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Roughening Interfaces in Deterministic Dynamics

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Abstract – Two deterministic processes leading to roughening interfaces are considered. It is shown that the dynamics of linear perturbations of turbulent regimes in coupled map lattices is governed by a discrete version of the Kardar-Parisi-Zhang equation. The asymptotic scaling behavior of the perturbation field is investigated in the case of large lattices. Secondly, the dynamics of an order-disorder interface is modelled with a simple two-dimensional coupled map lattice, possesing a turbulent and a laminar state. It is demonstrated, that in some range of parameters the spreading of the turbulent state is accompanied by kinetic roughening of the interface.

I. INTRODUCTION

Dynamics of growing and roughening interfaces have been intensively investigated recently[1]. This phenomenon is of importance in deposition, crystal growth, two-phase flow in porous media, etc. The universal equation governing the motion of the interface was derived by Kardar, Parisi and Zhang (KPZ)[2]:

$$\frac{\partial H}{\partial t} = \frac{\lambda}{2}(\frac{\partial H}{\partial x})^2 + \nu\frac{\partial^2 H}{\partial x^2} + \xi(x,t). \tag{1}$$

Here $H(x,t)$ is the local position of the interface, ν is viscosity (surface tension), λ is the nonlinearity parameter and ξ is the driving white Gaussian noise: $<\xi(x,t)\xi(x',t')>= D\delta(x-x')\delta(t-t')$. According to the KPZ equation (1), from the initially flat interface a rough surface develops, which width obeys a scaling law[2]

$$\xi^2 =< (H(x,t)- < H(x,t) >)^2 >\sim t^{2/3}. \tag{2}$$

This growth is faster than $\sim t^{1/2}$ which appears in a linearized KPZ equation (also called Edwards–Wilkinson equation[3]) resulting from (1) when $\lambda = 0$.

While the KPZ equation assumes external randomness, it has been recently demonstrated that deterministic systems with spatio-temporal chaos can have similar properties. In ref.[4] the Kuramoto-Sivashinsky equation

$$\frac{\partial H}{\partial t} = \frac{\lambda}{2}(\frac{\partial H}{\partial x})^2 - \frac{\partial^2 H}{\partial x^2} - \frac{\partial^4 H}{\partial x^4} \tag{3}$$

was investigated and it was shown that it has the same scaling properties as KPZ. Here the source of random roughening is the turbulent dynamics of the solution, which is effectively renormalized as noise for large scale modes[5]. A discrete dynamical analog of (1) was considered in ref.[6].

Here we consider two situations, where deterministic spatio-temporal chaos leads to roughening interfaces. In contrast to the approach of [6, 4], we do not model the KPZ equation itself, but demonstrate that it desribes the evolution of some deterministic nonlinear fields. In the first example, we do not even have an interface. In section II we study the dynamics of perturbations in a simple coupled map lattice model and show, that with a Hopf-Cole-type ansatz one obtains for the evolution of this perturbation field a discrete version of the KPZ equation[7]. In the second example (Section III) we consider a two-dimensional system, where both irregular (turbulent) and regular (laminar) regimes coexist. The dynamics of the interface between these states is investigated, and the regime of roughening is described.

II. DYNAMICS OF PERTURBATIONS IN ONE–DIMENSIONAL CML

The simplest model demonstrating spatial-temporal chaos is a coupled map lattice (CML) [8, 9, 10]. In this model a field $u(x,t)$ that depends on discrete space $x = 1, 2, ..., L$ and time $t = 0, 1, 2, ...$ obeys an evolution equation

$$u(x, t+1) = f(\hat{D}(\varepsilon)u(x,t)). \tag{4}$$

Here $f(\cdot)$ is a nonlinear transformation and $\hat{D}$ is a linear operator depending on the coupling parameter ε. A widely used choice for $\hat{D}$ corresponds to the nearest–neighbor interaction of diffusive type:

$$\hat{D}(\varepsilon)v(x) = \varepsilon v(x-1) + (1-2\varepsilon)v(x) + \varepsilon v(x+1). \tag{5}$$

If the mapping $u \to f(u)$ is chaotic, spatiotemporal chaos is typically observed in the distributed system (4) [8, 9, 10]. In order to study perturbations of a turbulent state $u^0(x,t)$, we linearize (4) and get for the evolution of the perturbation $w(x,t)$

$$\begin{aligned} w(x,t+1) &= a(x,t)\hat{D}(\varepsilon)w(x,t), \\ a(x,t) &= f'(\hat{D}(\varepsilon)u^0(x,t)). \end{aligned} \tag{6}$$

Our goal is to study the statistical properties of the perturbation field w for large system size L and time t.

The main idea is the similarity of eq.(6) to the KPZ equation. Indeed, eq. (6) may be considered as a discrete analog of the diffusion equation with multiplicative noise

$$\frac{\partial W}{\partial t} = \xi(x,t)W + R\frac{\partial^2 W}{\partial x^2}. \tag{7}$$

This equation with the ansatz $W = \exp(H)$ is transformed to the KPZ equation (1). If the KPZ equation is derived from eq. (7), one has $\lambda = 2R, \quad \nu = R$.

We now explore the analogy between the discrete equation (6) and the multiplicative noise equation (7) and apply the ansatz

$$w(x,t) = e^{h(x,t)}. \tag{8}$$

Then we get from eq.(6) a discrete analog of the KPZ equation:

$$\begin{aligned} h(x,t+1) - h(x,t) = \ln a(x,t) + \\ + \ln[1 - 2\varepsilon + \varepsilon \exp(h(x-1,t) - h(x,t)) + \varepsilon \exp(h(x+1,t) - h(x,t))]. \end{aligned} \tag{9}$$

It is worth noting that for the discrete case there is an important restriction in performing the ansatz (8), namely, $w(x,t)$ must be positive for all x, t. In the continuous case this can be ensured by a proper choice of the initial field, while in the discrete case also the condition $a(x,t) > 0$ must be fulfilled for all x, t.

It follows from (8) that the exponential growth of the field $w(x,t)$ in time corresponds to the linear motion of the interface position $h(x,t)$; the mean velocity is exactly the Lyapunov exponent. Except for this mean motion, the interface $h(x,t)$ also fluctuates (due to fluctuations of $a(x,t)$) and we now can investigate these fluctuations using the correspondence to the KPZ equation.

Because ε is an effective diffusion constant related to R in eq.(7), eq.(9) corresponds to the KPZ equation (1) with

$$\lambda = 2\varepsilon, \quad \nu = \varepsilon.$$

Note that the parameter ε is the diffusion constant both in the KPZ equation and in the discrete eq.(6). The parameter λ in the KPZ equation describes the change in the growth rate of the tilted interface. For the discrete eq. (6) this corresponds, because of the ansatz (8), to the change of the Lyapunov exponent when exponentially growing in space perturbations are considered; such generalized Lyapunov exponents have been introduced recently by Politi and Torcini [11]. The problem remains in finding a value for the noise strength D. The values of $a(x,t)$ are produced by chaotic motions in the CML (4) and, of course, are neither Gaussian nor δ – correlated. These differences are, however, not important if the asymptotic behavior coincides with that predicted by the KPZ equation. While a large number of models belong to the universality class of KPZ equation, we have to check this for the perturbation field in CML once more.

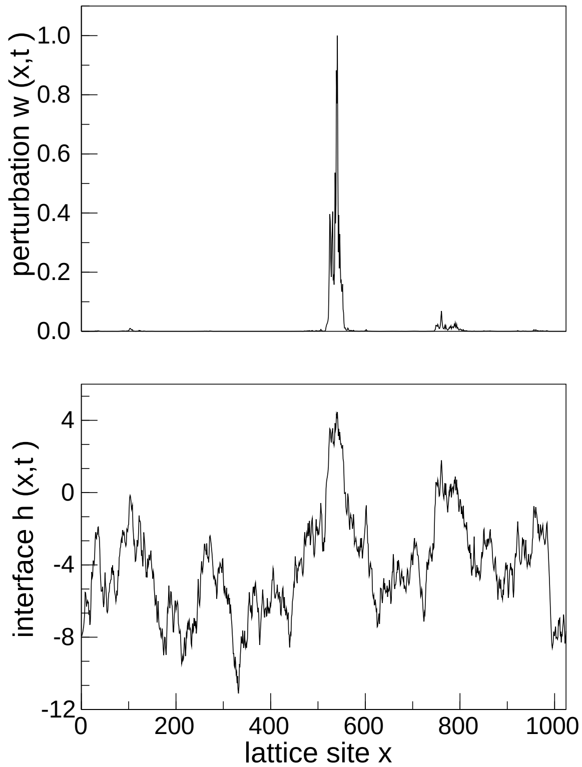

Figure 1: Snapshot of the fields $w(x,t)$ and $h(x,t)$ for the CML eqs. (4-6),(10) with $L = 1024$, $\varepsilon = 0.1$, $c = 4$.

We used in the numerical calculations the following "skewed" doubling transformation

$$f(u) = \begin{cases} bu & \text{for } \ 0 \leq u < c^{-1} \\ \frac{c}{c-1}u & \text{for } \ c^{-1} \leq u \leq 1 \end{cases} \tag{10}$$

In this transformation the local instantaneous expansion rate $a(x,t)$ takes the values c and $c(c-1)^{-1}$, so varying the parameter c we can consider both cases of weak ($c \approx 2$) and strong ($c >> 1$) noise. Numerical simulation shows that the CML (4) –(6), (10) indeed demonstrates properties of the KPZ equation. If a system of finite length L is considered, then for sufficiently large t a statistically stationary roughened interface appears (Fig. 1)[we consider here only statistical properties of the interface's fluctuations, thus its mean position is always subtracted]. The probability distribution density of h obeys a Gaussian law, and the spatial spectrum scales as k^{-2}, as is expected from the KPZ equation[12].

It is worth noting that the observed field $w(x,t)$ demonstrates highly intermittent properties, as one can see from Fig. 1. In fact, what is observed in the w vs. x graph is a narrow region near the maximum of the field $h(x,t)$, due to the exponent in (8). This narrow region moves in space irregulary, as is shown in Fig. 2.

For the KPZ equation a scaling growth of the width of the interface (starting from the flat one) is given by eq.(2). However, as was mentioned in [4], this scaling is observed only for large times

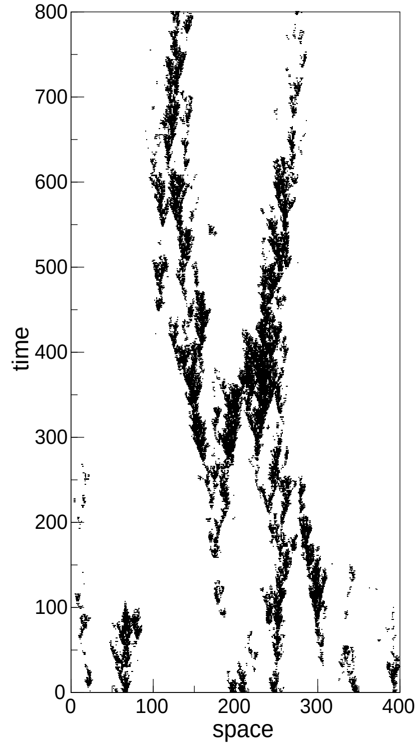

Figure 2: Spatio-temporal dynamics of the perturbation field. The sites, where the field exceeds a threshold, are marked with dark squares.

and for long systems, because only for large t and L the nonlinear term in the KPZ equation dominates. Applying the estimations of ref.[4] to the CML model (4) –(6), (10), we conclude that the scaling (2) may be observed only for systems with sufficiently large c (large noise) and small

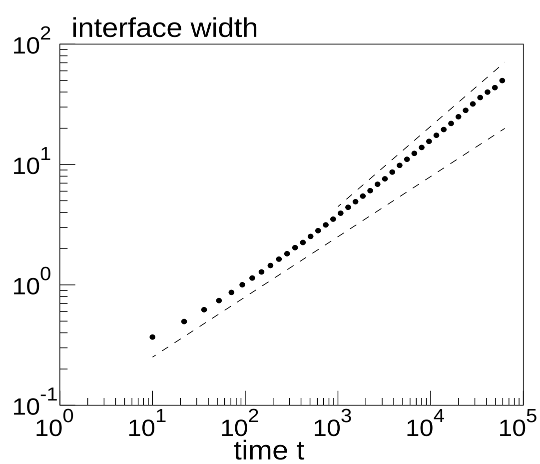

Figure 3: Growth of the "interface width" ξ^2 in the CML with $\varepsilon = 0.1, \quad c = 5, \quad L = 8000$. The broken lines have slopes 0.5 and 2/3. One can see a crossover to a nonlinear regime at $t \approx 10^3$.

ε (small diffusion). In Fig. 3 the results of the simulations with $c = 5$, $\varepsilon = 0.1$ are presented. The observed exponent is clearly larger than the value 0.5 predicted by linear theory[4], but still slightly less then the asymptotic KPZ value 2/3, probably due to still insufficient length of the system.

III. ROUGHENING OF ORDER–DISORDER INTERFACE IN TWO-DIMENSIONAL CML

In this section, the roughening of an interface in a two-dimensional CML is investigated. This system may be regarded as an example for the propagation of a turbulent regime into a laminar one. We analyse here the behavior of the interface between these two regimes. (The dynamics of complex interface between different laminar states has been recently described in [13].) As a model, we take a two-dimensional CML where each site is coupled diffusively with its eight nearest spatial neighbours:

$$u_{t+1}^{x,y} = (1-\frac{5}{4}\varepsilon)f(u_t^{x,y}) + \frac{\varepsilon}{4}[f(u_t^{x-1,y}) + f(u_t^{x+1,y}) + f(u_t^{x,y-1}) + f(u_t^{x,y+1})] + \frac{\varepsilon}{16}[f(u_t^{x-1,y-1}) + f(u_t^{x-1,y+1}) + f(u_t^{x+1,y-1}) + f(u_t^{x+1,y+1})].$$

For the map $f(u)$ we choose a somewhat modified tent-map, in order to be able to prepare an initial state with a well defined interface between an ordered and a disordered regime. The map is given by equation (11)

$$f(u) = \max\left[-b\,,\,\left(1-2\left|u-\frac{1}{2}\right|\right)\right] \quad , \qquad b > 0 \tag{11}$$

and is visualized in Fig. 4.

Two regimes are possible in this map: a steady state given by a superstable fixed point $u = -b$, and a chaotic state for which $0 < u < 1$ and the dynamics is governed by the usual

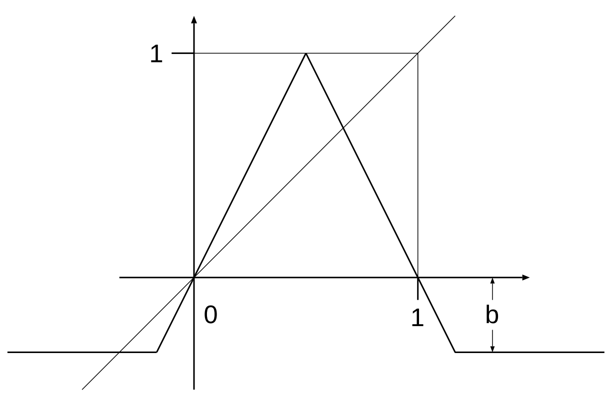

Figure 4: Visualization of the modified tent map.

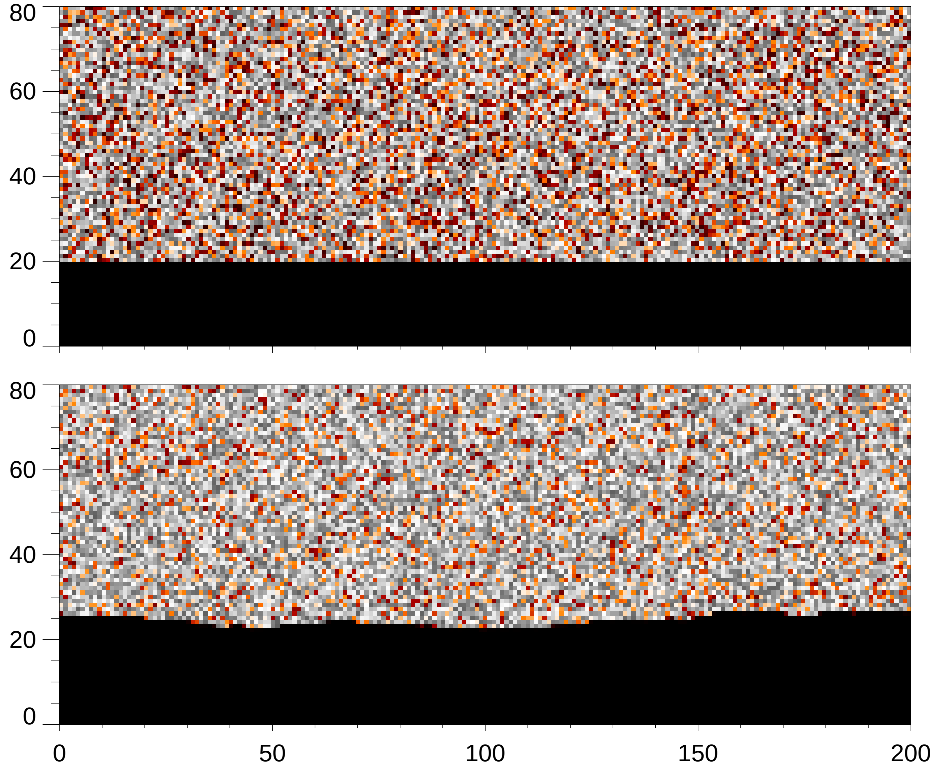

Figure 5: Top panel: Initially prepared smooth interface in a 2d CML. The black squares denote maps with values of $-b$, while the gray-scaled squares are randomly initialized with values between 0 and 1. In the figure only a part of a lattice with dimensions $L_x = 1000$ and $L_y = 80$ is shown. Bottom panel: The interface after $t = 10000$ iterations, $b = 0.05, \varepsilon = 0.0905$. It can be seen clearly, that the initially smooth interface has roughened.

tent map. Choosing initial conditions according to these states, we can easy prepare a flat order-disorder interface (see Fig. 5). The boundary conditions for all computations are periodic in the x direction parallel to the interface and open in "flow" direction y.

We have fixed the parameter $b = 0.05$ and investigated the dynamics of the interface for different lengths of the system and coupling constants ε. A typical situation of the evolution of the interface after some time is shown in Fig. 5. At each time t the local position of the interface $h(x)$ is defined as

$$h(x) = \min\{y : \ u(x,y) > 0\}.$$

The averaged position of the interface and the velocity are given by

$$< h_t >= \frac{1}{L_x}\sum_{x=1}^{L_x} h_t(x), \qquad v = \lim_{t\to\infty} \frac{|< h_t > -h_0|}{t}.$$

The width is defined as

$$\xi_t^2 = \frac{1}{L_x}\sum_{x=1}^{L_x}(h_t(x) - < h_t >)^2.$$

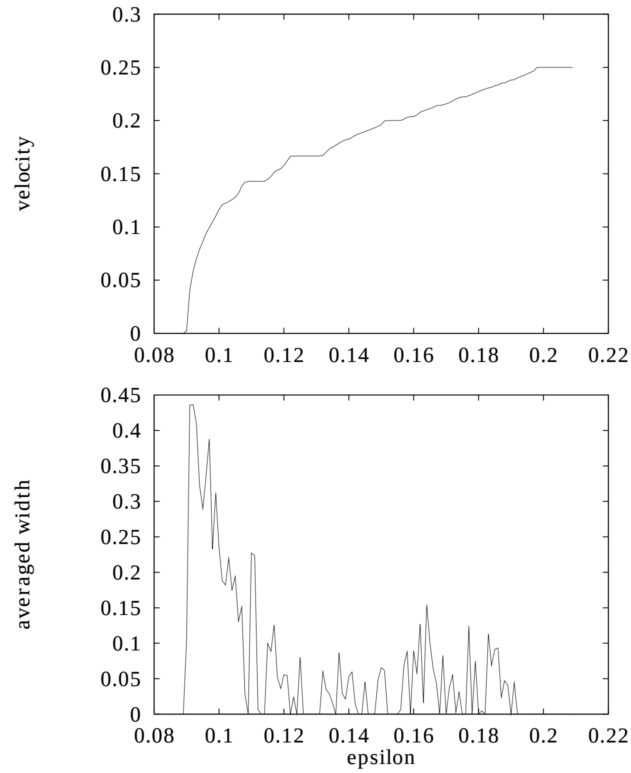

Figure 6: Velocity and width of the interface in dependence on the coupling strength ε, obtained at $t = 10^4$. It can be seen clearly that the velocity shows a step-like structure.

In Fig. 6 the velocity of the interface in dependence on the coupling ε is shown. The size of the lattice in these computations was $L_y = 50$ and $L_x = 100$. The curve shows averaging over 10 different initial conditions.

Below some critical value $\varepsilon_c \approx 0.09$ the interface does not move at all: the coupling is too small to make an excitation of an ordered state by neighbouring disordered states possible. The interface for $\varepsilon < \varepsilon_c$ remains flat. Above this threshold the velocity of the interface increases rapidly. Furthermore, it shows a staircase-like structure with well defined plateaus. These steps correspond to rational values of the velocity ($v = 1/4,\ 1/5$, etc.). It is not yet clear, however, whether the staircase is complete (like devil's staircase in the circle map) or the steps coexist with continuous regions[14].

The bottom panel of Fig. 6 presents the averaged width at $t = 10^4$. It can be seen that the roughening of the interface is most dominant just after ε_c where the interface starts to move. Furthermore one can see a window-like structure, which means that the interface shows no roughening in regions where the velocity belongs to a plateau. Above some particular value of ε, when the velocity becomes large enough, the interface shows no roughening.

Consider now the situation where the roughening of the interface is maximal, i.e. near the threshold ε_c. All the following calculations are therefore done with a value $\varepsilon = 0.0905$ of the coupling constant. In Fig. 7 the scaling of the width of the interface with time is shown. It

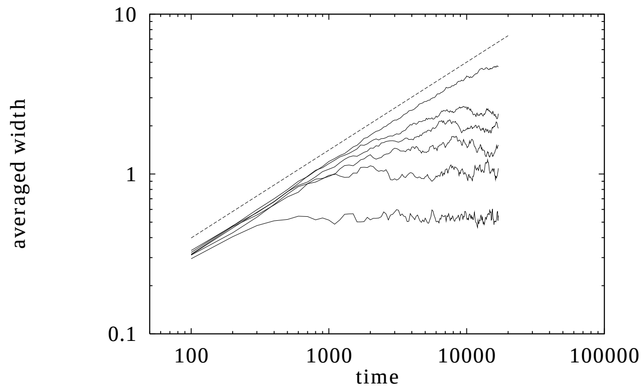

Figure 7: Growth of the interface width with time on a double logarithmic scale. The slope of the dashed curve is ≈ 0.55. The different curves (showing an averaging over 100 initial conditions) belong to lattices with different lenghts L_x, from bottom to top: $L_x = 100, 200, 300, 400, 500, 1000$.

can be seen that the width ξ^2 of the interface saturates for $t \to \infty$. This value of the saturated width ξ_s^2 is dependent on the size L_x of the lattice and grows linearly with the length L_x. The growth of the interface shows a power law scaling with an exponent ≈ 0.55 (see Fig. 7). This means that the model presented behaves nearly like the linearized version of the KPZ equation (the Edwards-Wilkinson equation). Indeed, the nonlinearity in the KPZ equation describes dependence of the local velocity on the tilting of the interface. In our case, even for the largest size $L_x = 1000$ the width was of order $\xi \approx 3$, so the tilt $\xi/L_x \approx 0.003$ is obviously too small to provide nonlinear effects. Nonlinear corrections could be relatively important for very large lattices and correspondingly large times, what is, however, beyond our computational facilities.

IV. DISCUSSION

In conclusion, we have presented two coupled map lattice models where roughening interfaces are observed. The first model describes the dynamics of the perturbation field in spatiotemporal chaos. The intermittency observed for this perturbation field seems to be a general phenomenon, not restricted to the particular CML considered in section II. To verify this hypothesis detailed investigations of other types of CMLs and PDEs are needed. In the second part we have studied the dynamics of order-disorder interfaces. An interesting feature is the alternating sequence of parameter intervals where the interface roughens or stays flat. This fine structure is caused, probably, by the anisotropy of correlations in the 2D CML due to the front motion. A detailed study of these correlations will be presented elsewhere. Finally we would like to mention that CMLs may be considered not only as toy models, but as promising tools to get a deeper understanding of the underlying mechanisms of complex structures in high dimensional systems.

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