int_0^{\pi/2\varrho}\sin^2(r\sqrt{1-(\frac{\gamma}{r})^2})\,d\gamma\,dr\\&\geq C\int_{\pi/2}^\infty r^{2(\alpha-1/2)-1}\int_0^{\pi/2\varrho}\sin^2(r\sqrt{1-(\frac{\gamma}{r})^2})\,d\gamma\,dr\ .\end{aligned}$$

Since the last integral diverges for all $\alpha\geq 1/2$ we see that $D_\varrho\sigma((1,0),\diamond)$ is no smoother than σ in terms of Sobolev regularity. If $\varphi\neq 0,\pi/2,\pi,3/4\pi$ it is possible to show that $\|D_{\varrho,\omega}\sigma\|_{H^\alpha}$ is finite for all $\alpha<1$ but without upper bound when φ approaches the critical values. Because these estimates are rather technical and give no deeper insight we omit them. ■

6.2. L^p-spaces and estimates

In the following section let μ be a positive measure on a domain Ω. We denote as usual by $L^1:=L^1(\Omega,\,d\mu)$ the set of all equivalence classes of measurable functions f with

$$\int_\Omega|f|\,d\mu=:\int|f|=:\|f\|_1<\infty\ .$$

* With $\sin\varphi\geq\frac{2}{\pi}\varphi,\ \varphi\in[0,\pi/2]$ and $\gamma=r\cos(\vartheta),\ \mathrm{sinc}\sqrt{r^2-\gamma^2}\geq\frac{\sin(r\sqrt{1-(\gamma/r)^2})}{r}$

Further, we denote by $L^p := L^p(\Omega, d\mu), 0 < p < \infty$ the set of all equivalence classes of measurable functions with

$$\|f\|_p := \|\,|f|^p\|_1^{1/p} < \infty$$

and by $L^\infty := L^\infty(\Omega, d\mu)$ the set of functions with

$$\|f\|_\infty := \inf\{\sup_{\omega\in\Omega\setminus N} |f(\omega)|; N \subset \Omega, \mu(N) = 0\} < \infty \ .$$

The space L^p is a Banach space only for $p \in [1,\infty]$, because the Minkowsky inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ holds only for $p \geq 1$. We note the simple rule

$$\|f^s\|_p = \|f\|_{sp}^s \ \forall\ p, s > 0 \ , \tag{18}$$

which will be used later. For any positive measure μ we have the well known estimate

$$|\int fg\, d\mu| \leq \|fg\|_{L^1(\mu)} \leq \|f\|_{L^p(\mu)} \cdot \|g\|_{L^q(\mu)} \quad \forall\ 1/p + 1/q = 1 \ .$$

An application of this to $(|f|\cdot|g|)^r, r > 0$ gives the generalized version

$$\|fg\|_r \leq \|f\|_p \|g\|_q \quad \forall p, q, r > 0,\ 1/p + 1/q = 1/r \ . \tag{19}$$

For a bounded measure μ, e.g. Lebesgue-measure on a bounded domain, we have $\mathrm{vol}(\Omega) := \|1\|_1 < \infty$ and get

$$\|f\|_r \leq \mathrm{vol}(\Omega)^{1/r-1/p}\|f\|_p \quad \forall\ p \geq r \ . \tag{20}$$

Thus, $L^p \subset L^r$ is a continuous embedding $\forall\ p \geq r$ with norm $\mathrm{vol}(\Omega)^{1/r-1/p}$ for finite measure μ. Another useful estimate is derived form the Hölder estimate (19).

Lemma: 6.1 *Let $f, g \in L^r(\mu)$, $r > 0$, $\lambda \in [0,1]$ for a positive measure μ and $f, g \geq 0$ μ-almost everywhere. Then*

$$\|f^\lambda \cdot g^{1-\lambda}\|_r \leq \|f\|_r^\lambda \cdot \|g\|_r^{1-\lambda} \ . \tag{21}$$

Proof: The estimate obviously holds for $\lambda = 0, 1$ and for $\lambda \in (0,1)$ we have the estimate

$$\|f^\lambda \cdot g^{1-\lambda}\|_r \leq \|f^\lambda\|_{r/\lambda}\|g^{1-\lambda}\|_{r/(1-\lambda)} = \|f\|_r^\lambda \cdot \|g\|_r^{1-\lambda} \ .$$

■

We get further for any positive measure μ and $f \in L^p \cap L^\infty, r \geq p > 0$ using $(x/\|x\|_\infty)^r \leq (x/\|x\|_\infty)^p$

$$\|f\|_r \leq \|f\|_\infty^{1-p/r}\|f\|_p^{p/r} \leq \|f\|_{L^p\cap L^\infty} \ \forall\ p \leq r \tag{22}$$

with $\|\diamond\|_{L^p\cap L^\infty} := \max(\|\diamond\|_p, \|\diamond\|_\infty)$. Thus, the embedding $L^p \cap L^\infty \subset L^r$ is continuous $\forall\ p \leq r$.

6.3. An interpolation result

Proof:(of Proposition 3.8) From (19) and with (18) the following estimates are derived for $\lambda \in (0,1)$; $q \geq p$ and $f \geq 0$

$$\begin{aligned}
\|f\|_{X\times Y,p} &= \|\,\|f(x,\diamond)\|_{Y,p}\|_{X,p} \\
&\leq \|\,\|f^\lambda(x,\diamond)\|_{Y,q} \cdot \|f^{1-\lambda}(x,\diamond)\|_{Y,(1/p-1/q)^{-1}}\|_{X,p} \\
&= \|\,\|f^{\lambda q}(x,\diamond)\|_{Y,1}^{1/q} \cdot \|f^{(1-\lambda)\frac{qp}{q-p}}(x,\diamond)\|_{Y,1}^{\frac{q-p}{qp}}\|_{X,p} \\
&\leq \|\,\|f^{\lambda q}(x,\diamond)\|_{Y,1}^{1/q}\|_{X,p} \cdot \|\|\|f^{(1-\lambda)\frac{qp}{q-p}}(x,\diamond)\|_{Y,1}^{\frac{q-p}{qp}}\|_{X,\infty} \\
&= \|\,\|f(x,\diamond)\|_{Y,\lambda q}^{\lambda p/p_0}\|_{X,p_0}^{p_0/p} \cdot \|\|\|f(x,\diamond)\|_{Y,(1-\lambda)\frac{qp}{q-p}}^{(1-\lambda)}\|_{X,\infty} \ .
\end{aligned}$$

For $p \geq p_o$ and $p \leq p_2$ we choose $\lambda = p_0/p < 1$ and $q = p_2/\lambda = pp_2/p_0 \geq p_2 \geq p$. Thus, we arrive with the given choice of p in the theorem at

$$\begin{aligned} \|f\|_{X\times Y,p} &\leq \| \, \|f(x,\diamond)\|_{Y,p_2}\|^{\lambda}_{X,p_0} \cdot \| \, \|f(x,\diamond)\|_{Y,\frac{(1-p_0/p)p_2 p}{p_2-p_0}}\|^{(1-\lambda)}_{X,\infty} \\ &= \| \, \|f(x,\diamond)\|_{Y,p_2}\|^{\lambda}_{X,p_0} \cdot \| \, \|f(x,\diamond)\|_{Y,\frac{p-p_0}{1-p_0/p_2}}\|^{(1-\lambda)}_{X,\infty} \\ &= \| \, \|f(x,\diamond)\|_{Y,p_2}\|^{p_0/p}_{X,p_0} \cdot \| \, \|f(x,\diamond)\|_{Y,p_1}\|^{(1-p_0/p)}_{X,\infty} \leq C_1^{p_0/p} \cdot C_2^{1-p_0/p} . \end{aligned}$$

■

6.4. Some remarks on composition operators

In this section we study continuity and differentiability properties of composition operators

$$(\phi\circ) : f \in L^r(\Omega) \mapsto \phi \circ f \in L^p(\Omega) \tag{23}$$

and further

$$(\phi \circ \cdot) : f \in L^r(\Omega) \mapsto (t \mapsto \phi \circ f \cdot t) \in L(L^q(\Omega), L^p(\Omega)) \tag{24}$$

for some suitable $\phi : \mathbb{R} \to \mathbb{R}$ or $\phi : \mathbb{C} \to \mathbb{C}$.

The results easily extend to higher derivatives of $(\phi\circ)$ where we have to study multi-linear maps of the form

$$(\phi\circ\cdot\Pi) : f \in L^r(\Omega) \mapsto ((t_1, \ldots, t_m) \mapsto \phi\circ f \cdot \prod_{j=1}^{m} t_j) \in L^m(L^{q_1} \times \ldots L^{q_m}, L^p(\Omega)) \,. \tag{25}$$

Proposition: 6.2 *Let $\phi : \mathbb{R} \to \mathbb{R}$ (resp. $\phi : \mathbb{C} \to \mathbb{C}$) satisfy the growth condition $|\phi(t)| \leq C(1+|t|)^s$, $\forall\, t \in \mathbb{R}$ (resp. $\mathbb{C}$) and a uniform Hölder estimate:*

$$|\phi(t+h) - \phi(t)| \leq C \begin{cases} |h|^{\varepsilon}(1+|t|+|h|)^{s-\varepsilon} & \forall\, t,h \in \mathbb{R} \ (\textit{resp. } \mathbb{C}) \ \textit{if } s \geq \varepsilon \\ |h|^{\varepsilon}(1+|h|)^{s-\varepsilon} & \forall\, t,h \in \mathbb{R} \ (\textit{resp. } \mathbb{C}) \ \textit{if } s \leq \varepsilon \end{cases} \tag{26}$$

for some $\varepsilon \in (0,1], s \geq 0$. Let μ be Lebesgue measure on a finite domain Ω (or some other suitable finite measure that admits all the estimates below, e.g. a weighted Lebesgue measure $d\mu = \zeta(\omega)\, d\omega$ with $\int 1\, d\mu < \infty$).
I. If $s \leq r/p$, $r,p \in [1,\infty)$ and $\varepsilon_* := \min(\varepsilon, r/p) > 0$, then the operator given by (23) is a (for $s < \varepsilon$ uniformly) Hölder continuous operator $\phi\circ : L^r(\mu) \to L^p(\mu)$ of order ε_*. For $(\phi\circ)$ the following estimate holds for all $f,h \in L^r$

$$\|\phi(f+h) - \phi(f)\|_p \leq C' \begin{cases} \|h\|_r^{\varepsilon_*}(1+\|h\|_r)^{s-\varepsilon_*} + \|h\|_r^{\varepsilon}\|f\|_r^{s-\varepsilon} & s \geq \varepsilon \\ \|h\|_r^{\varepsilon_*}(1+\|h\|_r)^{s-\varepsilon_*} & s \leq \varepsilon \end{cases} \tag{27}$$

II. If $0 \leq s \leq r(1/p - 1/q)$, $1 \leq p, r < \infty$, $p \leq q \leq \infty$ and $\varepsilon_* := \min(\varepsilon, r(1/p - 1/q)) > 0$, then the operator $\phi\circ\cdot$ given by (24) is a uniformly H-continuous operator $\phi\circ\cdot : L^r \to L(L^q, L^p)$ of order ε_*. The estimate (27) holds with $\|\phi(f+h) - \phi(f)\|_p$ replaced by $\||\phi(f+h)\cdot - \phi(f)\cdot\|_{L(L^q,L^p)}$, for all $f,h \in L^r$.

The results in I, II also hold for $r = \infty, p \in [1,\infty]$ and any s, with $\varepsilon_* = \varepsilon$ (if $1/p \geq 1/q$) in II). In case $p = r(= q) = \infty$ the measure μ need not be finite. The result for the cases with $r = \infty$ also hold if ϕ merely satisfies $|\phi(t+h) - \phi(t)| \leq C|h|^{\varepsilon} e^{(|h|+|t|)^s}$ when (27) is replaced by $\|\phi(f+h) - \phi(f)\|_\infty \leq C\|h\|^{\varepsilon} e^{(\|f\|+\|h\|)^s}$.

Proof:

Part I: The case with $r = p = \infty$ follows immediately from the Hölder estimate

$$\begin{aligned} \|\phi(f+h) - \phi(f)\|_\infty &\leq \sup_{\substack{|t|\leq\|f\|_\infty \\ |\vartheta|\leq\|h\|_\infty}} |\phi(t+\vartheta) - \phi(t)| \\ &\leq C \begin{cases} \|h\|^{\varepsilon}(1+\|f\|+\|h\|)^{s-\varepsilon} & s \geq \varepsilon \\ \|h\|^{\varepsilon}(1+\|h\|)^{s-\varepsilon} & s \leq \varepsilon \,. \end{cases} \end{aligned}$$

The cases $r=\infty,\ p\in[1,\infty)$ follow from $r=p=\infty$ and the continuity of the embedding $L^\infty\subset L^p$ for finite μ. The proof for $r=\infty$ is still valid (with the exponential estimate instead of (27)), if ϕ only satisfies the exponential estimate .

Now let $r,p\in[1,\infty)$ and $f,h\in L^r(\mu)$. Without loss of generality let $h\geq 0$. Due to the growth condition for $|h|\to\infty$ there exists $C_1<\infty$ such that $|\phi(f)(\omega)|\leq C_1(1+|f(\omega)|^{r/p})$. Thus, $\|\phi(f)\|_p\leq C_2(\|1\|_p+\||f(\omega)|^{r/p}\|_p)=C_2(\|1\|_p+\|f\|_r^{r/p})<\infty$ for some $C_2<\infty$ by Minkowsky's inequality. Hence, $\phi(f)\in L^p(\mu)$.

Let us define $\Lambda:=\{\omega;|h(\omega)|>1\}$ the set of large values of h and the characteristic function $\chi=\chi_\Lambda$. We further note the simple estimate

$$(\sum_1^n a_j)^x\leq\begin{cases} n^x(\sum_1^n a_j^x) & a_j\geq 0, x\geq 0\\ a_j^{-|x|} & \forall\ j,\ a_j\geq 0, x\leq 0\ .\end{cases}$$

With this it follows from (26) that there exists $C'<\infty$ with

$$|\phi(t+h)-\phi(t)|\leq C'\begin{cases} |h|^\varepsilon(1+|t|^{s-\varepsilon}) & |h|\leq 1,\ s\geq\varepsilon\\ |h|^\varepsilon & |h|\leq 1,\ s\leq\varepsilon\\ |h|^s & |h|\geq 1,\ s\leq\varepsilon\\ |h|^s+|h|^\varepsilon|t|^{s-\varepsilon} & |h|\geq 1,\ s\geq\varepsilon\end{cases}$$

for any $h\in\mathbb{R}$ (resp. $\mathbb{C}$). This yields

$$\|\phi(f+h)-\phi(f)\|_p\leq C'\begin{cases} \|(1-\chi)h^\varepsilon\|_p+\|\chi h^s\|_p+\|f^{s-\varepsilon}h^\varepsilon\|_p & s\geq\varepsilon\\ \|(1-\chi)h^\varepsilon\|_p+\|\chi h^s\|_p & s\leq\varepsilon\ .\end{cases}$$

We get with (20) and (22)

$$\|(1-\chi)h^\varepsilon\|_p=\|(1-\chi)h\|_{\varepsilon p}^\varepsilon\leq\begin{cases} \|(1-\chi)h\|_r^{r/p} & \forall\ r\leq\varepsilon p\\ (\mathrm{vol}(\Omega)^{1/\varepsilon p-1/r}\|(1-\chi)h\|_r)^\varepsilon & \forall\ r\geq\varepsilon p\ .\end{cases}$$

With the simple estimate $\|\chi h^{sp}\|_1\leq\|\chi h^r\|_1\ \forall\ r\geq sp$ we obtain

$$\|\chi h^s\|_p=\|\chi h^{sp}\|_1^{1/p}\leq\|\chi h^r\|_1^{1/p}=\|\chi h\|_r^{r/p}\leq\|h\|_r^{r/p}\ .$$

Further, we find from (21) and (18) for $s\geq\varepsilon$

$$\|f^{s-\varepsilon}h^\varepsilon\|_p\leq\|f^{\frac{s-\varepsilon}{s}}h^{\frac{\varepsilon}{s}}\|_{sp}^s\leq\|f\|_{sp}^{s-\varepsilon}\|h\|_{sp}^\varepsilon\leq(\mathrm{vol}(\Omega)^{1/sp-1/r})^s\|f\|_r^{s-\varepsilon}\|h\|_r^\varepsilon\ .$$

Therefore, we see

$$\|\phi(f+h)-\phi(f)\|_p\leq C'\begin{cases} \|h\|_r^{\min(\varepsilon,r/p)} & ;\|h\|_r\leq 1,\ s\leq\varepsilon\\ \|h\|_r^{\min(\varepsilon,r/p)}+\|f\|_r^{s-\varepsilon}\|h\|_r^\varepsilon & ;\|h\|_r\leq 1,\ s\geq\varepsilon\ .\end{cases}\tag{28}$$

Further, from (22) with $r\geq ps$

$$\|\chi h^s\|_p=\|\chi h\|_{sp}^s\leq\mathrm{vol}(\Omega)^{s/p-s/r}\|\chi h\|_r^s\leq\mathrm{vol}(\Omega)^{s/p-s/r}\|h\|_r^s\ .$$

Using $\|(1-\chi)h^\varepsilon\|_p\leq\|1\|_1^{\frac{1}{\varepsilon p}}<\infty$ we arrive at

$$\|\phi(f+h)-\phi(f)\|_p\leq C'\begin{cases} \|h\|_r^s & ;\|h\|_r\geq 1,\ s\leq\varepsilon\\ \|h\|_r^s+\|f\|_r^{s-\varepsilon}\|h\|_r^\varepsilon & ;\|h\|_r\geq 1,\ s\geq\varepsilon\end{cases}\ .\tag{29}$$

Putting the estimates (28) and (29) together yields the Hölder estimate (27) of order $\varepsilon_*=\min(\varepsilon,r/p)$ which implies (for $s\leq\varepsilon$ uniform) Hölder continuity of the composition operator between L^r and L^p whenever $r/p\geq s$. Any bounded uniformly Hölder continuous function ϕ satisfies the condition (26) with $s=0$.

Part II: Because of the growth condition we find $|\phi(f)(\omega)|\leq C+|f(\omega)|^s$. Using the Minkowsky inequality and the embedding $L^r\hookrightarrow L^{sp'}$ if $1\leq p'\leq r/s$ we see that for such p' the estimate $\|\phi(f)\|_{p'}\leq C'(\|1\|_{p'}+\||f(\omega)|^s\|_{p'})\leq C'(\|1\|_{p'}+C'\|f\|_r^{r/p'})$ holds. Let $1/p'+1/q\leq 1/p$ and $t\in L^q$. Since then $\phi(f)t\in L^p(\mu)$ the operator $\phi\circ\cdot$ maps into $L(L^q,L^p)$ whenever $s\leq r(1/p-1/q),\ p,q,r\in[1,\infty],q\geq p$.

We estimate for $t \in L^q, h \in L^r$

$$\|\phi(f+h)\cdot t - \phi(f)\cdot t\|_p \le \|\phi(f+h)-\phi(f)\|_{p'}\cdot \|t\|_q$$

with $1/p' = 1/p - 1/q$. Now we observe that p' satisfies the conditions of part I. Thus, we may estimate

$$\begin{aligned}\|\phi(f+h)\cdot - \phi(f)\cdot\|_{L(L^q,L^p)} &\le \sup_{\|t\|_q=1} \|\phi(f+h)\cdot t - \phi(f)\cdot t\|_p \\ &\le \|\phi(f+h)-\phi(f)\|_{p'}\end{aligned}$$

and therefore the estimate (27) holds for $\|\phi(f+h)\cdot - \phi(f)\cdot\|_{L(L^q,L^p)}$. ■

Example: 6.3 *We want to give a counterexample to show that even very well behaved functions need $r/p \ge \varepsilon$ over a bounded domain in order to be Hölder continuous of order ε, if ε is the maximal H-continuity order of ϕ. The map $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(t) = \sqrt{|t|}\cdot e^{-|t|^2}$ is obviously uniformly H-continuous of order at most 1/2. Take the zero-sequence $h_n = e_n \cdot \chi_n$ with $e_n \ll 1$ and χ_n a characteristic function of a set with volume 1/n. Then $\|h_n\|_r = e_n n^{-1/r}$ and $\|\phi\circ h_n\|_p \sim \sqrt{e_n} n^{-1/p}$. So, near any function z vanishing on the supports of the h_n we get*

$$\|\phi\circ(z+h_n) - \phi\circ(z)\|_p \sim \sqrt{e_n} n^{-1/p} = e_n^{r/p+1/2}\|h_n\|_r^{r/p}$$

if we choose $r/p = \varepsilon < 1/2$ and $e_n \ge e_{min} > 0$ we get $\|\phi\circ(z+h_n) - \phi\circ(z)\|_p > e_{min}^{r/p+1/2}\cdot\|h_n\|_r^{\varepsilon}$. This shows that r/p is the best possible order of H-continuity for $\phi\circ$.

Proposition: 6.4 *Let $\phi : \mathbb{R} \to \mathbb{R}$ (resp. $\phi : \mathbb{C} \to \mathbb{C}$) be differentiable and satisfy the growth condition $|\phi'(t)| \le C(1+|t|)^{s-1}$ (resp. $\|\phi'(t)\|_{M(\mathbb{C},2)} \le C(1+|t|)^{s-1}$) and for some $\varepsilon \in (1,2]$ and $s \ge 0$ a remainder estimate:*

$$|\phi(t+h)-\phi(t)-\phi'(t)h| \le C\begin{cases} |h|^\varepsilon(1+|t|+|h|)^{s-\varepsilon} & \forall\, t,h \in \mathbb{R}(resp.\ \mathbb{C})\ if\ s \ge \varepsilon \\ |h|^\varepsilon(1+|h|)^{s-\varepsilon} & \forall\, t,h \in \mathbb{R}(resp.\ \mathbb{C})\ if\ s \le \varepsilon\ . \end{cases} \tag{30}$$

Let μ be a finite measure (as in Proposition 6.2). The operators $\phi\circ$, $\phi\circ\cdot$ and $\phi\circ\cdot\Pi : L^r(\mu) \to L^p(\mu)$ given by (23), (24) and (25) are then everywhere Fréchet differentiable, provided $1/r < 1/p - \sum_{j=1}^m 1/q_j$ or $r = p = q_j = \infty$ and $s \le r/p - \sum r/q_j =: \varepsilon'$ and $\varepsilon' > 1$ (with the convention $\sum_\emptyset 1/q_j := 0$ in the case of $\phi\circ$). For $\phi\circ\prod^m$ to be Fréchet differentiable the conditions are $1/r < 1/p - \sum_{j=1}^m 1/q_j$ and $s \le r/p - 1 - \sum r/q_j$ and the derivative at f is given by $h \mapsto ((t_1,\dots t_m) \mapsto \phi'(f)\cdot h\cdot\prod t_j)$.

With $\varepsilon_* := \min(\varepsilon,\varepsilon')$ we have the estimate

$$\|(\phi\circ\cdot\Pi^m)(f+h) - (\phi\circ\cdot\Pi^m)(f) - (\phi'\circ\cdot\Pi^m)(f)\cdot h\|_{L(L^{q_1}\times\dots L^{q_m}\times L^r,L^p)} \le C\begin{cases} \|h\|_r^{\varepsilon_*}(1+\|h\|_r)^{s-\varepsilon_*} + \|h\|_r^\varepsilon\|f\|_r^{s-\varepsilon} & s \ge \varepsilon \\ \|h\|_r^{\varepsilon_*}(1+\|h\|_r)^{s-\varepsilon_*} & s \le \varepsilon\ . \end{cases} \tag{31}$$

In particular we have $(\phi\circ)' = (\phi'\circ\cdot)$ if $r > p$ and $(\phi\circ\cdot)' = (\phi'\circ\cdot\Pi^2)$ if $1/r < 1/p - 1/q$.

If the higher derivatives of the function ϕ satisfy (30) with $\phi^{(j)}$ in place of ϕ and s_j in place of s for all $j \le m$, then $(\phi\circ)$ is m times Fréchet differentiable provided that $s_j \le r/p - 1 - j$ and $r > mp$. The condition $r > mp$ is in general necessary for $(\phi\circ) : L^r \to L^p$ to be m times Fréchet differentiable if ϕ is not a polynomial.

Proof: The growth condition ensures that $\phi(t) \le C'(1+|t|)^s$ by the mean-value theorem. Thus, for $f \in L^r$ we find $\phi(f) \in L^u$ provided $s \le r/u$ and further $\phi'(f) \in L^u$ if $s \le r/u + 1$.

Let $t_j \in L^{q_j}, j = 1, \dots, m$ and $t = \prod t_j$. Than $t \in L^q, 1/q = \sum 1/q_j$ with $\|t\|_q \leq \prod \|t_j\|_{q_j}$. Thus, we get as in the proof of Proposition 6.2 (with $1/p' := 1/p - 1/q = 1/p - \sum 1/q_j$)

$$\begin{aligned}
\clubsuit &:= \|(\phi \circ \cdot\Pi^m)(f+h) - (\phi \circ \cdot\Pi^m)(f) - (\phi' \circ \cdot\Pi^m)(f) \cdot h\|_{L(L^{q_1}\times..L^{q_m}, L^p)} \\
&\leq \sup_{\substack{t_j \in L^{q_j} \\ \|t_j\|=1}} \left\|(\phi\circ)(f+h)\cdot\prod t_j - (\phi\circ)(f)\cdot\prod t_j - (\phi'\circ)(f)\cdot\prod t_j \cdot h\right\|_{L^p(\Omega)} \\
&\leq \sup_{\substack{t \in L^q \\ \|t\|=1}} \|(\phi\circ)(f+h)\cdot t - (\phi\circ)(f)\cdot t - (\phi'\circ)(f)\cdot t\cdot h\|_{L^p(\Omega)} \\
&\leq \sup_{\substack{t \in L^q \\ \|t\|=1}} \|(\phi\circ)(f+h) - (\phi\circ)(f) - (\phi'\circ)(f)\cdot h\|_{L^{p'}(\Omega)} \|t\|_q \\
&\leq \|(\phi\circ)(f+h) - (\phi\circ)(f) - (\phi'\circ)(f)\cdot h\|_{L^{p'}(\Omega)} \ .
\end{aligned}$$

Now $(\phi \circ \cdot\Pi^m)$ would be Fréchet differentiable if $\clubsuit = \|h\|_r \cdot o(\|h\|_r),\ \|h\|_r \to 0$. Since we assumed $1/r < 1/p - \sum 1/q_j$ we can conclude via arguments similar to (28) and (29) that led from (26) to (27) that (31) follows from (30) and hence $(\clubsuit) = \mathcal{O}(\|h\|_r^{\varepsilon_*}) = \|h\|_r \cdot o(\|h\|_r),\ \|h\|_r \to 0$. This proves the main statement.

When we take $m = 0$ and take the empty product equal to 1 we get the result $(\phi\circ)' = (\phi' \circ \cdot)$ if $r > p$.

Applying the first part to $(\phi\circ)' = (\phi' \circ \cdot)$ with $(\phi' \circ \cdot)(f) \in L^1(L^r, L^p)$ yields $(\phi'\circ\cdot)' = (\phi''\circ\cdot\Pi^2)$ provided $1/r < 1/p - 1/r$ i.e. $r > 2p$ and $s_1 < r/p - 2$. The result for $(\phi\circ)^{(m)}$ follows by induction. The condition on r becomes $1/r < 1/p - (m-1)/r$ i.e $r > mp$. That this is in general necessary for $\phi\circ : L^r \to L^p$ to be m times Fréchet differentiable follows from Example 6.5. ■

Example: 6.5 *We give a counterexample in order to demonstrate, that in general $r > mp$ is necessary for $\phi\circ$ to be m times Fréchet differentiable in the $L^r(\Omega) \to L^p(\Omega)$ topology. Let ϕ be a smooth function with all derivatives bounded and $\phi^{(m+2)}(0) = c > 0$, e.g $\phi(t) = t^{m+2}e^{-t^2}$. Now consider a zero sequence $h_n = e_n\chi_n$ with characteristic functions χ_n of set of volume $\|\chi_n\|_1 = 1/n$ and $0 < e_n \ll 1$. Then $\|h_n\|_r = e_n n^{-1/r}$ and for the zero function z we have*

$$\begin{aligned}
&\phi^{(m)}(z+h_n)(\omega)\cdot(h_n)^m(\omega) - \phi^{(m)}(z)(\omega)\cdot(h_n)^m(\omega) - \phi^{(m+1)}(z)(\omega)\cdot(h_n)^{m+1}(\omega) \\
&= \frac{1}{(m-1)!}\int_0^1 (1-t)^{m-1}\phi^{(m+2)}(z+th_n)(\omega)\cdot(h_n)^{m+2}(\omega) \\
&\qquad \simeq c_m e_n^{m+2}\chi_n(\omega)
\end{aligned}$$

for some $c_m \neq 0$. If the $m+1$ derivative of $\phi\circ$ exists it must be given by $(\phi\circ)^{m+1} : (t_1, \dots, t_m, h) \mapsto \phi^{(m+1)}(z) \cdot (\prod_1^m t_j) \cdot h$. Therewith we estimate

$$\begin{aligned}
&\|(\phi^{(m)}\circ\cdot\Pi^m)(z+h) - (\phi^{(m)}\circ\cdot\Pi^m)(z) - (\phi^{(m+1)}\circ\cdot\Pi^m)(z)\cdot h\|_{L(\otimes_1^m L^r, L^p)} \\
&\geq \frac{\|(\phi^{(m)}\circ\cdot\Pi^m)(z+h_n)(h_n)^m - (\phi^{(m)}\circ\cdot\Pi^m)(z)(h_n)^m - (\phi^{(m+1)}\circ\cdot\Pi^m)(z)(h_n)^m\cdot h_n\|_p}{\|h_n\|_r^m} \\
&\qquad \sim \frac{\|c_m e_n^{m+2}\chi_n\|_p}{\|h_n\|_r^m} \ = \ c_m e_n^{m+2} n^{-1/p+m/r}
\end{aligned}$$

if $r = mp$ and $e_n \geq e_{min} > 0$ is not a zero sequence we find

$$\begin{aligned}
&\|(\phi^{(m)}\circ\cdot\Pi^m)(z+h) - (\phi^{(m)}\circ\cdot\Pi^m)(z) - (\phi^{(m+1)}\circ\cdot\Pi^m)(z)\cdot h\|_{L(\otimes_1^m L^r, L^p)} \\
&\qquad \geq c_m e_{min}^{m+2} \neq o(\|h_n\|_r) \ .
\end{aligned}$$

This proves that $r > mp$ is a necessary requirement for $(\phi\circ) : L^r \to L^p$ to be m times Fréchet differentiable.

Example: 6.6 *Let* $\phi \in \mathcal{C}^{m+1,1}(\mathbb{R},\mathbb{R})$ *i.e. there is a* $C < \infty$ *such that for all* $\forall n \leq m+1,\ t,h \in \mathbb{R}$ *we have* $|\phi^{(n)}(t)| \leq C$ *and* $|\phi^{(m+1)}(t+h) - \phi^{(m+1)}(t)| \leq C|h|$, *then* $\phi\circ : L^r \to L^p$ *is Fréchet differentiable up to order* $m_* = \min(m, \max_j(j < r/p))$, *and* $(\phi\circ)^{(m_*)}$ *is Lipschitz continuous with Lipschitz constant less then* C *if* $r/p - m_* \geq 1$.

Proof: First we note that $|\phi^{(n)}(t)| \leq C(1+|t|)^0, n = 0,\ldots,m+1$, i.e. ϕ satisfies the growth conditions for m times differentiability for any combination of $r,p \in [1,\infty]$. By the mean value theorem we see that

$$|\phi^{(n)}(t+h) - \phi^{(n)}(t) - \phi^{(n+1)}(t)h| \leq \begin{cases} |\int_0^1 (\phi^{(n+1)}(t+\vartheta h) - \phi^{(n+1)}(t))h\,d\vartheta| & \leq \; C|h|^2 \\ |\phi^{(n)}(t+h)| + |\phi^{(n)}(t)| + |\phi^{(n+1)}(t)h| & \leq \; C(2+|h|) \end{cases}$$

thus the condition (30) is satisfied for all $s \geq 1$ with $\varepsilon = 2$. ■

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