Universität Potsdam

Olaf Schmidtmann

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Olaf Schmidtmann

Max-Planck-Gruppe Nichtlineare Dynamik, Universität Potsdam, PF 601553, D-14415 Potsdam, Germany

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Introduction

The present paper is related to the problem of approximating the exact solution to the magnetohydrodynamic equations (MHD). The behaviour of a viscous, incompressible and resistive fluid is exemined for a long period of time.
We follow the strategy introduced by several authors to approximate the solutions of the Navier-Stokes-Equations (NSE). The existence of a finite dimensional global attractor and a finite number of determining modes of the two-dimensional NSE (cf.[4],[5]) shows that the long time behaviour of these equations is determined by a finite number of parameters.
This justifies the approximation of the solutions by the linear Galerkin approximation associated to the eigenvectors of the Stokes operator.(cf.[4])
In order to obtain a better approximation it was pointed out that the long time behaviour of two-dimensional turbulent flows is mainly controlled by a finite number of modes related to the Stokes operator and that the higher modes remain small for a large time. (cf.[6])
Let u be the solution to the abstract evolution equation in an appropriate Hilbertspace

$$\frac{du}{dt} + Au + B(u) = f \ , \tag{0.1}$$

where A is the Stokes operator and B is a nonlinear quadratic operator.
(0.1) can be written as a coupled system of equations for p and q, where p denotes the projection of u onto the finite dimensional space, spanned by the

first m eigenvectors of A (lower modes) and q the projection of u onto the infinite dimensional space spanned by the remaining eigenvectors (higher modes),

$$\frac{dp}{dt} + Ap + PB(p+q) = Pf \ , \tag{0.2}$$

$$\frac{dq}{dt} + Aq + QB(p+q) = Qf \ . \tag{0.3}$$

P and Q are the projectors onto the finite and infinite dimensional spaces. The results of Foias, Manley and Temam (cf.[6]) show that a reasonable approximation of (0.3) is given by

$$Aq + QB(p) = Qf \ . \tag{0.4}$$

This led them to introduce the finite dimensional manifold M_0 by :

$$q = \Phi_0(p) = A^{-1}(Qf - QB(p)) \ . \tag{0.5}$$

Another type of approximate inertial manifolds for two-dimensional turbulent flows containing all stationary solutions has been examined by Titi. (cf.[12])
The fact that the two-dimensional MHD-equations have finite dimensional global, universal attractors and a finite number of determining parameters (cf.[1],[8],[9]), inspires us to justify the definition of an approximate inertial manifold similiar to (0.5) for two-dimensional MHD-equations.
The main purpose of this paper is to show that this manifold yields a better approximation than the linear Galerkin approximation.

The plan for our paper is the following.
In section 1 we describe the problem under consideration whereas in section 2 we give a precise functional setting of the problem. Section 3 is related to a few existence and uniqueness results. In section 4 and 5 the main results are presented: norm-estimates for higher modes of the solutions to the MHD-equations and estimates of the distance between the attractor and the approximate inertial manifold.

1 The magnetohydrodynamic equations

The equations, we are interested in, are given in a bounded domain $\Omega \subset I\!R^2$, occupied by the viscous, incompressible and resistive fluid. The unknown functions u, B and p denote the velocity of the fluid, the magnetic field and the pressure of the fluid, respectively.
We shall suppose that the density ρ at time $t = 0$ is equal to a prescribed constant ρ_0 . The incompressibility yields that the fluid is homogenouos at time $t \geq 0$. For the sake of simplicity we set $\rho_0 \equiv 1$.
The nondimensional form of the MHD-equations for viscous, incompressible flows is the following (cf.[2],[8]):

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{Re}\Delta u + \nabla p + S\nabla(\frac{1}{2}B^2) - S(B \cdot \nabla)B = f \quad \text{in } \Omega \quad (1.1)$$

$$\frac{\partial B}{\partial t} + (u \cdot \nabla)B - (B \cdot \nabla)u - \frac{1}{Rm}\Delta B = 0 \quad \text{in } \Omega \qquad (1.2)$$

$$\text{div } u = 0 \quad \text{in } \Omega \quad , \quad \text{div } B = 0 \quad \text{in } \Omega \qquad (1.3)$$

p, u, B are nondimensional quantities, f represents a nondimensional volume density force. The nondimensional numbers are:
- the Reynolds number Re ,
- the magnetic Reynolds number Rm ,
- the number $S = M^2/Re \cdot Rm$, where M is the Hartman number.

We complete (1.1) - (1.3) by initial and boundary conditions upon u and B. Let $\partial\Omega$ denote the boundary of Ω and η the unit outward normal along $\partial\Omega$.

$$u(x,0) = u_0(x) \quad , \quad B(x,0) = B_0(x) \quad x \in \Omega \qquad (1.4)$$

$$u = 0 \quad , \quad B \cdot \eta = 0 \quad , \quad \text{curl } B = 0 \quad \text{on } \partial\Omega \qquad (1.5)$$

Remark:
Since we consider the space dimension N = 2, we define the operator curl for every vector function $\varphi = (\varphi_1, \varphi_2)$ as follows:

$$\text{curl}\, \varphi \ = \ \frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} \ .$$

The two dimensional case means that the region is a cylinder $\Omega \times I\!R$, $\Omega \subset I\!R^2$, and all quantities are independent of x_3 , u and B beeing parallel to the x_1, x_2-plane.

2 Notations and precise functional setting of the problem

Let $\Omega \subset I\!R^2$ be a bounded domain with Lipschitz-boundary $\partial\Omega \in C^{0,1}$.By $L^p(\Omega), L^p(\Omega, I\!R^2), 1 \le p \le \infty$; $W^{k,p}(\Omega), W^{k,p}(\Omega, I\!R^2), 1 \le p \le \infty$, $k = 1, 2, \dots$; we denote the usual spaces of real valued or $I\!R^2$-valued functions defined on Ω.
$W_0^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega, I\!R^2)$ are subspaces of $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega, I\!R^2)$, respectively, consisting of functions vanishing on $\partial\Omega$.
If there is no danger of misunderstanding we shall write shortly L^p for $L^p(\Omega)$ as well as for $L^p(\Omega, I\!R^2)$ and $W^{k,p}$ for $W^{k,p}(\Omega)$ as well as for $W^{k,p}(\Omega, I\!R^2)$.
If E is any Banach space and (0,T) an intervall of the real axis we denote by $L^p(0, T; E), 1 \le p \le \infty$, the usual space of functions defined on $(0, T)$ with values in E. As usual let $E^\star$ be the dual space of the Banach space E.

Let

$$(\varphi, \psi) \ := \int\limits_\Omega \varphi(x) \cdot \psi(x) dx$$

denote the usual scalar product in $L^2(\Omega)$.

Next we introduce the spaces, used in the theory of MHD-equations (cf.[6],[9]). They are :

$\mathcal{V}_1 = \{ \ \varphi \in C_c^\infty(\Omega, I\!R^2) \ ; \ \text{div} \ \varphi = 0 \ \}$,

$V_1 = \{ \ \varphi \in W_0^{1,2}(\Omega, I\!R^2) \ ; \ \text{div} \ \varphi = 0 \ \}$ (the closure of $\mathcal{V}_1$ in $W^{1,2}$) ,

$\mathcal{V}_2 = \{ \ \Phi \in C^\infty(\bar{\Omega}, I\!R^2) \ ; \ \text{div} \ \Phi = 0 \ ; \ \Phi \cdot \eta = 0 \quad \text{on} \ \partial\Omega \ \},$

$V_2 = \{ \ \Phi \in W^{1,2}(\Omega, I\!R^2) \ ; \ \text{div} \ \Phi = 0 \ ; \ \Phi \cdot \eta = 0 \quad \text{on} \ \partial\Omega \ \},$
(the closure of $\mathcal{V}_2$ in $W^{1,2}$)

$H :=$ the closure of $\mathcal{V}_1$ in L^2 = the closure of $\mathcal{V}_2$ in L^2.

These spaces are equiped with the following scalar products. We define for all $\varphi, \psi \in V_1$:

$$((\varphi, \psi))_1 := \sum_{i=1}^{2} \left(\frac{\partial \varphi}{\partial x_i}; \frac{\partial \psi}{\partial x_i} \right) \ ,$$

which is a scalar product in V_1 and thanks to Friedrich's inequality in $W^{1,2}$. Further we define (cf.[8]) for all $\Phi, \Psi \in V_2$

$$((\Phi, \Psi))_2 := (\text{curl}\Phi, \text{curl}\Psi) \ ,$$

which is a scalar product in V_2. The norms $\| \cdot \|_{V_i} := \sqrt{((\cdot, \cdot))_i}$ in $V_i \ _{(i=1,2)}$ are equivalent to the norm of $W^{1,2}$. So we have :

$$\| \varphi \|_{W^{1,2}} \ \leq \ \alpha_1 \cdot \| \varphi \|_{V_1} \ , \ \| \varphi \|_{V_1} \ \leq \ \alpha_2 \cdot \| \varphi \|_{W^{1,2}} \qquad \forall \ \varphi \in V_1 \ ,$$

$$\| \varphi \|_{W^{1,2}} \ \leq \ \beta_1 \cdot \| \varphi \|_{V_2} \ , \ \| \varphi \|_{V_2} \ \leq \ \beta_2 \cdot \| \varphi \|_{W^{1,2}} \qquad \forall \ \varphi \in V_2 \ ,$$

$$\alpha_1 = \alpha_1(\Omega) \ , \ \alpha_2 = \alpha_2(\Omega) \ , \ \beta_1 = \beta_1(\Omega) \ , \ \beta_2 = \beta_2(\Omega) \ = \text{const} > 0 \ .$$

Finally we mention that the injections

$$V_1 \subset H \subset V_1^\star \quad , \quad V_2 \subset H \subset V_2^\star$$

are compact and continuous and each space is dense in the following.

Let us introduce the operators related to the MHD-equations.
We define two linear continuous operators $\mathcal{A}_1 \in \mathcal{L}(V_1, V_1^\star)$ and $\mathcal{A}_2 \in \mathcal{L}(V_2, V_2^\star)$ by setting

$$< \mathcal{A}_1 \varphi, \psi > \ := \ ((\varphi, \psi))_1 \quad \text{for all} \ \varphi, \psi \in V_1 \ ,$$

$$< \mathcal{A}_2 \Phi, \Psi > \ := \ ((\Phi, \Psi))_2 \quad \text{for all} \ \Phi, \Psi \in V_2 \ .$$

We can also consider $\mathcal{A}_1$ and $\mathcal{A}_2$ as unbounded operators on H whose domains are :

$$D(\mathcal{A}_1) := \{ \ \varphi \in V_1 \ \ , \ \ \mathcal{A}_1\varphi \in H \ \ \} \ ,$$

$$D(\mathcal{A}_2) := \{ \ \Phi \in V_2 \ \ , \ \ \mathcal{A}_2\Phi \in H \ \ \} \ .$$

In order to characterize $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$ we state the following well-known results :

Lemma 2.1:
Let $g \in V_1^\star$ be given. Then $\varphi \in V_1$ is a solution of $\mathcal{A}_1\varphi = g$ if and only if there exists a function $p \in L^2(\Omega)$ such that:

$$-\Delta\varphi + \text{grad } p \ = \ g \quad , \quad \text{div } \varphi \ = \ 0 \ .$$

Furthermore if $g \in H$ then there hold $\varphi \in V_1 \cap W^{2,2}$, $p \in W^{1,2}$ and there exists a constant $c_1 = c_1(\Omega)$ such that:

$$\| \ \varphi \ \|_{W^{2,2}} + \| \ p \ \|_{W^{1,2}} \ \leq \ c_1 \ \| \ g \ \|_{L^2} \ .$$

Thus we have $D(\mathcal{A}_1) = W^{2,2} \cap V_1$ (cf. [8]).

Lemma 2.2:
Let $g \in H$ be given. Then $\Phi \in V_2$ is a solution of $\mathcal{A}_2\Phi = g$ if and only if Φ satisfies:

$$\begin{aligned} -\Delta\Phi \ &= \ g && \text{in } \Omega, \\ \text{div } \Phi \ &= \ 0 && \text{in } \Omega, \\ \Phi \cdot \eta \ &= \ 0 && \text{on } \partial\Omega, \\ \text{curl } \Phi \ &= \ 0 && \text{on } \partial\Omega. \end{aligned}$$

Furthermore there exists a constant $c_2 = c_2(\Omega)$ such that :

$$\| \ \Phi \ \|_{W^{2,2}} \ \leq \ c_2 \ \| \ g \ \|_{L^2} \ .$$

Thus we have $D(\mathcal{A}_2) = W^{2,2} \cap V_2$ (cf.[8]).

The linear operators $\mathcal{A}_i : D(\mathcal{A}_i) \subset H \longrightarrow H$ $(i = 1, 2)$ are symmetric on the real Hilbertspace H with $cl(D(\mathcal{A}_i)) = H$ and $\mathcal{A}_i$ are strongly monotone, i.e:

$$(\mathcal{A}_i\varphi, \varphi) \;\geq\; c \parallel \varphi \parallel^2 \qquad \forall\; \varphi \in D(\mathcal{A}_i) \text{ and fixed } c > 0.$$

Since the operators $\mathcal{A}_i^{-1} : H \longrightarrow H$ are linear, compact and self-adjoint, there exist complete families of eigenvectors v_j of $\mathcal{A}_1$ and w_j of $\mathcal{A}_2$, which are orthonormal in H (cf. [13]):

$$\mathcal{A}_1 v_j = \lambda_j v_j \qquad j = 1, 2, \ldots$$

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \qquad \lambda_j \to \infty \quad \text{as} \quad j \to \infty,$$

$$\mathcal{A}_2 w_j = \mu_j w_j \qquad j = 1, 2, \ldots$$

$$0 < \mu_1 \leq \mu_2 \leq \ldots \qquad \mu_j \to \infty \quad \text{as} \quad j \to \infty.$$

Let P_m^1 and P_m^2 denote the projector in H onto the subspace of H spanned by the first m eigenvectors $v_1, \ldots, v_m$ of $\mathcal{A}_1$ and $w_1, \ldots, w_m$ of $\mathcal{A}_2$, respectively. We write $Q_m^1 = I - P_m^1$ and $Q_m^2 = I - P_m^2$.

There hold the following well-known estimates:

$$\parallel \varphi \parallel_{L^2} \;\leq\; \lambda_1^{-1/2} \parallel \varphi \parallel_{V_1} \quad , \quad \parallel \Phi \parallel_{L^2} \;\leq\; \mu_1^{-1/2} \parallel \Phi \parallel_{V_2}$$
$$\forall \varphi \in V_1 \;\; , \;\; \Phi \in V_2 \;\; ;$$

$$\parallel \varphi \parallel_{V_1} \;\leq\; \lambda_1^{-1/2} \parallel \mathcal{A}_1 \varphi \parallel_{L^2} \quad , \quad \parallel \Phi \parallel_{V_2} \;\leq\; \mu_1^{-1/2} \parallel \mathcal{A}_2 \Phi \parallel_{L^2}$$
$$\forall \;\; \varphi \in D(\mathcal{A}_1) \;\; , \;\; \Phi \in D(\mathcal{A}_2) \;\; ;$$

$$\parallel \varphi \parallel_{L^2} \;\leq\; \lambda_{m+1}^{-1/2} \parallel \varphi \parallel_{V_1} \quad , \quad \parallel \Phi \parallel_{L^2} \;\leq\; \mu_{m+1}^{-1/2} \parallel \Phi \parallel_{V_2}$$
$$\forall \;\; \varphi \in Q_m^1 H \;\; , \;\; \Phi \in Q_m^2 H \;\; ;$$

$$\| \varphi \|_{V_1} \leq \lambda_{m+1}^{-1/2} \| \mathcal{A}_1 \varphi \|_{L^2} \quad , \quad \| \Phi \|_{V_2} \leq \mu_{m+1}^{-1/2} \| \mathcal{A}_2 \Phi \|_{L^2}$$
$$\forall \ \varphi \in Q_m^1 H \ , \ \Phi \in Q_m^2 H \ .$$

We define now a trilinear continuous mapping b on $L^4(\Omega, I\!R^2) \times W^{1,2}(\Omega, I\!R^2) \times L^4(\Omega, I\!R^2)$ by setting:

$$b(\varphi, \psi, \xi) \ := \ \sum_{i,j=1}^{2} \int_\Omega \varphi_i \cdot \frac{\partial \psi_j}{\partial x_i} \cdot \xi_j \ dx \ . \tag{2.1}$$

Due to Hölders inequality this integral makes sense. We have (cf.[6],[8]):

$$b(\varphi, \psi, \psi) = 0 \quad \text{for all } \varphi \in V_2 \quad \psi \in W^{1,2}, \tag{2.2}$$
$$b(\varphi, \psi, \xi) = -b(\varphi, \xi, \psi) \quad \text{for all } \varphi \in V_2 \quad \psi, \xi \in W^{1,2}. \tag{2.3}$$

Thanks to (2.1) we can define a continuous bilinear operator $\tilde{\mathcal{B}}$ from $V_i \times V_i$ into $V_i^\star$ (i=1,2) by

$$< \tilde{\mathcal{B}}(\varphi, \psi), \xi > \ := \ b(\varphi, \psi, \xi),$$

which satisfies :

$$|< \tilde{\mathcal{B}}(\varphi, \psi), \xi >| \ \leq \ d_1 \cdot \| \varphi \|_{L^4} \cdot \| \psi \|_{V_i} \cdot \| \xi \|_{L^4} \tag{2.4}$$
$\forall \quad \varphi \in L^4, \quad \psi \in V_i, \quad \xi \in L^4 \ , \ d_1 = d_1(\Omega) \ ,$

$$|< \tilde{\mathcal{B}}(\varphi, \psi), \xi >| \ \leq \ d_2 \cdot \| \varphi \|_{L^\infty} \cdot \| \psi \|_{V_i} \cdot \| \xi \|_{L^2} \tag{2.5}$$
$\forall \ \varphi \in L^\infty, \quad \psi \in V_i, \quad \xi \in L^2, \ , \ d_2 = d_2(\Omega) \ .$

This is a simple consequence of Hölders inequality.

Definition:
Let Re, Rm, S, T be positive real numbers and $(0,T)$ an open intervall of the real axis.
Let $f \in L^2(0,T;V_1^\star)$ and u_0, $B_0 \in H$ be given. A tupel (u,B)
$u \in W^{1,2}(0,T;V_1^\star) \cap L^2(0,T;V_1)$, $B \in W^{1,2}(0,T;V_2^\star) \cap L^2(0,T;V_2)$
is called weak solution of (1.1) - (1.5) if:

$$\frac{d}{dt}(u(\tau),\varphi) \;+\; \frac{1}{Re} < \mathcal{A}_1 u(\tau),\varphi > \;+\; < \tilde{\mathcal{B}}(u(\tau),u(\tau)),\varphi > \;- \ldots \tag{2.6}$$

$$\ldots - \;\; S < \tilde{\mathcal{B}}(B(\tau),B(\tau)),\varphi > \;=\; (f(\tau),\varphi) \quad \forall \varphi \in V_1, \;\text{ f.a.a. } \tau \in (0,T)$$

$$\frac{d}{dt}(B(\tau),\Phi) \;+\; \frac{1}{Rm} < \mathcal{A}_2 B(\tau),\Phi > \;+ \ldots \tag{2.7}$$

$$\ldots + \;< \tilde{\mathcal{B}}(u(\tau),B(\tau)),\Phi > \;-\; < \tilde{\mathcal{B}}(B(\tau),u(\tau)),\Phi > \;=\; 0$$

$$\forall \Phi \in V_2, \;\text{ f.a.a. } \tau \in (0,T)$$

$$u(0) = u_0 \quad , \quad B(0) = B_0 \; . \tag{2.8}$$

(f.a.a. stands for " for allmost all ")

Remark:
Every function in $L^2(0,T;V_i) \cap W^{1,2}(0,T;V_i^\star)$ $i = 1,2$ is continuous on the closed intervall $[0,T]$ with values in H. Therefore (2.8) is meaningful.

3 Existence, uniqueness and regularity results

In this section we state an existence result of Temam and Sermange which we need in the following. Furthermore some useful estimates are given.

Theorem 3.1: (Temam , Sermange cf. [8])
(i) For f, u_0, B_0 given with $f \in L^2(0,T;V_1^\star)$, $u_0, B_0 \in H$ there exists a unique weak solution (u, B) of (1.1) - (1.5).
Furthermore u and B are analytic in t with values in $D(\mathcal{A}_1)$ and $D(\mathcal{A}_2)$, respectively $(t > 0)$ and the mapping $u_0, B_0 \longmapsto u(t), B(t)$ is continuous from $H \times H$ into $D(\mathcal{A}_1) \times D(\mathcal{A}_2)$, $\forall\ t \in (0,T)$.

(ii) If $f \in L^\infty(0,T;H)$ is defined by

$$f(t) \ := \ \tilde{f} \quad , \quad \tilde{f} \in H \quad \text{f.a.a. } t \in (0,T), \tag{3.1}$$

$u_0 \in V_1, B_0 \in V_2$, then the weak solution (u, B) satisfies:

$$\| u(t) \|_{V_1} \ \leq \ M_1 \quad , \quad \| B(t) \|_{V_2} \ \leq \ M_2 \quad \text{f.a.a. } t \in (0,T) \ ,$$

where $M_1, M_2 \in I\!R$, $M_1, M_2 = \text{const} > 0$; M_1,M_2 are independent of T.

(iii) Let $u_0 \in V_1$, $B_0 \in V_2$ be given and let f be defined as in (3.1) for $0 < T \leq \infty$. Then the weak solution to (1.1) - (1.5) satisfies, for every $\alpha_0 > 0$: $u \in L^\infty(\alpha_0, T; D(\mathcal{A}_1))$, $B \in L^\infty(\alpha_0, T; D(\mathcal{A}_2))$ and :

$$\| u(t) \|_{W^{2,2}} \ \leq \ M_3 \quad , \quad \| B(t) \|_{W^{2,2}} \ \leq \ M_4 \qquad \forall\ t \in (\alpha_0, T) \ ,$$

where M_3, $M_4 \in I\!R$; M_3, $M_4 = \text{const} > 0$; M_3 , M_4 are independent of T .

Before turning to the main theorem we note some a-priori estimates.

Lemma 3.2:
(i) The embedding $W^{1,2}(\Omega, I\!R^2)$ into $L^p(\Omega, I\!R^2)$ is continuous for $p \in I\!R$, $1 \leq p < \infty$ and

$$\| \varphi \|_{L^p} \ \leq \ \gamma_1 \ \| \varphi \|_{W^{1,2}} \quad \forall\ \varphi \ \in \ W^{1,2}(\Omega, I\!R^2) \ ,$$

(γ_1 does not depend on φ).

(ii) The embedding $W^{2,2}(\Omega, I\!R^2)$ into $L^\infty(\Omega, I\!R^2)$ is continuous and

$$\| \varphi \|_{L^\infty} \;\leq\; \gamma_2 \;\; \| \varphi \|_{W^{2,2}} \qquad \forall \;\; \varphi \in W^{2,2}(\Omega, I\!R^2) \;\; ,$$

(γ_2 does not depend on φ).

Lemma 3.3: (Gronwall's Lemma)
Let y be a positive locally integrable function on $(0, \infty)$ such that dy/dt is locally integrable on $(0, \infty)$, and wich satisfy:

$$\frac{dy}{dt}(\tau) \;\; \leq a \cdot y(\tau) + b \qquad \forall \tau \geq 0 \quad ; \quad a, b = \text{const} > 0$$

$$\text{then :} \qquad y(\tau) \;\; \leq \;\; y(0) \cdot \exp(a \cdot \tau) + \frac{b}{a} \qquad \forall \tau \geq 0 \;\; .$$

Lemma 3.4: (Young's inequality)

$$a \cdot b \;\; \leq \;\; \frac{\varepsilon}{p} a^p \;\; + \;\; \frac{1}{q \cdot \varepsilon^{q/p}} b^q \;\; , \;\; \forall a, b, \varepsilon > 0, \;\; 1 < p, q < \infty, \;\; q = \frac{p}{p-1}.$$

4 Statement and Proof of the main theorem

From now on we assume f to be defined as in (3.1), $u_0 \in V_1$ and $B_0 \in V_2$. To state the main theorem we fix an integer $m \in I\!N$ and set for every solution (u, B) to (1.1) - (1.5)

$$u = p + q \qquad p = P^1_m u \qquad q = Q^1_m u \;\; ,$$

$$B = P + Q \qquad P = P^2_m B \qquad Q = Q^2_m B \;\; ,$$

and we show that after a transient period q is small in comparison with u and Q is small in comparison with B, supposing that m is sufficiently large.

Theorem 4.1:
If m is sufficiently large then for any solution to (1.1) - (1.5) there hold the estimate:

$$\| q(t) \|_{L^2} + \| Q(t) \|_{L^2} \ \leq \ \Big(\ \| q(0) \|_{L^2} + \| Q(0) \|_{L^2}^2 \ \Big) \exp(-k_0 t) \ + \ldots$$

$$\ldots + \ L_0(\lambda_{m+1}, \mu_{m+1}) \qquad \forall \ t \geq 0 \ ,$$

$$\text{where } k_0 = \mu_{m+1} \cdot \min\left(\frac{1}{Re} \cdot \frac{\beta_1}{\alpha_1 \beta_2}, \frac{1}{Rm} \right) \text{ and } L_0 \to 0 \text{ as } m \to \infty \ .$$

Proof:
We consider (2.6) and (2.7) with test functions $\varphi = q(t)$ and $\Phi = Q(t)$ $\forall \ t \in (0, T)$ and obtain because of the symmetrie of the projectors:

$$\frac{1}{2} \frac{d}{dt} \| q \|_{L^2}^2 \ + \ \frac{1}{Re} \| q \|_{V_1}^2 \ = \ \ldots \tag{4.1}$$

$$\ldots = \ - \ < \tilde{\mathcal{B}}(u, u), q > \ + \ S < \tilde{\mathcal{B}}(B, B), q > \ + \ < Q_m^1 f, q > \ ,$$

$$\frac{1}{2} \frac{d}{dt} \| Q \|_{L^2}^2 \ + \ \frac{1}{Rm} \| Q \|_{V_2}^2 \ = \ldots \tag{4.2}$$

$$\ldots = \ - < \tilde{\mathcal{B}}(u, B), Q > \ + \ < \tilde{\mathcal{B}}(B, u), Q > \ .$$

Next we add (4.1) and (4.2) and by virtue of (2.4) and Lemma 3.2 we estimate the right hand side :

$$\frac{1}{2}\frac{d}{dt}\parallel q\parallel_{L^2}^2 \;+\; \frac{1}{2}\frac{d}{dt}\parallel Q\parallel_{L^2}^2 \;+\; \frac{1}{Re}\parallel q\parallel_{V_1}^2 \;+\; \frac{1}{Rm}\parallel Q\parallel_{V_2}^2 \;\leq \ldots$$

$$\ldots \leq d_1(\parallel u\parallel_{L^4}\cdot\parallel u\parallel_{W^{1,2}} \;+\; S\cdot\parallel B\parallel_{L^4}\cdot\parallel B\parallel_{W^{1,2}})\cdot\parallel q\parallel_{L^4} \;+\ldots$$

$$\ldots + \; d_1(\parallel u\parallel_{L^4}\cdot\parallel B\parallel_{W^{1,2}} \;+\; \parallel B\parallel_{L^4}\cdot\parallel u\parallel_{W^{1,2}})\cdot\parallel Q\parallel_{L^4} \;+\ldots$$

$$\ldots + \parallel Q_m^1 f\parallel_{L^2}\cdot\parallel q\parallel_{L^2} \;\leq \ldots$$

$$\ldots \leq D_1 \;\parallel q\parallel_{W^{1,2}} \;+\; D_2\;\parallel Q\parallel_{W^{1,2}} \;+\; \parallel Q_m^1 f\parallel_{L^2}\cdot\parallel q\parallel_{W^{1,2}} \;\leq \ldots$$

$$\ldots \leq D_1\cdot\alpha_1\cdot\parallel q\parallel_{V_1} \;+\; D_2\cdot\beta_1\cdot\parallel Q\parallel_{V_2} \;+\; \alpha_1\cdot\parallel Q_m^1 f\parallel_{L^2}\cdot\parallel q\parallel_{V_1} \;\leq \ldots$$

(Young's inequality)

$$\ldots \leq \frac{1}{2}\;\mu_{m+1}^{1/2}\cdot Re\cdot D_1^2\cdot\alpha_1^2 \;+\; \frac{1}{2\cdot Re\cdot\mu_{m+1}^{1/2}}\parallel q\parallel_{V_1}^2 \;+\; \frac{1}{2}\cdot\mu_{m+1}^{1/2}\cdot Rm\cdot D_2^2\cdot\beta_1^2 \;+\ldots$$

$$\ldots + \;\frac{1}{2\cdot Rm\cdot\mu_{m+1}^{1/2}}\parallel Q\parallel_{V_2}^2 \;+\; \frac{1}{2}\cdot\mu_{m+1}^{1/2}\cdot Re\cdot\alpha_1^2\cdot\parallel Q_m^1 f\parallel_{L^2}^2 \;+\; \frac{1}{2\cdot Re\cdot\mu_{m+1}^{1/2}}\parallel q\parallel_{V_1}^2 \;,$$

where

$$D_1 \;=\; 2\cdot d_1\cdot\gamma_1^2\cdot\Big(\;M_1^2 \;+\; S\cdot M_1\cdot M_2\;\Big)\;,$$

$$D_2 \;=\; 4 \cdot d_1 \cdot S \cdot \gamma_1^2 \cdot M_1 \cdot M_2 \ .$$

From (4.1) and (4.2) we conclude:

$$\frac{d}{dt} \parallel q \parallel^2_{L^2} \;+\; \frac{d}{dt} \parallel Q \parallel^2_{L^2} \;+ \ldots$$

$$\ldots + \;\parallel q \parallel^2_{V_1} \left(\frac{2}{Re} - \frac{2}{Re} \mu_{m+1}^{-1/2} \right) \;+\; \parallel Q \parallel^2_{V_2} \left(\frac{2}{Rm} - \frac{1}{Rm} \mu_{m+1}^{-1/2} \right) \leq \ldots$$

$$\ldots \leq \; \mu_{m+1}^{1/2} \,(\; \alpha_1^2 \cdot Re \cdot D_1^2 \;+\; \beta_1^2 \cdot Rm \cdot D_2^2 \cdot \;+\; \alpha_1^2 \cdot Re \cdot \parallel Q_m^1 f \parallel^2_{L^2} \;) \ .$$

Hence, assuming that m is sufficiently large the last inequality yields:

$$\frac{d}{dt} \parallel q \parallel^2_{L^2} \;+\; \frac{d}{dt} \parallel Q \parallel^2_{L^2} \;+ \ldots$$

$$\ldots + \; \mu_{m+1} \cdot \min \left(\frac{1}{Re} \cdot \frac{\beta_1}{\alpha_1 \cdot \beta_2} \;,\; \frac{1}{Rm} \right) \left(\parallel q \parallel^2_{L^2} + \parallel Q \parallel^2_{L^2} \right) \;\leq \ldots$$

$$\ldots \leq \; \mu_{m+1}^{1/2} \;(\; \alpha_1^2 \cdot Re \cdot D_1^2 \;+\; \beta_1^2 \cdot Rm \cdot D_2^2 \;+\; \alpha_1^2 \cdot Re \cdot \parallel Q_m^1 f \parallel^2_{L^2} \;) \ .$$

Applying Gronwall's Lemma we obtain:

$$\parallel q(t) \parallel^2_{L^2} \;+\; \parallel Q(t) \parallel^2_{L^2} \;\leq\; \Big(\; \parallel q(0) \parallel^2_{L^2} + \parallel Q(0) \parallel^2_{L^2} \;\Big) \exp(-k_0 t) \;+ \ldots$$

$$\ldots + \frac{1}{k_0} \cdot \mu_{m+1}^{1/2} \left(\alpha_1^2 \cdot Re \cdot D_1^2 \; + \; \beta_1^2 \cdot Rm \cdot D_2^2 \; + \; \alpha_1^2 \cdot Re \cdot \| Q_m^1 f \|_{L^2}^2 \right)$$

$$\forall \ t \geq \alpha_0 > 0 \ , \text{ where } k_0 \; = \; \mu_{m+1} \cdot \min \left(\frac{1}{Re} \cdot \frac{\beta_1}{\alpha_1 \cdot \beta_2} \ , \ \frac{1}{Rm} \right) \ .$$

□

Theorem 4.2 :
If m is sufficiently large then for any weak solution to (1.1) - (1.5) there hold the estimates:
(i)

$$\| q(t) \|_{V_1}^2 \; \leq \; M_1^2 \cdot \exp(-t \cdot \lambda_{m+1}) \; + \; \frac{1}{k_1} \cdot L_1(\lambda_{m+1})$$

$\forall \ t \geq \alpha_0 > 0$, where $L_1 \longrightarrow 0$ as $m \longrightarrow \infty$,

(ii)

$$\| Q(t) \|_{V_2}^2 \; \leq \; M_2^2 \cdot \exp(-t \cdot \mu_{m+1}) \; + \; \frac{1}{k_2} \cdot L_2(\mu_{m+1})$$

$\forall \ t \geq \alpha_0 > 0$, where $L_2 \longrightarrow 0$ as $m \longrightarrow \infty$.

Proof:
ad (i) : Since u (and so q) are analytic in t with values in $D(\mathcal{A}_1)$ (2.6) yields :

$$\frac{d}{dt}u(t) \; + \; \frac{1}{Re}\mathcal{A}_1 u(t) \; + \; \hat{\mathcal{B}}(u(t), u(t)) \; - \; S \cdot \hat{\mathcal{B}}(B(t), B(t)) \; = \; f \text{ in } L^2. \tag{4.3}$$

Taking the scalar product of (4.3) with $\mathcal{A}_1 q(t)$ in L^2 for all $t \in (\alpha_0, T)$ we find:

$$\frac{1}{2}\frac{d}{dt} \parallel q \parallel_{V_1}^2 \ + \ \frac{1}{Re} \parallel \mathcal{A}_1 q \parallel_{L^2}^2 \ = \ldots \tag{4.4}$$

$$\ldots = - \ (\hat{\mathcal{B}}(u,u), \mathcal{A}_1 q) \ + \ S \cdot (\hat{\mathcal{B}}(B,B), \mathcal{A}_1 q) \ + \ (Q_m^1 f, \mathcal{A}_1 q) \ .$$

By virtue of (2.4), (2.5) we obtain :

$$\frac{1}{2}\frac{d}{dt} \parallel q \parallel_{V_1}^2 \ + \ \frac{1}{Re} \parallel \mathcal{A}_1 q \parallel_{L^2}^2 \ \leq \ldots$$

$$\ldots \leq d_1 \cdot \parallel u \parallel_{L^\infty} \cdot \parallel u \parallel_{V_1} \cdot \parallel \mathcal{A}_1 q \parallel_{L^2} \ + \ldots$$

$$\ldots + d_1 \cdot S \cdot \parallel u \parallel_{L^\infty} \cdot \parallel B \parallel_{V_2} \cdot \parallel \mathcal{A}_1 q \parallel_{L^2} \ + \ \parallel Q_m^1 f \parallel_{L^2} \cdot \parallel \mathcal{A}_1 q \parallel_{L^2} \ \leq \ldots$$

(Young's inequality)

$$\ldots \leq \frac{1}{2} D_3^2 \cdot \lambda_{m+1}^{1/2} \cdot Re \ + \ \frac{1}{2 \cdot \lambda_{m+1}^{1/2} \cdot Re} \parallel \mathcal{A}_1 q \parallel_{L^2}^2 \ ,$$

where

$$D_3 \ = \ d_1 \cdot \gamma_2 (M_3 \cdot M_1 \ + \ S \cdot M_3 \cdot M_2) \ + \ \parallel Q_m^1 f \parallel_{L^2} \ .$$

We conclude now :

$$\frac{d}{dt} \parallel q \parallel_{V_1}^2 \ + \ \parallel \mathcal{A}_1 q \parallel_{L^2}^2 \cdot \left(\frac{2}{Re} - \frac{1}{Re} \cdot \lambda_{m+1}^{-1/2} \right) \ \leq \ D_3^2 \cdot \lambda_{m+1}^{1/2} \cdot Re \ ,$$

and if m is sufficiently large :

$$\frac{d}{dt} \parallel q \parallel_{V_1}^2 \ + \ \lambda_{m+1} \cdot \parallel q \parallel_{V_1}^2 \cdot \frac{1}{Re} \ \leq \ D_3^2 \cdot \lambda_{m+1}^{1/2} \cdot Re \ .$$

Applying Gronwalls Lemma we obtain :

$$\| q(t) \|_{V_1}^2 \ \leq \ \| q(\alpha_0) \|_{V_1}^2 \cdot \exp(-t \cdot \lambda_{m+1}) \ + \ \frac{D_3^2 \cdot Re}{\lambda_{m+1}^{1/2}} \ \leq \ldots$$

$$\ldots \leq \ M_1^2 \ \exp(-t \cdot \lambda_{m+1}) \ + \ \frac{D_3^2 \cdot Re}{\lambda_{m+1}^{1/2}} \qquad \forall \ t \geq \alpha_0 > 0.$$

ad (ii) : Since B (and so Q) are analytic in t with values in $D(\mathcal{A}_2)$ we infer from (2.7) :

$$\frac{d}{dt} B(t) \ + \ \frac{1}{Rm} \mathcal{A}_2 B(t) \ + \ \hat{B}(u(t), B(t)) \ - \ \hat{B}(B(t), u(t)) \ = \ 0 \quad \text{in } Ł^2. \tag{4.5}$$

Taking the scalar product of (4.5) with $\mathcal{A}_2 Q(t)$ for all $t \in (\alpha_0, T)$ we find :

$$\frac{1}{2} \frac{d}{dt} \| Q \|_{V_2}^2 \ + \ \frac{1}{Rm} \| \mathcal{A}_2 Q \|_{L^2}^2 \ = \ldots$$

$$\ldots - \ (\hat{\mathcal{B}}(u, B), \mathcal{A}_2 Q) \ + \ (\hat{\mathcal{B}}(B, u), \mathcal{A}_2 Q) \ .$$

By virtue of (2.5) Lemma 3.2 we obtain :

$$\frac{1}{2} \frac{d}{dt} \| Q \|_{V_2}^2 \ + \ \frac{1}{Rm} \| \mathcal{A}_2 Q \|_{L^2}^2 \ \leq \ d_2 \cdot \| u \|_{L^\infty} \cdot \| B \|_{V_2} \cdot \| \mathcal{A}_2 Q \|_{L^2} \ + \ldots$$

$$\ldots + \ d_2 \cdot \| B \|_{L^\infty} \cdot \| u \|_{V_1} \cdot \| \mathcal{A}_2 Q \|_{L^2} \ \leq \ldots$$

(Young's inequality)

$$\ldots \leq \frac{1}{2} \mu_{m+1}^{1/2} \cdot Rm \cdot D_4^2 \ + \ \frac{1}{2 \cdot \mu_{m+1}^{1/2} \cdot Rm} \| \mathcal{A}_2 Q \|_{L^2} \ ,$$

where $\qquad D_4 = d_2 \cdot \gamma_2 \cdot (M_3 \cdot M_2 \ + \ M_4 \cdot M_1) \ .$

Therefore :

$$\frac{d}{dt} \parallel Q \parallel_{V_2}^2 \ + \ \parallel \mathcal{A}_2 Q \parallel_{L^2}^2 \cdot \left(\frac{2}{Rm} - \frac{1}{Rm \cdot \mu_{m+1}^{1/2}} \right) \ \leq \ \mu_{m+1}^{1/2} \cdot Re \cdot D_4^2$$

and if m is sufficiently large :

$$\frac{d}{dt} \parallel Q \parallel_{V_2}^2 \ + \ \mu_{m+1} \parallel Q \parallel_{V_2}^2 \cdot \frac{1}{Rm} \ \leq \ \mu_{m+1}^{1/2} \cdot Re \cdot D_4^2$$

Applying Gronwalls Lemma we obtain :

$$\parallel Q(t) \parallel_{V_2}^2 \ \leq \ \parallel Q(\alpha_0) \parallel_{V_2}^2 \cdot \exp(-t \cdot \mu_{m+1}) \ + \ \frac{Re \cdot D_4^2}{\mu_{m+1}^{1/2}} \ \leq \dots$$

$$\dots \leq M_2^2 \cdot \exp(-t \cdot \mu_{m+1}) \ + \ \frac{Re \cdot D_4^2}{\mu_{m+1}^{1/2}} \quad \forall \ t \geq \alpha_0 > 0 \ .$$

□

5 The approximate inertial manifold

As mentioned in the introduction we are going to improve the linear Galerkin-approximation from section 4 by definition of a finite dimensional manifold $\mathcal{M}_0$.
We consider the projection $u = p + q$ and $B = P + Q$ for any solution to (1.1) - (1.5) and write these equations equivalently as a coupled system of equations for p, q and P, Q :

$$\frac{d}{dt} p(t) \ + \ \frac{1}{Re} \mathcal{A}_1 p(t) \ + \ P_m^1 \tilde{\mathcal{B}}(p(t) + q(t), p(t) + q(t)) \ - \dots \tag{5.1}$$

$$\dots - \ S \cdot P_m^1 \tilde{\mathcal{B}} \left(P(t) + Q(t), P(t) + Q(t) \right) \ = \ P_m^1 f \quad \text{in } L^2 \ \forall \, t \in (0, T) \ ,$$

$$\frac{d}{dt} q(t) \ + \ \frac{1}{Re} \mathcal{A}_1 q(t) \ + \ Q_m^1 \tilde{\mathcal{B}} \left(p(t) + q(t), p(t) + q(t) \right) \ - \dots \tag{5.2}$$

$$\ldots - \quad S\cdot Q^1_m\tilde{\mathcal{B}}\left(P(t)+Q(t),P(t)+Q(t)\right) \;=\; Q^1_m f \quad \text{in } L^2 \;\forall\; t\in(0,T) \;,$$

$$\frac{d}{dt}P(t) \;+\; \frac{1}{Rm}\mathcal{A}_2P(t) \;+\; P^2_m\tilde{\mathcal{B}}\left(p(t)+q(t),P(t)+Q(t)\right) \;+\ldots \qquad (5.3)$$
$$\ldots + \;\; P^2_m\tilde{\mathcal{B}}\left(P(t)+Q(t),p(t)+q(t)\right) \;=\; 0 \quad \text{in } L^2 \;\forall\, t\in(0,T) \;,$$

$$\frac{d}{dt}Q(t) \;+\; \frac{1}{Rm}\mathcal{A}_2Q(t) \;+\; Q^2_m\tilde{B}\left(p(t)+q(t),P(t)+Q(t)\right) \;+\ldots \qquad (5.4)$$
$$\ldots + \;\; Q^2_m\tilde{\mathcal{B}}\left(P(t)+Q(t),p(t)+q(t)\right) \;=\; 0 \quad \text{in } L^2 \;\forall\, t\in(0,T) \;.$$

The results of chapter 4 show that an acceptable approximation to (5.2) and (5.4) is given by :

$$\frac{1}{Re}\mathcal{A}_1q(t) \;+\; Q^1_m\tilde{\mathcal{B}}(p(t),p(t)) \;-\; S\cdot Q^1_m\tilde{B}(P(t),P(t)) \;=\; Q^1_m f \;, \quad (5.5)$$

and

$$\frac{1}{Rm}\mathcal{A}_2Q(t) \;+\; Q^2_m\tilde{\mathcal{B}}(p(t),P(t)) \;+\; Q^2_m\tilde{\mathcal{B}}(P(t),p(t)) \;=\; 0 \;. \qquad (5.6)$$

So we are able to introduce in $H\times H$ the finite dimensional manifold $\mathcal{M}_0$. We define a mapping $\Phi_0 : P^1_mH\times P^2_mH \longrightarrow Q^1_mH\times Q^2_mH$ by setting :

$$(q_m,Q_m) \;=\; \Phi_0(p,P) \;:=\; \ldots$$

$$\ldots = \;\mathcal{A}_1^{-1}\left(Re[Q^1_m f \;+\; S\cdot Q^1_m\tilde{\mathcal{B}}(P,P) \;-\; Q^1_m\tilde{\mathcal{B}}(p,p)]\right) \;,$$

$$\mathcal{A}_2^{-1}\left(Rm[-Q^2_m\tilde{\mathcal{B}}(P,p) \;-\; Q^2_m\tilde{\mathcal{B}}(p,P)]\right) \;.$$

The manifold $\mathcal{M}_0$ is defined by :

$$\mathcal{M}_0 \;\; := \;\; \{ \;\; (p,P) \;\; + \;\; \Phi_0(p,P) \;\; ; \qquad p \in P_m^1 H \;\; , \;\; P \in P_m^2 H \;\; \}.$$

From (5.2) and (5.4) we infer :

$$\| \frac{d}{dt} q(t) \|_{L^2} \;\leq\; M_5 \qquad \forall \;\; t \in [0,T] \;\; ,$$

$$\| \frac{d}{dt} Q(t) \|_{L^2} \;\leq\; M_6 \qquad \forall \;\; t \in [0,T] \;\; .$$

M_5 , M_6 = const > 0 ; M_5 and M_6 are independent of T .

Estimating the distance of $(u(t), B(t))$ to $\mathcal{M}_o$ means to estimate the distance of (q,Q) to (q_m, Q_m) . To end this we subtract (q,Q) from (q_m, Q_m), using (5.2) , (5.4) and the definition of Φ_0 and obtain:

$$\frac{1}{Re} \cdot \lambda_{m+1} \cdot \| \, q_m(t) - q(t) \, \|_{L^2} \;\leq \ldots$$

$$\frac{1}{Re} \cdot \lambda_{m+1} \cdot \| \, q_m(t) - q(t) \, \|_{V_1} \;\leq\; \frac{1}{Re} \cdot \| \, \mathcal{A}_1 q_m(t) \;\; - \;\; \mathcal{A}_1 q(t) \, \|_{L^2} \;= \ldots$$

$$\ldots = \;\; S \cdot \| \, Q_m^1 \tilde{\mathcal{B}}(P(t), P(t)) - Q_m^1 \tilde{\mathcal{B}}(B(t), B(t)) \, \|_{L^2} \;\; + \ldots$$

$$\| \, Q_m^1 \tilde{\mathcal{B}}(p(t), p(t)) \;\; - \;\; Q_m^1 \mathcal{B}(u(t), u(t)) \, \|_{L^2} \;\; + \;\; \| \, \frac{d}{dt} q(t) \, \|_{L^2} \;\leq \ldots$$

$$\ldots \leq S \cdot d_1 \cdot \gamma_1 \cdot 2M_2^2 + d_1 \cdot \gamma_1 \cdot 2M_1^2 + M_5 =: D_5 \;\; ;$$

$$\frac{1}{Rm} \cdot \mu_{m+1} \cdot \| Q_m(t) - Q(t) \|_{L^2} \quad \leq$$

$$\leq \frac{1}{Rm} \cdot \mu_{m+1} \cdot \| Q_m(t) - Q(t) \|_{V_2} \leq \frac{1}{Rm} \cdot \| \mathcal{A}_2 Q_m(t) - \mathcal{A}_2 Q(t) \|_{L^2} = \dots$$

$$\dots = \| Q_m^2 \tilde{\mathcal{B}}(P(t), p(t)) - Q_m^2 \mathcal{B}(B(t), u(t)) \|_{L^2} + \dots$$

$$\dots + \| Q_m^2 \mathcal{B}(p(t), P(t)) - Q_m^2 \mathcal{B}(u(t), B(t)) \|_{L^2} + \| \frac{d}{dt} Q(t) \|_{L^2} \leq$$

$$\dots \leq d_1 \gamma_1 2 M_2 \cdot M_1 + d_1 \cdot \gamma_1 2 M_1 \cdot M_2 + M_6 =: D_6 \ .$$

So we have :

$$\| q_m(t) - q(t) \|_{L^2} \leq \lambda_{m+1}^{-1} \cdot Re \cdot D_5 \ ,$$

and

$$\| Q_m(t) - Q(t) \|_{L^2} \leq \mu_{m+1}^{-1} \cdot Rm \cdot D_6 \ .$$

6 Summary

A new method of approximating the solutions of the magnetohydrodynamic-equations for a long time by means of approximate inertial manifolds has been proposed.
This approximation scheme has been derived directly from the MHD-equations without any phenomenological considerations.
The last two inequalities of section 5 show that the distance between any solution to the MHD-equations and the approximate inertial manifold is smaller than the distance to the flat space $q = 0$ by a factor $\lambda_{m+1}^{-1/2}$ for the

velocity and $\mu_{m+1}^{-1/2}$ for the magnetic field.
Our arguments have been yielded an improvement of the distance to the manifold $\mathcal{M}_0$ in the L^2-norm. The estimates in the $W^{1,2}$-norm will be one of our subjects for further investigation.

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