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Semiclassical asymptotics for the scattering amplitude in the presence of focal points at infinity

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Contents

Introduction and statement of the main result

The subject of this work is potential scattering in $\mathbb{R}^n$. Quantum-mechanically, we seek solutions of the Schrödinger equation

$$P\psi = P(h)\psi := -h^2\Delta\psi + V\psi = \lambda\psi, \qquad x \in \mathbb{R}^n,\ 0 < h < 1,\ n \geq 2, \tag{1}$$

while classically we seek integral curves (phase trajectories) of the vector field H_p generated by the hamiltonian

$$p(x,\xi) = |\xi|^2 + V(x), \qquad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2}$$

In either case we will take V to be a smooth function on configuration space $\mathbb{R}^n_x$ as follows.

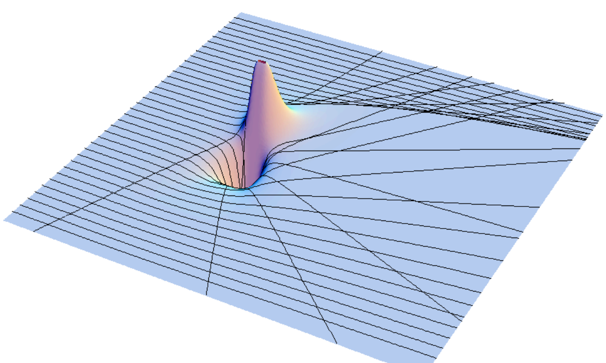
Potential Hypothesis The potential $V \in C^\infty(\mathbb{R}^n)$ satisfies

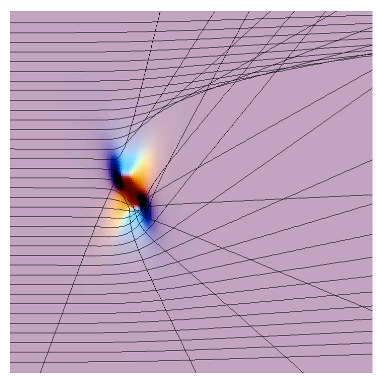
$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\varrho-|\alpha|}, \qquad \varrho > 1, \tag{3}$$

for any multi–index $\alpha \in \mathbb{N}^n$ and a corresponding constant $C_\alpha > 0$, where $\langle x\rangle := \sqrt{1+|x|^2}$.

Thus V is a short-range potential in the sense of Agmon [1]. In particular, $V(x)$ decays faster than the Coulomb potential as $|x| \to \infty$.

We will interest ourselves in the semiclassical asymptotics of the scattering amplitude, an object which we will describe in detail below. In the classical picture, we first fix an "incoming direction" $\omega_- \in \mathbb{R}^n$ and a hyperplane $\mathbb{H}$ orthogonal to ω_-, the "impact plane". For $V \equiv 0$, the trajectories of the hamiltonian vector field are straight lines in phase space, and we refer to them as "free". We will consider the ensemble of trajectories that are asymptotic, for time $s \to -\infty$, to free trajectories that intersect $\mathbb{H}$ orthogonally. It turns out that this association is bijective, and we hence obtain a parametrisation of all trajectories by points in $\mathbb{H}$, denoting such trajectories by $\mathcal{T}_z$, $z \in \mathbb{H}$.





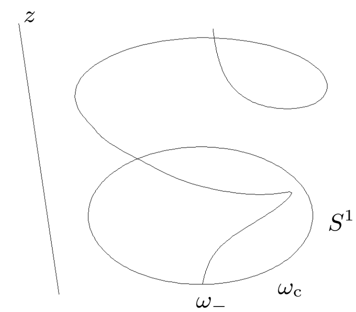
The image on the left shows some classical configuration-space trajectories for scattering in $\mathbb{R}^2$, superimposed on the graph of the potential

$$V(x_1,x_2) = \frac{20x_1}{10 + x_1^4 + x_2^4}. \tag{4}$$

For $s \to -\infty$, the trajectories approach parallel straight lines. On the right, a bird's-eye view of the same image is shown. As time $s \to +\infty$, the trajectories approach straight lines,

$$x(s,z) \sim \omega_+(z)s + r_+(z) \qquad \text{in configuration space.}$$

where $z \in \mathbb{H}$ parametrises the trajectories and $s \in \mathbb{R}$ is the time (flow), coordinate of the integral curve. We call $\omega_+(z) \in S^{n-1}$ the "outgoing direction" of the trajectory $\mathcal{T}_z$, where $S^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^n$. Of particular interest are points for which the differential of the map $z \mapsto \omega_+(z)$ has rank less than $n-1$ (recall that $\dim \mathbb{H} = n-1$).



For V given by (4), the right-hand figure shows the graph in $S^1 \times \mathbb{R}$ of the relation $z \mapsto \omega_+(z)$, $z \in [-14, 14]$, drawn as a line over S^1. The incoming direction ω_- is indicated, and the projection onto S^1 of points of the graph approaches ω_- as $|z|$ becomes large. At ω_c the projection onto S^1 is singular, i.e., the rank of the differential of the projection is zero.
An outgoing direction $\omega_0 \in \omega_+(\mathbb{H}) \subset S^{n-1}$, $\omega_0 \neq \omega_-$, is called regular (as in [24]) if for all $z \in \mathbb{H}$ with $\omega_+(z) = \omega_0$ the differential $d\omega_+|_z$ has rank $n-1$. Hence all outgoing directions except ω_c and ω_- are regular in this example.

We now review the definition of the scattering amplitude. We denote by $H^2(\mathbb{R}^n)$ the Sobolev space of second order and define the operators

$$P_0 = -h^2\Delta, \qquad P = P_0 + V, \qquad \operatorname{dom} P_0 = \operatorname{dom} P = H^2(\mathbb{R}^n), \qquad 0 \le h \le 1. \tag{5}$$

The Potential Hypothesis guarantees (cf. Agmon [1, Theorem 7.1]) that the wave operators

$$W_\pm := \operatorname*{s-lim}_{t \to \pm\infty} e^{i\frac{t}{h}P} e^{-i\frac{t}{h}P_0} \tag{6}$$

exist and are complete, i.e.,

$$\operatorname{ran} W_+ = \operatorname{ran} W_- = L^2(\mathbb{R}^n)_{\mathrm{ac}}, \tag{7}$$

where $L^2(\mathbb{R}^n)_{\mathrm{ac}}$ denotes the absolute continuity subspace with respect to P. The scattering operator can then be defined as the unitary operator

$$S(h) \colon L^2(\mathbb{R}^n_x) \to L^2(\mathbb{R}^n_x), \qquad S(h) := W_+^* W_-. \tag{8}$$

Using the $1/h$–Fourier transform $\mathcal{F}_h$ of (D.102), we define for $\gamma > \frac{1}{2}$,

$$F_0(\lambda, h) \colon L^2_\gamma(\mathbb{R}^n) \to L^2(S^{n-1}), \qquad (F_0(\lambda, h)g)(\omega) := \lambda^{(n-2)/4} (\mathcal{F}_h g)(\sqrt{\lambda}\omega). \tag{9}$$

Here $L^2_\gamma(\mathbb{R}^n)$ denotes the space of weighted square integrable functions, i.e., all functions f such that $\|\langle\,\cdot\,\rangle^\gamma f\|_{L^2} < \infty$. The smoothness of V allows us to find a unitary operator $S(\lambda, h)$ on $L^2(S^{n-1})$ such that

$$F_0(\lambda, h) S(h) g = S(\lambda, h) F_0(\lambda, h) g, \qquad \text{for all } \lambda > 0,\ g \in L^2_\gamma(\mathbb{R}^n_x),\ \gamma > \frac{1}{2}. \tag{10}$$

The operator $S(\lambda, h)$ is called the scattering matrix. The transition matrix $T(\lambda, h)$, defined through

$$S(\lambda, h) = T(\lambda, h) - 2\pi i I, \tag{11}$$

(I denotes the unit operator) is a compact operator on $L^2(S^{n-1})$. Furthermore, under the Potential Hypothesis, it can be shown (Isozaki and Kitada, [16, Theorem 0.1]) that $T(\lambda, h)$ is an integral operator on $L^2(S^{n-1})$,

$$(T(\lambda, h)g)(\omega) = \int T(\omega, \omega'; \lambda, h) g(\omega')\, d\omega', \tag{12}$$

where the kernel $T(\omega, \omega'; \lambda, h)$ is smooth for $\omega \neq \omega'$ and $\lambda > 0$. We define the scattering amplitude $f(\omega_-, \omega_+; \lambda, h)$ as

$$f(\omega_-, \omega_+; \lambda, h) := c_{n,k,h} T(\omega_-, \omega_+; \lambda, h), \tag{13}$$

with $c_{n,\lambda,h} = -2\pi(\sqrt{\lambda}/2\pi h)^{(n-1)/2} e^{-i(n-3)\frac{\pi}{4}}$.

In accordance with the classical picture described above, we fix ω_- and analyse the semiclassical asymptotics as $h \to 0$ of $f(\omega_-, \omega_+; \lambda, h)$ for varying ω_+. Our goal will be to give the leading term of these asymptotics, i.e., the behaviour of f modulo $O(h)$, where here and throughout $O(h)$ refers to function of ω_+ whose modulus is bounded by a constant multiplied by h when $h \to 0$. Before stating our result, we first review the available literature on this subject.

The problem of finding a semiclassical expansion for the scattering amplitude in this setting was considered by Vainberg [26, 27] in the case of a compactly supported potential V and regular scattering directions. Protas [21] then expanded Vainberg's approach to include non-regular points in the setting of compact potentials, but without giving the leading term of the asymptotics as $h \to 0$ explicitly, see [12].

Using the phase functions introduced by Isozaki and Kitada [14, 15, 16] for Schrödinger operators, Robert and Tamura [24] gave a formula for the scattering amplitude for short-range potentials, generalising Vainberg's result but valid only for regular outgoing directions. Robert and Tamura's techniques have been used by Brummelhuis and Nourrigat [9] to give an analogous formula for regular scattering directions in the case of a Dirac operator with compact potential.

All of the above approaches use the theory of the Maslov operator developed by Maslov, presented, e.g., in [18], [27] and [20]. Following the idea of Protas, we define the lagrangian manifold $\mathcal{L}_+ \subset T^*S^{n-1}$ consisting of the asymptotic angles and angular momenta of the trajectories as time $s \to +\infty$. We build on Robert and Tamura's general representation formula for the scattering amplitude in the short-range case and show that in this case the scattering amplitude can be expressed in leading order as a simple Maslov operator on $\mathcal{L}_+$.

Since non-regular outgoing directions correspond to lagrangian singularities in $\mathcal{L}_+$ (see Lemma 3.3.19), this allows for a full consideration of generically occurring caustics, which in sufficiently small dimensions have been classified by Arnol'd [5, 6, 7].[1] In fact, lagrangian singularities and the asymptotics of oscillating integrals with corresponding phase functions (especially in the simplest case of a fold singularity and the Airy function) have been extensively studied, see for instance the textbooks by Arnol'd [8], Guillemin and Sternberg [11] or Taylor [25, Chapter 6.7]. This has, however, heretofore not been the case in conjunction with asymptotics of the quantum mechanical scattering amplitude defined in (13), and we give a short summary of the situation in the physically relevant cases of scattering in $\mathbb{R}^2$ and $\mathbb{R}^3$. This is motivated by the fact that the effect of caustics on the scattering amplitude has heretofore been treated confusedly or simply been avoided in the physics literature. For example, Jung and Pott [17] mention the problem posed caustics for the scattering amplitude rather clearly, but then apply a "uniformization" close to caustics and apparently avoid discussing what the precise effects of caustics are.

There has recently been renewed interest in this problem. Alexandrova [2] has given a generalisation of Vainberg's and Protas' work for compactly supported perturbations of the Laplacian. Michel [19] has generalised [24, Theorem 1] to hold for a slightly weaker non-trapping condition than Robert and Tamura had assumed, under additional assumptions on the resonances of the Schrödinger operator (see Section 3.4 for more details).

We will work directly from the article of Robert and Tamura [24], which gives the leading-order semiclassical term of $f(\omega_-, \omega_+; \lambda, h)$ for fixed ω_- and ω_+ in the case that $\omega_+ \neq \omega_-$ is regular. Numerous results from [24] will be cited in our text.

In Chapter 1 we establish some basic definitions, notation and the setting, introducing the impact plane $\mathbb{H}$ (orthogonal to ω_-) and the hamiltonian $p \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ associated to $P(h)$. We prove some important estimates concerning the behaviour of the integral curves of the hamiltonian vector field H_p as the time $s \to \pm\infty$. These estimates in Propositions 1.2.7 and 1.2.10 are crucial to the geometrical constructions in Propositions 2.1.4 and 2.4.6. In Section 1.3 we take a more geometrical viewpoint, showing that the integral curves of the hamiltonian system form a lagrangian manifold $\Lambda \subset T^*\mathbb{R}^n$ (Theorem 1.3.3). The main difficulty there is that instead of the usual initial conditions, we have "initial conditions at $s \to -\infty$".

In Chapter 2, we first show that the limit as $s \to +\infty$ of the angle and angular momentum of points on trajectories exists. The ensemble of the asympotic angles and angular momenta (one for each trajectory) form a lagrangian manifold $\mathcal{L}_+ \subset T^*S^{n-1}$ (Theorem 2.1.4). In Section 2.2 we review the definition of phase functions of Isozaki and Kitada [14] and some results of Robert and Tamura, who apply the phase functions in our setting. In Section 2.3, we obtain generating functions on Λ and $\mathcal{L}_+$ (Lemmas 2.3.7 and 2.3.8), which we will use in the construction of Maslov operators on Λ and $\mathcal{L}_+$ (see Appendix D). In Section 2.4 we analyse the relationship between caustics in $\mathcal{L}_+$ and caustics in Λ, culminating in Proposition 2.4.6.

[1] By a slight abuse of language, we will refer to both $p \in \mathcal{L}_+$ and πp as caustics if rank $d\pi|_p$ is not maximal, where π denotes the canonical projection onto the base.

After these preparatory results, we commence in Chapter 3 with the generalisation of Robert and Tamura's representation formula for $f(\omega_-, \omega_+; \lambda, h)$. After reviewing their results in Section 3.1, we use a Maslov operator to approximate the action of $e^{\frac{i}{h}Pt}$ for $t \in [0, T_0]$ for some $T_0 > 0$. These estimates, using the extended phase space, follow ideas of Maslov [18] which we further refine for our situation, obtaining Proposition 3.2.16. In Section 3.3 we apply this result to the integral formula for the scattering amplitude, proceeding with some elaborate constructions before applying the method of stationary phase. In principle, this construction follows the ideas of Robert and Tamura [24], but is made much more complicated by the presence of caustics. We complete the proof of Theorem 1, which we state below.

Section 3.4 contains a short discussion of the results obtained, as well as some open question, principally on the structure and presence of caustics. In Section 3.5 we review the well-known results on lagrangian singularities and discuss their application to our scattering problem. Finally, the Appendices generally serve as a repository of results referred to in the text, Appendix D contains details on the construction of Maslov operators on lagrangian manifolds.

Definition 1 *Let $\mathcal{L}_+ \subset T^*S^{n-1}$ with global coordinate map S^+_λ be the lagrangian manifold of Theorem 2.1.4 and F_+ the global generating function of Lemma 2.3.8. Denote by $\pi_\omega \colon T^*S^{n-1} \to S^{n-1}$ the canonical projection onto the base and let $\{(\Sigma_j, \chi_j)\}$, $\Sigma_j \subset S^{n-1}$, $\chi_j \colon S^{n-1} \to \mathbb{R}^{n-1}$, be an atlas on S^{n-1}.*

Let $\{(\Gamma_k, \pi_{\Gamma_k, I_k})\}_{k\geq 0}$ denote some lagrangian atlas on $\mathcal{L}_+$ with $I_k \subset \{1, \dots, n-1\}$ as in Definition 2.3.2 such that for all $k \geq 0$, $\Gamma_k \subset T^*\Sigma_{i_k}$ for some i_k and Γ_0 is simply connected and well-projected.

Let $\{e_k\}$, $e_k \in C^\infty_0(\Gamma_k)$ be some partition of unity subordinate to $\{\Gamma_k\}$ and define functions $\{g_k\}$, $g_k \in C^\infty_0(\Sigma_{i_k})$, such that $g_k = 1$ on $\pi_\omega\Gamma_k$. Define a Maslov operator on $\mathcal{L}_+$ using these data following Definitions D.1, D.5. In particular,

$$K_{\mathcal{L}_+} \colon C^\infty(\mathcal{L}_+) \to C^\infty(S^{n-1}), \qquad K_{\mathcal{L}_+}[\varphi] := \sum_k e^{i\frac{\pi}{2}\gamma_k} K_{\Gamma_k, I_k}[e_k \varphi], \tag{14}$$

where K_{Γ_k, I_k} denotes a local Maslov operator on Γ_k as in (D.105) and γ_k is the index of the chain of charts joining Γ_k to Γ_0, cf. Definition D.3.

Theorem 1 *Let $K_{\mathcal{L}_+}$ be a Maslov operator constructed on $\mathcal{L}_+$ as in Definition 1 and let μ_0 denote the Keller-Maslov-Morse path index of some trajectory $\mathcal{T}_z$ with $S^+_\lambda(z) \in \Gamma_0$. Let $\lambda > 0$ satisfy the Energy Hypothesis. Then*

$$f(\omega_-, \omega_+; \lambda, h) = e^{i\mu_0\frac{\pi}{2}} \cdot K_{\mathcal{L}_+}[1](\omega_+) + O(h) \tag{15}$$

Remark 1 *Note that fixing Γ_0 and S^+_λ determines a Maslov operator up to $O(h)$, see Remark D.6. The constant μ_0 essentially compensates for the choice of Γ_0, cf. Remark 3.3.17.*

Remark 2 *The representation* [24, Theorem 1] *is a special case of (15) for the case when ω_+ is not a caustic, i.e.,* $\operatorname{rank} d(\pi_\omega \circ S^+_\lambda)^{-1} = n-1$. *In that case the Maslov operator at ω_+ can be represented simply as an exponential function. This is discussed more explicitly at the end of Section 3.3.*

The author wishes to thank Prof. Markus Klein for supervising this work with limitless patience in countless discussions and scrupulous proofreading. Much gratitude is due to Dr. Elke Rosenberger, whose willingness to engage in innumerable conversations regarding fine points of the proofs and whose moral support was a major help in the completion of this work. However, this work could not have been completed without the loving support of Quanbo Xie, who in a million little and a thousand large ways has helped to make it happen.

CHAPTER 1

The scattering problem in classical phase space

We start this chapter by giving a few essential definitions in Section 1.1 and setting the stage for the study of asymptotics of the scattering amplitude. In Section 1.2 we will study the classical phase trajectories in euclidean phase space $T^*\mathbb{R}^n$ with "initial conditions at $t \to -\infty$", which imply an "incoming direction" and a "non–trapping energy". We will obtain essential estimates for the convergence of trajectories to asymptotically "free" trajectories. A union over all trajectories with given incoming direction will yield a submanifold of $T^*\mathbb{R}^n$, which is the main result of Section 1.3.

1.1. The classical setting

In this section we give an introduction to the classical scattering problem, fixing definitions and notations for later use and formulating some crucial estimates. We will analyse the symbol $p \in C^\infty(T^*\mathbb{R}^n)$ associated to the operator $P(h)$ of (1), where the cotangent bundle on configuration space $\mathbb{R}^n_x$ is the classical euclidean phase space. It will turn out that the crucial objects in the analysis of scattering, the scattering angle and the angular momentum at infinity, can be regarded as elements of T^*S^{n-1}, which we regard as a natural subspace of $T^*\mathbb{R}^n$.

1.1.1. Convention *For euclidean space $\mathbb{R}^n$ we have a natural basis of the tangent space $T_p\mathbb{R}^n$ given by the partial derivatives $\frac{\partial}{\partial x_j}$ evaluated at $p \in \mathbb{R}^n$. We then write*

$$T_p\mathbb{R}^n = \{X_p(v) \colon v \in \mathbb{R}^n\}, \qquad X_p(v) := \sum_{j=1}^n v_j \frac{\partial}{\partial x_j}\Big|_p, \quad v \in \mathbb{R}^n. \tag{1.1.1a}$$

We identify $X_p(v)$ with $(p, X_p(v))$. Often, the subscript p is redundant, and to shorten our notation we will omit it, writing instead

$$(p, X(v)) \in T\mathbb{R}^n, \qquad \text{where } p, v \in \mathbb{R}^n \text{ and } X(v) \equiv X_p(v). \tag{1.1.1b}$$

For $f \colon \mathbb{R}^n \to \mathbb{R}^m$, we write $f_* \colon T\mathbb{R}^n \to \mathrm{T}\mathbb{R}^m$ for the the push-forward of f. Analogously, using the dual basis of 1-forms,

$$T^*_p\mathbb{R}^n = \left\{X^*_p(\xi) \colon \xi \in \mathbb{R}^n\right\}, \qquad X^*_p(\xi) := \sum_{j=1}^n \xi_j \, dx_j\big|_p, \quad \xi \in \mathbb{R}^n. \tag{1.1.2a}$$

and

$$(p, X^*(\xi)) \in T^*\mathbb{R}^n, \qquad \text{where } p, \xi \in \mathbb{R}^n \text{ and } X^*(\xi) = X^*_p(\xi). \tag{1.1.2b}$$

We will regard (x, ξ) as canonical coordinates of $T^*\mathbb{R}^n$. For $(p, X^*(\eta)) \in T^*\mathbb{R}^n$ we write

$$T_{(p,X^*(\eta))}(T^*\mathbb{R}^n) = \left\{ \sum_{j=1}^n u_j \frac{\partial}{\partial x_j}\Big|_{(p,X^*(\eta))} + \sum_{j=1}^n v_j \frac{\partial}{\partial \xi_j}\Big|_{(p,X^*(\eta))}, \quad u, v \in \mathbb{R}^n \right\} \tag{1.1.3}$$

and thus obtain natural coordinates $(u, v) = (u_1, \dots, u_n, v_1, \dots, v_n)$ on $T_{(p,X^*_p(\eta))}(T^*\mathbb{R}^n)$.

In the present section, we will generally identify $T^*\mathbb{R}^n \simeq \mathbb{R}^n_x \times \mathbb{R}^n_\xi$; all expressions are thus understood to be in canonical coordinates (x, ξ). The Hamiltonian associated to $P(h)$ is then given by

$$p(x, \xi) = |\xi|^2 + V(x), \qquad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{1.1.4}$$

which is a smooth function in the classical phase space. The hamiltonian vector field is given by

$$H_p = 2\sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial V(x)}{\partial x_i} \frac{\partial}{\partial \xi_i}. \tag{1.1.5}$$

The integral curves of (1.1.5) (which we also call phase trajectories) can be regarded as solutions of the system of ordinary differential equations

$$\frac{d\boldsymbol{x}}{ds} = 2\boldsymbol{\xi}, \qquad \frac{d\boldsymbol{\xi}}{ds} = -(\nabla V)(\boldsymbol{x}), \qquad \boldsymbol{x}, \boldsymbol{\xi} \in C^\infty(\mathbb{R}, \mathbb{R}^n). \tag{1.1.6}$$

1.1.2. Definition *We denote by* $(\boldsymbol{x}(\,\cdot\,; y, \eta), \boldsymbol{\xi}(\,\cdot\,; y, \eta))$ *a solution of* (1.1.6) *with given initial state*

$$(\boldsymbol{x}(0; y, \eta), \boldsymbol{\xi}(0; y, \eta)) = (y, \eta). \tag{1.1.7}$$

The Hamiltonian $p(x, \xi)$ being constant along an integral curve of the vector field (1.1.5), we define the energy of an integral curve by

$$\lambda(y, \eta) := p(\boldsymbol{x}(s; y, \eta), \boldsymbol{\xi}(s; y, \eta)) = |\boldsymbol{\xi}(s; y, \eta)|^2 + V(\boldsymbol{x}(s; y, \eta)) \qquad \text{for any } s \in \mathbb{R}. \tag{1.1.8}$$

The hamiltonian flow g is a map $\mathbb{R} \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, defined by

$$g(t, (y, X^*(\eta))) := g_t(y, X^*(\xi)) := (\boldsymbol{x}(t; y, \eta), X^*(\boldsymbol{\xi}(t; y, \eta))) \tag{1.1.9}$$

While the existence of an injective local flow, i.e., a map $g_t(y, X^*(\eta))$ for all $(y, X^*(\eta)) \in T^*\mathbb{R}^n$ at least for $|t| < T$ some small $T = T(y, \eta)$ is just standard existence and uniqueness theory of ordinary differential equations, cf., e.g., [25], the existence of an injective global flow is a more complicated question. The existence of a global flow g in our context will be discussed in Remark 1.2.8 below.

We will consider only phase trajectories that "go out to infinity" as $s \to \infty$; more precisely we formulate the following property of integral curves.

1.1.3. Definition *The energy* $\lambda > 0$ *is called "non–trapping for* p*" if for any* R *there exists a time* $T(R)$ *so that if* $|s| > T$*, then* $|\boldsymbol{x}(s; y, \eta)| > R$ *for all integral curves* $(\boldsymbol{x}(\,\cdot\,; y, \eta), \boldsymbol{\xi}(\,\cdot\,; y, \eta) \in p^{-1}(\lambda)$ *with* $|y| < R$*.*

Energy Hypothesis ("Non–trapping Condition") We fix an energy $\lambda > 0$ that is non–trapping for p.

1.1.4. Remark We note for later use that the Energy Hypothesis implies that the hamiltonian vector field does not vanish on $p^{-1}(\lambda)$, since $(\partial_s \boldsymbol{x}(t; y, \eta), \partial_s \boldsymbol{\xi}(t; y, \eta))_{t=0} = 0$ would imply $\eta = \boldsymbol{\xi}(t; y, \eta)_{t=0} = 0$ by (1.1.6) and so $(\boldsymbol{x}(t; y, \eta), \boldsymbol{\xi}(t; y, \eta)) = (y, 0)$ for all $t > 0$.

1.2. Phase trajectories defined at infinity

We will assume throughout this and all subsequent sections that the Energy and Potential Hypotheses hold. We will parametrise the phase trajectories of H_p through an "impact plane" $\mathbb{H} \subset \mathbb{R}^n$ which is perpendicular to a certain "incoming direction" $\omega_- \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. For notational convenience, and without any ensuing loss of generality, we make the following basic assumption which we also assume to hold throughout this text:

Impact Plane Hypothesis We fix

$$\omega_- := (0, \ldots, 0, 1), \qquad \mathbb{H} := \{x \in \mathbb{R}^n : x \perp \omega_-\} = \{x \in \mathbb{R}^n : x_n = 0\}. \tag{1.2.1}$$

1.2.1. Convention *We generally denote points in the impact plane by the letter* z*. Since* $\mathbb{H}$ *is canonically isomorphic to* $\mathbb{R}^{n-1}$ *via the map* $\mathbb{R}^{n-1} \ni z \mapsto (z, 0) \in \mathbb{R}^n$*, we will often use* $\mathbb{H}$ *and* $\mathbb{R}^{n-1}$ *interchangeably. In cases where it becomes necessary to distinguish between* $\mathbb{R}^{n-1}$ *and* $\mathbb{H}$*, we employ the notation*

$$\check{z} := (z, 0) \in \mathbb{R}^n \quad \text{for } z \in \mathbb{R}^{n-1} \quad \text{and} \quad x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \quad \text{for } x \in \mathbb{R}^n \tag{1.2.2}$$

to avoid confusion.

It will be useful to consolidate the constants C_α of (3) by setting

$$\max_{|\alpha| \le m} |(\nabla \partial_x^\alpha V)(x)| \le C_{m+1} \langle x \rangle^{-\varrho-1-m}, \qquad C_m := \sqrt{n} \max_{|\alpha| \le m} C_\alpha. \tag{1.2.3}$$

1.2.2. Lemma *Take* $x, y \in \mathbb{R}^n$ *such that* $|x|, |y| > R$ *for some* $R > 0$*. Then*

$$\begin{aligned} \max_{|\alpha| \le m} |(\nabla \partial^\alpha V)(x) - (\nabla \partial^\alpha V)(y)| &\le 2|x - y| \max_{|\alpha| \le m+1} \sup_{|z| \ge R} |(\nabla \partial^\alpha V)(z)| \\ &\le 2C_{m+2}|x - y| \langle R \rangle^{-\varrho-2-m}. \end{aligned} \tag{1.2.4}$$

PROOF. For x and y given as supposed, we can find a semi–circle $\mathcal{C} = \gamma([0,1])$, $\gamma\colon [0,1] \to \mathbb{R}^n$, centered at $(x+y)/2$ with endpoints $\gamma(0) = y$ and $\gamma(1) = x$, such that $|\gamma(t)| > R$ for $t \in [0,1]$. Then for any $j \in \mathcal{N} = \{1, \dots, n\}$,

$$(\partial^{\delta_{jn}} \partial^\alpha V)(x) - (\partial^{\delta_{jn}} \partial^\alpha V)(y) = \int_0^1 \frac{d}{ds}(\partial^{\alpha+\delta_{jn}} V)(\gamma(s))\, ds = \int_0^1 \langle (\nabla \partial^{\alpha+\delta_{jn}} V)(\gamma(s)), \partial_s \gamma(s)\rangle\, ds$$

where $\delta_{jn} = 1$ for $j = n$, zero otherwise. It then follows from (3) that

$$\begin{aligned} \max_{|\alpha|\le m} |(\nabla\partial^\alpha V)(x) - (\nabla\partial^\alpha V)(y)| &\le \max_{|\alpha|\le m+2} \sup_{w\in\mathcal{C}} |(\partial_x^\alpha V)(w)| \int_0^1 |\partial_s \gamma(s)|\, ds \\ &\le 2 \max_{|\alpha|\le m+2} \sup_{w>R} |(\partial_x^\alpha V)(w)| \cdot |x-y| \\ &\le 2C_{m+2} |x-y| \langle R\rangle^{-\varrho-2-m} \end{aligned}$$

□

We define "free" trajectories in $T^*\mathbb{R}^n$ (see Convention 1.1.1) via

$$\mathcal{T}_z^0 := \{(x, X^*(\xi)) \in T^*\mathbb{R}^n \colon (x,\xi) = (2\sqrt{\lambda}\omega^- \cdot s + \check{z}, \sqrt{\lambda}\omega_-),\ s\in\mathbb{R}\}, \qquad z \in \mathbb{R}^{n-1}. \tag{1.2.5}$$

In other words, a free trajectory has constant momentum in the direction of ω_- and at $s = 0$ the configuration–space projection intersects the impact plane at $\check{z} = (z, 0) \in \mathbb{H}$. Following [23, Chapter XI.1], we introduce trajectories asymptotic to $\mathcal{T}_z^0$ as $s \to -\infty$.

1.2.3. DEFINITION *For any $z \in \mathbb{H}$ we define by $(\boldsymbol{x}_\infty(\,\cdot\,, z;\lambda), \boldsymbol{\xi}_\infty(\,\cdot\,, z;\lambda))$ the unique integral curve of H_p such that*

$$\begin{aligned} \lim_{s\to-\infty} |\boldsymbol{x}_\infty(s,z;\lambda) - 2\sqrt{\lambda}\omega_- s - \check{z}| &= 0, \\ \lim_{s\to-\infty} |\boldsymbol{\xi}_\infty(s,z;\lambda) - \sqrt{\lambda}\omega_-| &= 0. \end{aligned} \tag{1.2.6}$$

We further define $\omega_+(z;\lambda) \in S^{n-1}$ and $r_+(z;\lambda) \in \mathbb{R}^n$ as the unique vectors such that

$$\lim_{s\to+\infty} |\boldsymbol{x}_\infty(s,z;\lambda) - 2\sqrt{\lambda}\omega_+(z;\lambda)s - r_+(z;\lambda)| = 0, \tag{1.2.7a}$$

$$\lim_{s\to+\infty} |\boldsymbol{\xi}_\infty(s,z;\lambda) - \sqrt{\lambda}\omega_+(z;\lambda)| = 0. \tag{1.2.7b}$$

The existence and uniqueness of phase trajectories $(\boldsymbol{x}_\infty(\,\cdot\,, z;\lambda), \boldsymbol{\xi}_\infty(\,\cdot\,, z;\lambda))$ is proven in [23, Theorem XI.1] (where the estimates of Lemma 1.2.2 are used), while the existence of $\omega_+(z;\lambda)$ and $r_+(z;\lambda)$ such that (1.2.7) holds is shown in [23, Theorem XI.3].

1.2.4. CONVENTION *Using the conventions of Definition 1.2.3, we set*

$$\begin{aligned} \mathcal{T}_z &:= \{(\boldsymbol{x}_\infty(s,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s,z;\lambda))) \in T^*\mathbb{R}^n \colon s \in \mathbb{R}\}, \\ \mathcal{T}_U &:= \bigcup_{z\in U} \mathcal{T}_z. \end{aligned} \tag{1.2.8}$$

$$\begin{aligned} \mathcal{T}_{z,T}^\pm &:= \{(\boldsymbol{x}_\infty(s,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s,z;\lambda))) \in T^*\mathbb{R}^n \colon s \gtrless T\}, \\ \mathcal{T}_{U,T}^\pm &:= \bigcup_{z\in U} \mathcal{T}_{z,T}^\pm. \end{aligned} \tag{1.2.9}$$

For future reference we reiterate the existence and uniqueness statements made in Definition 1.2.3.

1.2.5. PROPOSITION [23] *The correspondences*

$$\mathbb{R}^{n-1} \ni z \mapsto \mathcal{T}_z \subset T^*\mathbb{R}^n \qquad \text{and} \qquad \mathbb{R}^{n-1} \ni z \mapsto \big(\omega_+(z;\lambda), r_+(z;\lambda)\big) \in S^{n-1}\times\mathbb{R}^n \tag{1.2.10}$$

are one-to-one, i.e., bijective on their image.

We will now state two propositions that give more precise estimates on the limits (1.2.6) and (1.2.7). In particular, it turns out that the map $(s,z) \mapsto \big(\boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda)\big)$ is smooth in s and z and that the convergence as $s \to \pm\infty$ is uniform for $z \in \mathbb{H}$.

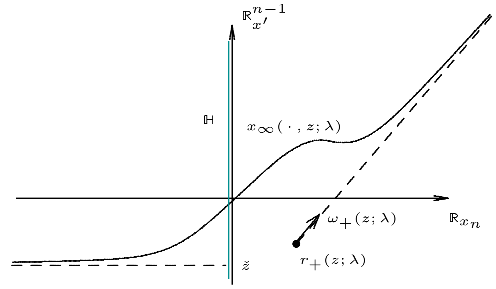


FIGURE 1. The projection onto $\mathbb{R}^n_x$ of a phase trajectory $\mathcal{T}_z$ for $\omega_- = (0, \dots, 0, 1)$.

1.2.6. REMARK The crucial estimates of Propostions 1.2.7 and 1.2.10 are (to the author's knowledge) not covered elsewhere in the literature. In [23, Chapter XI.1] a straightforward fixed-point argument is given for the existence and uniqueness of continuous functions $\boldsymbol{g}_\pm$ in (1.2.11) and (1.2.15), but without any considerations of the dependence of integral curves on parameters.

In [10, Section 1.10] the smoothness of trajectories with respect to parameters is analysed, and the estimates in [10, Theorem 10.1] are similar to (1.2.13) below. However, only derivatives with respect to an asymptotic momentum parameter (corresponding to $z \in \mathbb{H}$) and an initial position are considered, and no information on the decay with respect to the parameter of the derivative is given. Proposition 1.2.7, by comparison, yields time– and parameter–decay information for time– and parameter–derivatives of the trajectories. The same is true of (1.2.13) and the results of Propostion 1.2.10.

1.2.7. PROPOSITION *The integral curves of Definition 1.2.3 are smooth, $\boldsymbol{x}_\infty(\,\cdot\,,\cdot\,;\lambda)$, $\boldsymbol{\xi}_\infty(\,\cdot\,,\cdot\,;\lambda) \in C^\infty(\mathbb{R}\times\mathbb{H}, \mathbb{R}^n)$. Writing*

$$\boldsymbol{x}_\infty(s,z;\lambda) = 2\sqrt{\lambda}\,\omega_- s + \check z + 2\boldsymbol{g}_-(s,z;\lambda), \tag{1.2.11a}$$

$$\boldsymbol{\xi}_\infty(s,z;\lambda) = \sqrt{\lambda}\,\omega_- + \partial_s \boldsymbol{g}_-(s,z;\lambda), \tag{1.2.11b}$$

the function $\boldsymbol{g}_-$ can be estimated uniformly away from the origin in the $\mathbb{R}\times\mathbb{H}$ coordinate plane, i.e., for any multi–index $(k,\beta) \in \mathbb{N}\times\mathbb{N}^{n-1}$ there exist constants $C_{\mathbb{R},R;k,\beta}, C_{S,\mathbb{H};k,\beta} > 0$ and $S_{k,\beta}, R_{k,\beta} > 0$ such that

$$|\partial_z^\beta \partial_s^k \boldsymbol{g}_-(s,z;\lambda)| \le C_{\mathbb{R},R;k,\beta} \cdot \langle z\rangle^{-\varrho-|\beta|} \langle 2\sqrt{\lambda}\omega_- s + \check z\rangle^{1-k} \qquad \text{if } |z| > R_{k,\beta} \tag{1.2.12}$$

and

$$|\partial_z^\beta \partial_s^k \boldsymbol{g}_-(s,z;\lambda)| \le C_{S,\mathbb{H};k,\beta} \cdot \langle 2\sqrt{\lambda}\omega_- s + \check z\rangle^{1-\varrho-k-|\beta|} \qquad \text{if } s < S_{k,\beta}. \tag{1.2.13}$$

1.2.8. REMARK The fact that for any $z \in \mathbb{H}$ the map $s \mapsto (\boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda))$ is smooth implies the existence of a global flow $g(\,\cdot\,,\,\cdot\,)$ on $\mathbb{R}\times\Lambda$, where

$$\Lambda = \bigcup_{z\in\mathbb{H}} \mathcal{T}_z$$

(see Theorem 1.3.3 below).

1.2.9. DEFINITION *We denote by g the restriction of the hamiltonian flow to $\mathbb{R}\times\Lambda$, i.e., the map*

$$g\colon \mathbb{R}\times\Lambda \to \Lambda, \quad g(t, (\boldsymbol{x}_\infty(s,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s,z;\lambda))) = (\boldsymbol{x}_\infty(s+t,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s+t,z;\lambda)). \tag{1.2.14}$$

For short, we write $g_t(\,\cdot\,) := g(t,\,\cdot\,)$.

1.2.10. PROPOSITION *The functions r_+ and ω_+ of Definition 1.2.3 are smooth in z, $r_+(\,\cdot\,;\lambda)$, $\omega_+(\,\cdot\,;\lambda) \in C^\infty(\mathbb{H}, \mathbb{R}^n)$. Writing*

$$\boldsymbol{x}_\infty(s,z;\lambda) = 2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda) + 2\boldsymbol{g}_+(s,z;\lambda), \tag{1.2.15a}$$

$$\boldsymbol{\xi}_\infty(s,z;\lambda) = \sqrt{\lambda}\omega_+(z;\lambda) + \partial_s \boldsymbol{g}_+(s,z;\lambda), \tag{1.2.15b}$$

for any multi–index $(k,\beta)\in\mathbb{N}\times\mathbb{N}^{n-1}$ *there exist positive constants* $C_{\omega;\beta}$, $C_{r;\beta}$, $C_{T,\mathbb{H};k,\beta}$ *and* $T_{k,\beta}$ *such that*

$$|\partial_z^\beta(\check{z}-r_+(z;\lambda))|\le C_{r;\beta}\cdot\langle z\rangle^{1-\varrho-|\beta|} \tag{1.2.16a}$$

$$|\partial_z^\beta(\omega_- -\omega_+(z;\lambda))|\le C_{\omega;\beta}\cdot\langle z\rangle^{-\varrho-|\beta|}, \tag{1.2.16b}$$

$$|\partial_z^\beta\partial_s^k\boldsymbol{g}_+(s,z;\lambda)|\le C_{T,\mathbb{H};k,\beta}\cdot\langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{1-\varrho-k}\qquad \text{for } s>T_{k,\beta}. \tag{1.2.16c}$$

1.2.11. Convention *While the indices "+" and "∞" of the functions of Proposition 1.2.10 are useful mnemonics, we will often need to refer to the components of (say)* $\boldsymbol{x}_\infty(s,z;\lambda)\in\mathbb{R}^n$*, where they become cumbersome. We thus write, e.g.,*

$$\boldsymbol{x}_\infty(s,z;\lambda)=(\boldsymbol{x}_i(s,z;\lambda))_{i=1,\dots,n},\qquad \boldsymbol{g}_+(s,z;\lambda)=(\boldsymbol{g}_i(s,z;\lambda))_{i=1,\dots,n}, \tag{1.2.17}$$

i.e., we drop the index "+" and "∞" when refering to the components of $\boldsymbol{x}_\infty$, $\boldsymbol{\xi}_\infty$, ω_+, r_+, $\boldsymbol{g}_+$ *and* L_+ *(the last introduced below in (2.1.14)), all of which are functions mapping* $\mathbb{R}^n\to\mathbb{R}^n$*, in cases where no confusion might arise.*

1.2.12. Corollary *For any* $z\in\mathbb{H}$ *and* $\varepsilon>0$ *there exists some* $T(z,\varepsilon)>0$ *such that*

$$(1-\varepsilon)2\sqrt{\lambda}s<|\boldsymbol{x}_\infty(s,z;\lambda)|<(1+\varepsilon)2\sqrt{\lambda}s\qquad \text{for } s\ge T(z,\varepsilon). \tag{1.2.18}$$

There exists some $T>0$ *such that*

$$\sqrt{\lambda}s<|\boldsymbol{x}_\infty(s,z;\lambda)|\qquad \text{for all } s>T \text{ and } z\in\mathbb{H}. \tag{1.2.19}$$

Furthermore, for any $\beta\in\mathbb{N}^{n-1}$ *and any* $\varepsilon>0$ *there exists some* $T(\varepsilon)>0$ *such that for all* $z\in\mathbb{H}$ *and* $s,s'\ge T(\varepsilon)$

$$|\partial_z^\beta(\boldsymbol{x}_\infty(s,z;\lambda)-\boldsymbol{x}_\infty(s',z;\lambda))-2\sqrt{\lambda}\partial_z^\beta\omega_+(z;\lambda)(s-s')|\le\varepsilon|s-s'|. \tag{1.2.20}$$

Moreover, for any multi–index $\alpha\in\mathbb{N}^n$ *there exist constants* $C'_\alpha>0$ *such that*

$$|(\partial_x^\alpha V)(\boldsymbol{x}_\infty(s,z;\lambda))|\le C'_\alpha\langle s\rangle^{-\varrho-|\alpha|}\qquad \text{for all } z\in\mathbb{R}^{n-1}. \tag{1.2.21}$$

Proof. The assertion (1.2.18) follows directly from (1.2.15a) with (1.2.16c). In order to show (1.2.19), we set

$$\begin{aligned}\boldsymbol{x}_\infty(s,z;\lambda)&=2\sqrt{\lambda}s(\omega_+(z)+R(s,z)),\\ |\boldsymbol{x}_\infty(s,z;\lambda)|&=2\sqrt{\lambda}s\sqrt{1+2\langle\omega_+(z),R(s,z)\rangle+|R(s,z)|^2}\end{aligned} \tag{1.2.22}$$

with

$$R(s,z):=\frac{r_+(z;\lambda)+2\boldsymbol{g}_+(s,z;\lambda)}{2\sqrt{\lambda}s}. \tag{1.2.23}$$

We first claim that for any $(k,\beta)\in\mathbb{N}^n$

$$\left|\partial_s^k\partial_z^\beta\langle\omega_+(z;\lambda),R(s,z)\rangle\right|\le C_s\cdot\langle z\rangle^{1-\varrho-|\beta|}\cdot s^{-1-k}. \tag{1.2.24}$$

Then it follows from (1.2.24) that

$$1+2\langle\omega_+(z;\lambda),R(s,z)\rangle+|R(s,z)|^2>1-2|\langle\omega_+(z;\lambda),R(s,z)\rangle|>\frac{1}{2} \tag{1.2.25}$$

for sufficiently large s and all $z\in\mathbb{H}$. Then (1.2.22) and (1.2.25) imply (1.2.19).

We first prove

$$\left|\partial_s^k\partial_z^\beta\Big(R(s,z)-\frac{\check{z}}{2\sqrt{\lambda}s}\Big)\right|\le C_1\cdot\langle z\rangle^{1-\varrho-|\beta|}\cdot s^{-1-k}. \tag{1.2.26}$$

(1.2.27)

By (1.2.23),

$$R(s,z)-\frac{\check{z}}{2\sqrt{\lambda}s}=\frac{1}{2}\lambda^{-\frac{1}{2}}s^{-1}(r_+(z;\lambda)-\check{z})+\lambda^{-\frac{1}{2}}s^{-1}\boldsymbol{g}_+(s,z;\lambda) \tag{1.2.28}$$

so (1.2.26) follows from (1.2.16a), (1.2.16c) and Corollary A.2.

We now show (1.2.24). Since $\langle \omega_-, \check{z} \rangle = 0$,

$$\begin{aligned}\langle \omega_+(z), R(s,z)\rangle &= \langle \omega_+(z) - \omega_-, R(s,z)\rangle + \langle \omega_-, R(s,z)\rangle \\ &= (2\sqrt{\lambda}s)^{-1}\langle \omega_+(z) - \omega_-, \check{z}\rangle + \langle \omega_+(z) - \omega_-, R(s,z) - (2\sqrt{\lambda}s)^{-1}\check{z}\rangle \\ &\quad + \langle \omega_-, R(s,z) - (2\sqrt{\lambda}s)^{-1}\check{z}\rangle \end{aligned} \tag{1.2.29}$$

and (1.2.24) follows from (1.2.29) using the estimates (1.2.16b), (1.2.16c) and (1.2.26) with the product rule. This completes the proof of(1.2.19).

We will show (1.2.20). By (1.2.15a),

$$\partial_z^\beta(\boldsymbol{x}_i(s,z;\lambda) - \boldsymbol{x}_i(s',z;\lambda)) = 2\sqrt{\lambda}\partial_z^\beta\omega_i(z;\lambda)(s-s') + 2\partial_z^\beta\boldsymbol{g}_+(s,z;\lambda) - 2\partial_z^\beta\boldsymbol{g}_+(s',z;\lambda). \tag{1.2.30}$$

We will show that for any $\varepsilon > 0$ there exists some $T(\varepsilon) > 0$ so that for $s, s' > T(\varepsilon)$

$$2|\partial_z^\beta\boldsymbol{g}_+(s,z;\lambda) - \partial_z^\beta\boldsymbol{g}_+(s',z;\lambda)| < \varepsilon. \tag{1.2.31}$$

By the mean-value theorem, for $s, s' > T$,

$$|\partial_z^\beta\boldsymbol{g}_+(s,z;\lambda) - \partial_z^\beta\boldsymbol{g}_+(s',z;\lambda)| \le \sup_{t\in(T,\infty)} |\partial_t\partial_z^\beta\boldsymbol{g}_+(t,z;\lambda)||s-s'|.$$

By (1.2.16c), there exists some $C_\beta > 0$ so that

$$\sup_{t\in(T,\infty)} |\partial_z\partial_t\boldsymbol{g}_+(t,z;\lambda)| \le C_\beta \cdot |T|^{-\varrho},$$

hence we can take T sufficiently large to ensure (1.2.31). □

Note that Definition 1.2.3 also yields (1.2.18) and thereby (1.2.21), but without the stated uniformity in $z \in \mathbb{H}$.

In Section 1.3, we will see that the union over $z \in \mathbb{H}$ of the integral curves $\mathcal{T}_z$ forms a lagrangian manifold. This geometrical object is crucial for the introduction of the canonical Maslov operator (see Appendix D), which in turn allows the construction of asymptotic solutions to (1). The procedure is discussed in detail in [27] or [18], and we will make use of it in Section 3.2 below.

Proof of Proposition 1.2.7. We first note that phase trajectories solving (1.1.6) with (1.2.6) have the form (1.2.11) with $\boldsymbol{g}_-(\,\cdot\,,z;\lambda) \in C^\infty(\mathbb{R})$ satisfying the integral equation

$$\begin{aligned}\boldsymbol{g}_-(s,z;\lambda) &= -\int_{-\infty}^{s}\int_{-\infty}^{t} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau\,dt \\ &= -\int_{-\infty}^{s}\int_{-\infty}^{t} (\nabla V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2\boldsymbol{g}_-(\tau,z;\lambda))\,d\tau\,dt\end{aligned} \tag{1.2.32}$$

with

$$|\boldsymbol{g}_-(s,z;\lambda)|,\ |\partial_s\boldsymbol{g}_-(s,z;\lambda)| \to 0 \qquad \text{as } s \to -\infty.$$

1.2.13. Remark Referring to the proof of [23, Theorem XI.1], we recall that there exists some $T \ll 0$ such that for any $z \in \mathbb{H}$ there exists $T(z) < T$ such that the map

$$(\mathcal{F}_{\lambda,z}u)(s) := -\int_{-\infty}^{s}\int_{-\infty}^{t} (\nabla V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2u(\tau))\,d\tau\,dt, \tag{1.2.33}$$

is a strict contraction on

$$\mathcal{M}_{T(z)} := \Big\{u \in C((-\infty,T(z)),\mathbb{R}^n) \colon \sup_{s<T(z)} |u(s)| < 1\Big\}, \tag{1.2.34}$$

where $C((-\infty,T(z)),\mathbb{R}^n)$ denotes the space of continuous functions on $(-\infty,T)$ with values in $\mathbb{R}^n$. Then for $z \in \mathbb{H}$ we choose $T(z)$ so that $\boldsymbol{g}_-(\,\cdot\,,z;\lambda)|_{(-\infty,T(z))} \in \mathcal{M}_{T(z)}$ and the existence and uniqueness of $\boldsymbol{g}_-(\,\cdot\,,z;\lambda)|_{(-\infty,T(z))}$ follows from the existence and uniqueness of the fixed point of $\mathcal{F}_{\lambda,z}|_{\mathcal{M}_{T(z)}}$. By the existence and uniqueness of solutions to ordinary differential equations, we obtain for each $z \in \mathbb{H}$ a unique function $\boldsymbol{g}_-(\,\cdot\,,z;\lambda) \in C^\infty(\mathbb{R})$ (while the continuity of $\boldsymbol{g}_-$ follows from the contraction mapping principle, the smoothness is implied by the continuation of $(\boldsymbol{x}_\infty,\boldsymbol{\xi}_\infty)$ as an integral curve of the smooth vector field (1.1.6)).

Since the aforementioned time T does not depend on z, the arguments of the proof of [23, Theorem XI.1] imply that the map

$$(\mathcal{F}_\lambda u)(s,z) := -\int_{-\infty}^{s}\int_{-\infty}^{t}(\nabla V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2u(\tau,z))\,d\tau\,dt, \tag{1.2.35}$$

is a strict contraction on

$$\mathcal{M}_{T,\mathbb{H}} := \big\{u \in C((-\infty,T)\times\mathbb{H},\mathbb{R}^n)\colon \sup_{\substack{s<T\\ z\in\mathbb{H}}}|u(s,z)| <\leq 1\big\}. \tag{1.2.36}$$

Again the existence and uniquenees of a fixed point implies that $\boldsymbol{g}_-(\,\cdot\,,\,\cdot\,;\lambda)|_{(-\infty,T)\times\mathbb{H}} \in \mathcal{M}_{T,\mathbb{H}}$ and since $\boldsymbol{g}_-(\,\cdot\,,z;\lambda)$ is smooth, $\boldsymbol{g}_-(\,\cdot\,,\,\cdot\,;\lambda)|_{(-\infty,T)\times\mathbb{H}}$ is smooth in s and a continuous function of $z \in \mathbb{H}$. Our present goal is to refine these arguments and prove that $\boldsymbol{g}_-(\,\cdot\,,\,\cdot\,;\lambda)$ is an element of $C^\infty(\mathbb{R}\times\mathbb{H};\mathbb{R}^n)$ satisfying the estimates (1.2.13) and (1.2.12).

For $N \in \mathbb{N}$ and $T \in \mathbb{R}$ we introduce the Banach space

$$\mathcal{B}_{T,\mathbb{H};N} = \Big\{u \in C^N((-\infty,T)\times\mathbb{H},\mathbb{R}^n)\colon \lim_{s\to-\infty}|u(s,z)| = 0, \|u\|_{T,\mathbb{H};N} < \infty\Big\} \tag{1.2.37}$$

with the norm

$$\|u\|_{T,\mathbb{H};N} := \max_{\substack{(k,\beta)\in\mathbb{N}\times\mathbb{N}^{n-1}\\ k+|\beta|\leq N}} \sup_{\substack{z\in\mathbb{H}\\ s<T}}\big|\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{k+|\beta|}\partial_s^k\partial_z^\beta u(s,z)\big|. \tag{1.2.38}$$

We will consider the convex subset

$$\mathcal{M}_{T,\mathbb{H};N} = \Big\{u \in \mathcal{B}_{T,\mathbb{H};N}\colon \|u\|_{T,\mathbb{H};N} \leq 1\Big\}, \tag{1.2.39}$$

which is a complete metric space. The map $\mathcal{F}_\lambda$ of (1.2.35) will turn out (see Lemma 1.2.15 below) to be a contraction on $\mathcal{M}_{T,\mathbb{H};N}$ for any N if T is small enough, yielding a unique fixed point, which is just $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}$. Before proceeding, we need a technical result.

1.2.14. Lemma *Let $u \in \mathcal{M}_{T,\mathbb{H};N}$ for $T < -2/\sqrt{\lambda}$ and let V satisfy the Potential Hypothesis. Then for any $N \in \mathbb{N}$ there exists a constant $C(N) > 0$ such that for any $k \in \mathbb{N}$ and any multi–index $\beta \in \mathbb{N}^{n-1}$ with $k + |\beta| \leq N$ and any $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq 2$ the estimate*

$$\big|\partial_s^k\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z))\big| \leq C(N)\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-\varrho-|\alpha|-k-|\beta|} \tag{1.2.40}$$

holds.

Proof. We will apply Lemma A.4 with $\Omega = (-\infty,T)\times\mathbb{H}$; the assertion (1.2.40) follows directly from the validity of (A.35) for any $\alpha \in \mathbb{N}^n$ and $k + |\beta| \leq N$, and

$$\psi(s,z) := 2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z) \qquad \text{with } u \in \mathcal{M}_{T,\mathbb{H};N},\ T < -2/\sqrt{\lambda}, \tag{1.2.41a}$$

$$\rho_1(s,z) = \rho_2(s,z) := \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle, \tag{1.2.41b}$$

The condition (A.33) holds for any α by the Potential Hypothesis. We need to verify (A.34) and first note that the orthogonality of $\check{z}$ and ω_- yields

$$|2\sqrt{\lambda}\omega_- s + \check{z}|^2 = 4\lambda s^2 + |z|^2. \tag{1.2.42}$$

Since $|2\sqrt{\lambda}s| > 4$ on Ω by (1.2.41a), we have

$$\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle > |2\sqrt{\lambda}\omega_- s + \check{z}| > 4. \tag{1.2.43}$$

Using $\langle x + y\rangle \geq \langle x\rangle - |y|$ for $x, y \in \mathbb{R}^n$, we have

$$\langle\psi(s,z)\rangle = \langle 2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z)\rangle \geq \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle - 2|u(s,z)| \tag{1.2.44}$$

and by (1.2.39) we have $|u(s,z)| < 1$ on Ω, so

$$\langle\psi(s,z)\rangle \geq \frac{1}{2}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \geq 2. \tag{1.2.45}$$

It follows from (1.2.41b) and (1.2.45) that

$$1 \leq 2\langle\psi(s,z)\rangle\rho_2(s,z)^{-1}. \tag{1.2.46}$$

Furthermore, by (1.2.38), $u \in \mathcal{M}_{T,\mathbb{H};N}$ implies

$$|\partial_s^j \partial_z^\delta u(s,z)| \leq \rho_2(s,z)^{-j-|\delta|}. \tag{1.2.47}$$

We verify (A.34) for all $j + |\delta| \leq N$. For $j + |\delta| = 0$ there is nothing to show, so we assume $j + |\delta| \geq 1$. For $j + |\delta| = 1$ we now have, using (1.2.41a),

$$|\partial_s^j \partial_z^\delta \psi(s,z)| \leq 1 + 2\sqrt{\lambda} + 2|\partial_s^j \partial_z^\delta u(s,z)| \leq (1 + 2\sqrt{\lambda}) 2\rho_2(s,z)^{-1} \langle \psi(s,z) \rangle + 2\rho_2(s,z)^{-1}. \tag{1.2.48}$$

where we have applied (1.2.46) and (1.2.47). Since $2 \leq \langle \psi \rangle$ by (1.2.45),

$$|\partial_s^j \partial_z^\delta \psi(s,z)| \leq (3 + 4\sqrt{\lambda}) \rho_2(s,z)^{-1} \langle \psi(s,z) \rangle. \tag{1.2.49}$$

For $2 \leq j + |\delta| \leq N$, we have

$$|\partial_s^j \partial_z^\delta \psi(s,z)| \leq 2|\partial_s^j \partial_z^\delta u(s,z)| \leq \rho_2(s,z)^{-j-|\delta|} \langle \psi(s,z) \rangle \tag{1.2.50}$$

by (1.2.45) and (1.2.47). Thus (A.34) holds for all k, β with $k + |\beta| \leq N$ and (A.35) yields

$$\left|\partial_s^k \partial_z^\beta (\partial_x^\alpha V)(\psi(s,z))\right| \leq C(\alpha, k + |\beta|) \cdot \rho_2(s,z)^{-k-|\beta|} \langle \psi(s,z) \rangle^{-\varrho-|\alpha|} \tag{1.2.51}$$

with some constant $C(\alpha, N) > 0$. Using $\langle \psi \rangle^{-1} \leq 2\rho_2^{-1}$ by (1.2.46), we can thus deduce the existence of a single constant $C(N)$ such that (1.2.40) holds. □

1.2.15. Lemma *For $T = T(N) \ll -2/\sqrt{\lambda}$ small enough, $\mathcal{F}_\lambda$ is a strict contraction on $\mathcal{M}_{T,\mathbb{H};N}$.*

Proof. For $u \in \mathcal{M}_{T,\mathbb{H};N}$ it follows from Lemmas 1.2.14 and A.6 that $|\mathcal{F}_\lambda u(s,z)| \to 0$ as $s \to -\infty$ for any $z \in \mathbb{H}$. With the definition (1.2.38) of $\|\cdot\|_{T,\mathbb{H};N}$ we further see that $\|\mathcal{F}_\lambda u\|_{T,\mathbb{H};N}$ can be made arbitrarily small by choosing $|T|$ large enough. Thus for some $T \ll -2/\sqrt{\lambda}$, $\|\mathcal{F}_\lambda u\|_{T,\mathbb{H};N} < 1$ if $u \in \mathcal{M}_{T,\mathbb{H};N}$, so $\mathcal{F}_\lambda$ maps $\mathcal{M}_{T,\mathbb{H};N}$ onto itself. Furthermore, direct calculation gives

$$\frac{d}{dt}\mathcal{F}_\lambda(u + tv)\bigg|_{t=0}(s,z) = -2\int_{-\infty}^{s}\int_{-\infty}^{t} (\langle (\nabla \partial_{x_i} V)(2\sqrt{\lambda}\omega_- \tau + \check{z} + 2u(\tau, z; \lambda)), v(\tau, z; \lambda) \rangle)_{i=1}^n \, d\tau \, dt. \tag{1.2.52}$$

We need to verify that $\frac{d}{dt}\mathcal{F}_\lambda(u + tv)|_{t=0}$ is continuous in $u \in \mathcal{M}_{T,\mathbb{H};N}$ to ensure that $\mathcal{F}_\lambda$ is C^1 and $\frac{d}{dt}\mathcal{F}_\lambda(u + tv)|_{t=0} = D\mathcal{F}_\lambda|_u v$. It suffices to show that for $v, u, u_m \in \mathcal{M}_{T,\mathbb{H};N}$ ($m \in \mathbb{N}$),

$$\lim_{m\to\infty}\left\|\frac{d}{dt}\mathcal{F}_\lambda(u + tv)|_{t=0} - \frac{d}{dt}\mathcal{F}_\lambda(u_m + tv)|_{t=0}\right\|_{T,\mathbb{H};N} = 0 \qquad \text{if } \lim_{k\to\infty}\|u_m - u\|_{T,\mathbb{H};N} = 0. \tag{1.2.53}$$

Note that $\lim_{m\to\infty}\|u_m - u\|_{T,\mathbb{H};N} = 0$ implies

$$\lim_{m\to\infty}\sup_{\Omega}|\partial_z^\beta \partial_s^k (u_m - u)| = 0, \qquad \Omega = (-\infty, T) \times \mathbb{H},\ k + |\beta| \leq N. \tag{1.2.54}$$

By the smoothness of V and the chain rule (A.3), for any $\alpha \in \mathbb{N}^n$, $k + |\beta| \leq N$, (1.2.54) implies

$$\lim_{m\to\infty}\sup_{\Omega}|\partial_s^k \partial_z^\beta (\partial^\alpha V)(2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z)) - \partial_s^k \partial_z^\beta (\partial^\alpha V)(2\sqrt{\lambda}\omega_- s + \check{z} + 2u_m(s,z))| = 0. \tag{1.2.55}$$

Now by (1.2.38) and (1.2.52),

$$\begin{aligned}
&\lim_{m\to\infty}\left\|\frac{d}{dt}\mathcal{F}_\lambda(u + tv)|_{t=0} - \frac{d}{dt}\mathcal{F}_\lambda(u_m + tv)|_{t=0}\right\|_{T,\mathbb{H};N} \\
&\leq 2\lim_{m\to\infty}\sup_{\substack{z\in\mathbb{H}\\ s<T}}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{k+|\beta|}\int_{-\infty}^{s}\int_{-\infty}^{t}\max_{\substack{|\alpha|=2\\ k+|\beta|\leq N}}\left|\partial_s^k \partial_z^\beta \big\langle (\partial^\alpha V)(2\sqrt{\lambda}\omega_- \tau + \check{z} + 2u(\tau, z; \lambda))\right. \\
&\qquad \left. - (\partial^\alpha V)(2\sqrt{\lambda}\omega_- \tau + \check{z} + 2u_m(\tau, z; \lambda)), v(\tau, z; \lambda)\big\rangle\right| d\tau \, dt
\end{aligned} \tag{1.2.56}$$

The product rule (A.30) with (1.2.55) then gives (1.2.53) if we apply the theorem of dominated convergence to take the limit under the integral. The estimate (1.2.40) with Lemma A.6 yield the integrable dominating function.

Hence $\mathcal{F}_\lambda$ is C^1 and $\frac{d}{dt}\mathcal{F}_\lambda(u+tv)|_{t=0} = D\mathcal{F}_\lambda|_u v$. We use (1.2.38), (1.2.52) and the product rule (A.30) to see that

$$\begin{aligned}
\|D\mathcal{F}_\lambda|_u(v)\|_{T,\mathbb{H};N} &\leq 2\sup_{\substack{z\in\mathbb{H}\\ s<T}}\int_{-\infty}^{s}\int_{-\infty}^{t}\max_{k+|\beta|\leq N}\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{k+|\beta|}\\
&\quad\times|\partial_\tau^k\partial_z^\beta(\langle(\nabla\partial_{x_i}V)(2\sqrt{\lambda}\omega_-\tau+\check{z}+2u(\tau,z)),v(\tau,z)\rangle)_{i=1}^n|\,d\tau\,dt\\
&\leq 2\cdot 2^N\sup_{\substack{z\in\mathbb{H}\\ s<T}}\int_{-\infty}^{s}\int_{-\infty}^{t}\max_{\substack{k+|\beta|\leq N\\ \alpha=2}}|\partial_\tau^k\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_-\tau+\check{z}+2u(\tau,z))|\\
&\quad\times\max_{k+|\beta|\leq N}|\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{k+|\beta|}\partial_\tau^k\partial_z^\beta v(\tau,z)|\,d\tau\,dt\\
&\leq 2^{N+1}\|v\|_{T,\mathbb{H};N}\cdot\sup_{\substack{z\in\mathbb{H}\\ s<T}}\int_{-\infty}^{s}\int_{-\infty}^{t}\max_{\substack{k+|\beta|\leq N\\ \alpha=2}}|\partial_\tau^k\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_-\tau+\check{z}+2u(\tau,z))|\,d\tau\,dt
\end{aligned}\tag{1.2.57}$$

where we have used that $\max_{|\alpha|\leq N}\partial^\alpha(f\cdot g)\leq 2^N(\max_{|\alpha|\leq N}\partial^\alpha f)(\max_{|\alpha|\leq N}\partial^\alpha g)$. It follows that

$$\begin{aligned}
\|D\mathcal{F}_\lambda|_u\| &\leq 2^{N+1}\sup_{\substack{z\in\mathbb{H}\\ s<T}}\max_{\substack{k+|\beta|\leq N\\ \alpha=2}}\int_{-\infty}^{s}\int_{-\infty}^{t}|\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_-\tau+\check{z}+2u(\tau,z))|\\
&\leq 2^{N+1}\sup_{\substack{z\in\mathbb{H}\\ s<T}}\max_{\substack{k+|\beta|\leq N\\ \alpha=2}}C(N)\int_{-\infty}^{s}\int_{-\infty}^{t}\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{-\varrho-2-k-|\beta|}\,d\tau\,dt.
\end{aligned}\tag{1.2.58}$$

by (1.2.40). Again Lemma A.6 implies that $\|D\mathcal{F}_\lambda|_u\|<1$ if $T\ll -2/\sqrt{\lambda}$ is chosen sufficiently small. By Lemma B.2, $\mathcal{F}_\lambda$ is a strict contraction. □

By Lemma B.2 there exists a unique fixed point of $\mathcal{F}_\lambda$ in $\mathcal{M}_{T,\mathbb{H};N}$ for any $N\in\mathbb{N}$. Since $\mathcal{F}_\lambda$ is also a contraction on $\mathcal{M}_{T,\mathbb{H}}$ with fixed point $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}$ and $\mathcal{M}_{T,\mathbb{H};N}\subset\mathcal{M}_{T,\mathbb{H}}$ we can conclude that $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}\in\mathcal{M}_{T,\mathbb{H};N}$ for any N (note that $T=T(N)$). In particular, we obtain $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}\in C^N((-\infty,T)\times\mathbb{H})$ for any N. From (1.2.11), we conclude that $(\boldsymbol{x}_\infty,\boldsymbol{\xi}_\infty)|_{(-\infty,T)\times\mathbb{H}}\in C^N((-\infty,T)\times\mathbb{H},\mathbb{R}^n\times\mathbb{R}^n)$. Then the general theory of ordinary differential equations (smoothness of integral curves with respect to initial conditions) implies that $(\boldsymbol{x}_\infty,\boldsymbol{\xi}_\infty)$ (and hence $\boldsymbol{g}_-$) is a C^N–map $\mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}^n\times\mathbb{R}^n$. Since this is true for any N, the integral curves $(\boldsymbol{x}_\infty,\boldsymbol{\xi}_\infty)$ are actually smooth.

Now $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}\in\mathcal{M}_{T,\mathbb{H};N}$ for any $N\in\mathbb{N}$ with $T=T(N)$ implies

$$|\partial_s^k\partial_z^\beta\boldsymbol{g}_-(s,z;\lambda)|\leq\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{-k-|\beta|}\qquad\text{for } z\in\mathbb{H}\text{ and } s<T(k+|\beta|).\tag{1.2.59}$$

We will improve this preliminary estimate to obtain (1.2.13).

Proof of (1.2.13). We note that $\boldsymbol{g}_-|_{(-\infty,T)\times\mathbb{H}}\in\mathcal{M}_{T,\mathbb{H};N}$, (1.2.40) and (A.53) imply for $k+|\beta|\leq N$,

$$\begin{aligned}
|\partial_s^k\partial_z^\beta\boldsymbol{g}_-(s,z;\lambda)| &\leq\int_{-\infty}^{s}\int_{-\infty}^{t}\left|\partial_\tau^k\partial_z^\beta(\nabla V)(2\sqrt{\lambda}\omega_-\tau+\check{z}+2\boldsymbol{g}_-(\tau,z;\lambda))\right|\,d\tau\,dt\\
&\leq C(N)\int_{-\infty}^{s}\int_{-\infty}^{t}\langle 2\sqrt{\lambda}\omega_-\tau+\check{z}\rangle^{-\varrho-1-k-|\beta|}\,d\tau\,dt\\
&\leq C(N)C_{\varrho+1,k+|\beta|}\cdot\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{1-\varrho-k-|\beta|},\qquad z\in\mathbb{H}, s<T(N).
\end{aligned}$$

□

The proof of the estimate (1.2.12) stems from the same fixed–point argument as that of (1.2.13). We set $B_R:=\{z\in\mathbb{H}\colon |z|>R\}$ and introduce the Banach space

$$\mathcal{B}_{\mathbb{R},R;N}=\left\{u\in C^N(\mathbb{R}\times B_R,\mathbb{R}^n)\colon\lim_{s\to-\infty}|u(s,z)|=0,\|u\|_{\mathbb{R},R;N}<\infty\right\}\tag{1.2.60}$$

with the norm

$$\|u\|_{\mathbb{R},R;N}:=\sup_{\substack{|z|>R\\ s\in\mathbb{R}}}\max_{|\beta|\leq N}|\langle 2\sqrt{\lambda}\omega_- s+\check{z}\rangle^{-1}\langle z\rangle^{|\beta|}\partial_z^\beta u(s,z)|.\tag{1.2.61}$$

We consider the convex subset

$$\mathcal{M}_{\mathbb{R},R;N} = \Big\{ u \in \mathcal{B}_{\mathbb{R},R;N} \colon \|u\|_{\mathbb{R},R;N} \leq 1 \Big\}, \tag{1.2.62}$$

which is a complete metric space and show that for suitable $R > 0$ the map $\mathcal{F}_\lambda$ is actually a contraction on $\mathcal{M}_{\mathbb{R},R;N}$, yielding a unique fixed point, which will turn out to be $\boldsymbol{g}_-$.

1.2.16. Lemma *Let $u \in \mathcal{M}_{\mathbb{R},R;N}$ for $R > 4$ and let V satisfy the Potential Hypothesis. Then for any $N \in \mathbb{N}$ there exists a constant $C(N) > 0$ such that for any $k \in \mathbb{N}$ and any multi–index $\beta \in \mathbb{N}^{n-1}$ with $k + |\beta| \leq N$ and any $\alpha \in \mathbb{N}^n$ the estimate*

$$\big|\partial_z^\beta (\partial^\alpha V)(2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z))\big| \leq C_{|\alpha|}(N) \langle z\rangle^{-|\beta|} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-\varrho - |\alpha|} \tag{1.2.63}$$

holds. We set $C(N) = \max\{C_{|\alpha|}(N) \colon 1 \leq |\alpha| \leq 2\}$

Proof. We will apply Lemma A.4; specifically, we will show that the estimate (A.35) holds on $\Omega = B_R \times \mathbb{R}$ for

$$\psi(s,z) := 2\sqrt{\lambda}\omega_- s + \check{z} + 2u(s,z) \qquad \text{with } u \in \mathcal{M}_{\mathbb{R},R;N},\ R > 4, \tag{1.2.64a}$$

and

$$k = 0, \qquad |\delta| \leq |\beta|, \qquad \rho_1(s,z) := 1, \qquad \rho_2(s,z) := \langle z\rangle. \tag{1.2.64b}$$

The condition (A.33) holds for any α by the Potential Hypothesis. We note that for $|\delta| = 0$ there is nothing to show, so we assume $|\delta| \geq 1$. As in (1.2.45) and (1.2.46), $\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle > 4$ and $|u(s,z)| < 1$ on Ω implies

$$\langle \psi(s,z)\rangle \geq \frac{1}{2}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \geq \frac{1}{2}\langle z\rangle \geq 2, \qquad 1 \leq 2\langle\psi(s,z)\rangle \rho_1(s,z)^{-1}. \tag{1.2.65}$$

Now $u \in \mathcal{M}_{\mathbb{R},R;N}$ implies

$$|\partial_z^\delta u(s,z)| \leq \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \rho_2(s,z)^{-|\delta|} \tag{1.2.66}$$

by (1.2.61). For $|\delta| = 1$ we now have with (1.2.64a), (1.2.66) and (1.2.65)

$$\begin{aligned} |\partial_z^\delta \psi(s,z)| &\leq 1 + 2|\partial_z^\delta u(s,z)| \\ &\leq 1 + 2\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \rho_2(s,z)^{-1} \\ &\leq 6\langle\Psi\rangle \rho_2(s,z)^{-1}. \end{aligned} \tag{1.2.67}$$

For $1 < |\delta| \leq N$, we have with (1.2.64a) and (1.2.65),

$$|\partial_z^\delta \psi(s,z)| \leq 2|\partial_z^\delta u(s,z)| \leq \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \rho_2(s,z)^{-|\delta|} \leq 2\rho_2(s,z)^{-|\delta|}\langle\psi(s,z)\rangle. \tag{1.2.68}$$

Thus (A.34) holds and (A.35) yields

$$\big|\partial_z^\beta (\partial_x^\alpha V)(\psi(s,z))\big| \leq C(\alpha, |\beta|) \cdot \rho_2(s,z)^{-|\beta|} \langle\psi(s,z)\rangle^{-\varrho - |\alpha|} \tag{1.2.69}$$

with some constant $C(\alpha, N) > 0$. Together with (1.2.65) we can thus deduce the existence of a single constant $C(N)$ such that (1.2.63) holds. □

1.2.17. Lemma *For $R = R(N) > 4$ large enough, $\mathcal{F}_\lambda$ is a strict contraction on $\mathcal{M}_{\mathbb{R},R;N}$.*

Proof. The proof proceeds in an anlogous way as that of Lemma 1.2.15. For $u \in \mathcal{M}_{\mathbb{R},R;N}$ it follows from Lemmas 1.2.16 and (A.6), with the definition (1.2.38) of $\|\cdot\|_{T,\mathbb{H};N}$ that $\|\mathcal{F}_\lambda u\|_{T,\mathbb{H};N}$ can be made arbitrarily small by choosing R large enough. Thus for some $R \gg 4$, $\|\mathcal{F}_\lambda u\|_{T,\mathbb{H};N} < 1$ if $u \in \mathcal{M}_{\mathbb{R},R;N}$, so $\mathcal{F}_\lambda$ maps $\mathcal{M}_{\mathbb{R},R;N}$ onto itself. We ca repeat the arguments in the proof of Lemma 1.2.15 to see that $\mathcal{F}_\lambda$ is

C^1 and $D\mathcal{F}_\lambda|_u v$ is given by (1.2.52). Then similarly to (1.2.57), (1.2.61), the chain rule and (1.2.63) give

$$\begin{aligned}
&\|D\mathcal{F}_\lambda|_u(v)\|_{\mathbb{R},R;N} \\
&\le 2^{N+1} \sup_{\substack{|z|>R\\ s\in\mathbb{R}}} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-1} \\
&\quad \times \int_{-\infty}^{s}\int_{-\infty}^{t} \max_{\substack{|\beta|\le N\\ \alpha=2}} |\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2u(\tau,z))|\langle z\rangle^{|\beta|}\cdot \max_{|\beta|\le N}|\partial_z^\beta v(\tau,z)|\,d\tau\,dt \\
&\le 2^{N+1}C(N) \sup_{\substack{|z|>R\\ s\in\mathbb{R}}} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-1} \\
&\quad \times \int_{-\infty}^{s}\int_{-\infty}^{t} \max_{\substack{|\beta|\le N\\ \alpha=2}} \langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-2} \max_{|\beta|\le N}|\langle z\rangle^{|\beta|}\partial_z^\beta v(\tau,z)|\,d\tau\,dt \\
&\le 2^{N+1}C(N)\|v\|_{T,\mathbb{H};N}\cdot \sup_{\substack{|z|>R\\ s\in\mathbb{R}}} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-1} \int_{-\infty}^{s}\int_{-\infty}^{t} \max_{\substack{|\beta|\le N\\ \alpha=2}} \langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-1}\,d\tau\,dt \quad (1.2.70)
\end{aligned}$$

Hence it follows that

$$\|D\mathcal{F}_\lambda\| \le 2^{N+1}C(N) \sup_{\substack{|z|>R\\ s\in\mathbb{R}}} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-1} \max_{\substack{\beta\in\mathbb{N}^{n-1}\\ |\beta|\le N\\ \alpha=2}} \langle z\rangle^{-|\beta|} \int_{-\infty}^{s}\int_{-\infty}^{t} \langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-1}\,d\tau\,dt. \quad (1.2.71)$$

Again Lemma A.6 allows us to ensure $\|D\mathcal{F}_\lambda|_u\| < 1$ by choosing $R \gg 4$ sufficiently large. By Lemma B.2, $\mathcal{F}_\lambda$ is then a strict contraction. ☐

It follows that $\boldsymbol{g}_-|_{\mathbb{R}\times B_R} \in \mathcal{M}_{\mathbb{R},R;N}$ and hence there exists a constant $C_{0,\beta} > 0$ so that

$$|\partial_z^\beta \boldsymbol{g}_-(s,z;\lambda)| = |\partial_z^\beta \mathcal{F}\boldsymbol{g}_-(s,z;\lambda)| \le C_{0,\beta}\langle z\rangle^{-\varrho-|\beta|}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle \qquad \text{for } s\in\mathbb{R} \text{ and } |z|>R.$$

Now by (1.2.63)

$$\begin{aligned}
|\partial_z^\beta\partial_s \boldsymbol{g}_-(s,z;\lambda)| &\le \int_{-\infty}^{s} |\partial_z^\beta(\nabla V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2\boldsymbol{g}_-(\tau,z;\lambda))|\,d\tau \\
&\le C_1(|\beta|)\langle z\rangle^{-|\beta|}\int_{-\infty}^{s}\langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-1}\,d\tau. \quad (1.2.72)
\end{aligned}$$

Applying (A.54) and substituting in the integral we obtain

$$\begin{aligned}
|\partial_z^\beta\partial_s \boldsymbol{g}_-(s,z;\lambda)| &\le C(N)\langle z\rangle^{-\varrho-|\beta|}\int_{-\infty}^{2\sqrt{\lambda}\langle z\rangle^{-1}s}\langle\tau\rangle^{-\varrho-1}\,d\tau \\
&\le C(N)\langle z\rangle^{-\varrho-|\beta|}\int_{-\infty}^{\infty}\langle\tau\rangle^{-\varrho-1}\,d\tau \\
&=: C_{1,\beta}\cdot\langle z\rangle^{-\varrho-|\beta|}.
\end{aligned}$$

Now since

$$\partial_s^k \boldsymbol{g}_-(s,z;\lambda) = -\partial_s^{k-2}(\nabla V)(2\sqrt{\lambda}\omega_-\tau + \check{z} + 2\boldsymbol{g}_-(\tau,z;\lambda)) \qquad \text{for } \alpha\ge 2,$$

we can inductively apply (1.2.63) with the chain rule (A.32) and the previously obtained estimates for $\partial_s^{k-2}\partial_z^\beta \boldsymbol{g}_-$ to obtain estimates for $\partial_s^k\partial_z^\beta \boldsymbol{g}_-$ for $k>2$. This procedure gives (1.2.12). ☐

Proof of Proposition 1.2.10. Similarly to the representation (1.2.32), the phase trajectories solving (1.1.6) with (1.2.7) have the form (1.2.15) with

$$\begin{aligned}
\boldsymbol{g}_+(s,z;\lambda) &= \int_s^\infty\int_t^\infty (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau\,dt \\
&= \int_s^\infty\int_t^\infty (\nabla V)(2\sqrt{\lambda}\omega_+(z;\lambda)\tau + r_+(z;\lambda) + 2\boldsymbol{g}_+(\tau,z;\lambda))\,d\tau\,dt
\end{aligned} \quad (1.2.73)$$

and

$$|\boldsymbol{g}_+(s,z;\lambda)|, \quad |\partial_s \boldsymbol{g}_+(s,z;\lambda)| \to 0 \qquad \text{as } s \to +\infty. \tag{1.2.74}$$

The existence and uniqeness of $\boldsymbol{g}_+$ in (1.2.15) follows from the existence and uniqueness of ω_+ and r_+, cf. (1.2.7). By (1.2.73), $\boldsymbol{g}_+$ is smooth in s and z. Note that in particular

$$\partial_s \boldsymbol{g}_-(s,z;\lambda) = -\int_{-\infty}^{s} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau, \quad \partial_s \boldsymbol{g}_+(s,z;\lambda) = \int_{s}^{\infty} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau. \tag{1.2.75}$$

and $\boldsymbol{g}_+$ is a smooth function of s and z. We next verify the estimates (1.2.16), whose proof will incidentally yield the stated smoothness of ω_+ and r_+.

Proof of (1.2.16b). Comparing (1.2.11b) and (1.2.15b),

$$\omega_- - \omega_+(z;\lambda) = \lambda^{-\frac{1}{2}}(\partial_s \boldsymbol{g}_+(s,z;\lambda) - \partial_s \boldsymbol{g}_-(s,z;\lambda)) = \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau. \tag{1.2.76}$$

We obtain immediately that $\omega(\,\cdot\,;\lambda) \in C^\infty(\mathbb{R}^{n-1})$. It is sufficient to show the estimate (1.2.16b) for $|z| > R$ for some $R > 0$. We may therefore use the estimate (1.2.12) of Proposition 1.2.7; for $|z| > R$, R as in Proposition 1.2.7, there exists constants $C_R(\beta) > 0$ such that

$$|\partial_z^\beta (\nabla V)(\boldsymbol{x}_\infty(s,z;\lambda))| = |\partial_z^\beta \partial_s \boldsymbol{\xi}_\infty(s,z;\lambda)| \le C_R(\beta)\langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-\varrho-1} \tag{1.2.77}$$

It follows directly that

$$\begin{aligned} |\partial_z^\beta(\omega_- - \omega_+(z;\lambda))| &= \frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} |\partial_z^\beta (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))|\,d\tau \\ &\le \frac{C_R(\beta)}{\sqrt{\lambda}} \langle z\rangle^{-|\beta|} \int_{-\infty}^{\infty} \langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-1}\,d\tau \end{aligned}$$

Applying (A.54) and substituting in the integral,

$$\begin{aligned} |\partial_z^\beta(\omega_- - \omega_+(z;\lambda))| &\le \frac{C_R(\beta)}{\sqrt{\lambda}} \langle z\rangle^{-\varrho-1-|\beta|} \int_{-\infty}^{\infty} \langle 2\sqrt{\lambda}\langle z\rangle^{-1}\tau\rangle^{-\varrho-1}\,d\tau \\ &\le \frac{C_R(\beta)}{2\lambda} \langle z\rangle^{-\varrho-|\beta|} \int_{-\infty}^{\infty} \langle \tau\rangle^{-\varrho-1}\,d\tau \end{aligned}$$ □

Proof of (1.2.16a). The representations (1.2.11a) and (1.2.15a) at $s = 0$ yield $r_+(\,\cdot\,;\lambda) \in C^\infty(\mathbb{R}^{n-1})$, via

$$\begin{aligned} \check{z} - r_+(z;\lambda) &= 2(\boldsymbol{g}_+(0,z;\lambda) - \boldsymbol{g}_-(0,z;\lambda)) \\ &= 2\int_{-\infty}^{0}\int_{-\infty}^{t} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau\,dt - 2\int_{0}^{\infty}\int_{t}^{\infty} (\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))\,d\tau\,dt \end{aligned}$$

where we have used the representations (1.2.32), (1.2.73). Again we assume $|z| > R$ and apply (1.2.12) to obtain (1.2.77). Thus

$$\begin{aligned} |\partial_z^\beta(\check{z} - r_+(z;\lambda))| &\le 2\int_{-\infty}^{0}\int_{-\infty}^{t} |\partial_z^\beta(\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))|\,d\tau\,dt + 2\int_{0}^{\infty}\int_{t}^{\infty} |\partial_z^\beta(\nabla V)(\boldsymbol{x}_\infty(\tau,z;\lambda))|\,d\tau\,dt \\ &\le 4C_R(\beta)\langle z\rangle^{-|\beta|} \int_{-\infty}^{0}\int_{-\infty}^{t} \langle 2\sqrt{\lambda}\omega_-\tau + \check{z}\rangle^{-\varrho-1}\,d\tau\,dt. \end{aligned}$$

Applying (A.54) and substituting in the integral,

$$\begin{aligned} |\partial_z^\beta(\check{z} - r_+(z;\lambda))| &\le 4C_R(\beta)\langle z\rangle^{-1-\varrho-|\beta|} \int_{-\infty}^{0}\int_{-\infty}^{t} \langle 2\sqrt{\lambda}\langle z\rangle^{-1}\tau\rangle^{-\varrho-1}\,d\tau\,dt \\ &\le \frac{C_R(|\beta|)}{\lambda} \langle z\rangle^{1-\varrho-|\beta|} \int_{-\infty}^{0}\int_{-\infty}^{t} \langle \tau\rangle^{-\varrho-1}\,d\tau\,dt. \end{aligned}$$ □

PROOF OF (1.2.16c). With the same arguments as in Remark 1.2.13, we obtain that for sufficiently large $T > 0$

$$(\mathcal{F}_\lambda^+ u)(s,z) := \int_s^\infty \int_t^\infty (\nabla V)(2\sqrt{\lambda}\omega_- \tau + \check{z} + 2u(\tau, z))\, d\tau\, dt, \tag{1.2.78}$$

is a strict contraction on

$$\mathcal{M}_{T,\mathbb{H}}^+ := \big\{u \in C((T,\infty)\times\mathbb{H}, \mathbb{R}^n)\colon \sup_{\substack{s>T\\ z\in\mathbb{H}}} |u(s,z)| < 1\big\}. \tag{1.2.79}$$

By (1.2.73) and (1.2.74), its fixed point in $\mathcal{M}_{T,\mathbb{H}}^+$ (for sufficiently large T) is $\boldsymbol{g}_+(\,\cdot\,,\,\cdot\,;\lambda)|_{(T,\infty)\times\mathbb{H}}$. We will pursue the same strategy as in the proof of Proposition 1.2.7.

For $N \in \mathbb{N}$ and $T \in \mathbb{R}$ we introduce the Banach space

$$\mathcal{B}_{T,\mathbb{H};N,+} = \Big\{u \in C^N((T,\infty)\times\mathbb{H}, \mathbb{R}^n)\colon \lim_{s\to\infty} |u(s,z)| = 0, \|u\|_{T,\mathbb{H};N,+} < \infty\Big\} \tag{1.2.80}$$

with the norm

$$\|u\|_{T,\mathbb{H};N,+} := \max_{\substack{\beta\in\mathbb{N}^{n-1}\\ |\beta|\le N}} \sup_{\substack{z\in\mathbb{H}\\ s>T}} \big|\langle z\rangle^{|\beta|}\partial_z^\beta u(s,z)\big|. \tag{1.2.81}$$

We will consider the convex subset

$$\mathcal{M}_{T,\mathbb{H};N,+} = \Big\{u \in \mathcal{B}_{T,\mathbb{H};N,+}\colon \|u\|_{T,\mathbb{H};N,+} \le 1\Big\}, \tag{1.2.82}$$

which is a complete metric space.

1.2.18. LEMMA *Let $u \in \mathcal{M}_{T,\mathbb{H};N,+}$ with $T > 0$ large enough and let V satisfy the Potential Hypothesis. Then for any $N \in \mathbb{N}$ there exists a constant $C(N) > 0$ such that for any multi–index $\beta \in \mathbb{N}^{n-1}$ with $|\beta| \le N$ and any $\alpha \in \mathbb{N}^n$ the estimate*

$$\big|\partial_z^\beta(\partial^\alpha V)(2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda) + 2u(s,z))\big| \le C_\alpha(N)\langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{-\varrho-|\alpha|} \tag{1.2.83}$$

holds. We set $C(N) := \max_{|\alpha|\le 2} C_\alpha(N)$

PROOF. We will apply Lemma A.4; specifically, we will show that on $\Omega = (T,\infty)\times\mathbb{H}$ the estimate (A.35) holds for

$$\psi(s,z) := 2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda) + 2u(s,z) \qquad \text{with } u \in \mathcal{M}_{T,\mathbb{H};N,+},\ T \text{ large enough}, \tag{1.2.84a}$$

$$\rho_1(s,z) := 1, \quad \rho_2(s,z) := \langle z\rangle, \qquad \text{and } j = 0,\ |\beta| \le N. \tag{1.2.84b}$$

The condition (A.33) holds for any α by the Potential Hypothesis. We first remark that

$$\begin{aligned}
&1 + |2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda)|^2\\
&= 1 + |2\sqrt{\lambda}\omega_+(z;\lambda)s + \check{z} + (r_+(z;\lambda) - \check{z})|^2\\
&\ge 1 + 4\lambda s^2 - 4\sqrt{\lambda}|r_+(z;\lambda) - \check{z}|s + |z|^2 + |r_+(z;\lambda) - \check{z}|^2 - 2|z|\cdot|r_+(z;\lambda) - \check{z}| - 2\sqrt{\lambda}|z|s\\
&\ge \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^2 - 2\sqrt{\lambda}|z|s - C_{r;0}\langle z\rangle^{1-\varrho}(4\sqrt{\lambda}s + 2|z|),
\end{aligned} \tag{1.2.85}$$

where we have used the estimate (1.2.16a). For some sufficiently large $T > 0$ we thus have

$$\langle 2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda)\rangle^2 = 1 + |2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda)|^2 \ge \frac{9}{16}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^2 \tag{1.2.86}$$

for $s > T$, $z \in \mathbb{H}$. Then using $\langle x + y\rangle \ge \langle x\rangle - |y|$ and $u \in \mathcal{M}_{T,\mathbb{H};N,+}$, we can find some $T > 0$ so that

$$\langle\psi(s,z)\rangle \ge \langle 2\sqrt{\lambda}\omega_+(z;\lambda) + r_+(z;\lambda)\rangle - 2|u(s,z)| \ge \frac{3}{4}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle - 2 \ge \frac{1}{2}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle > 1 \tag{1.2.87}$$

for $s > T > 0$ and $z \in \mathbb{H}$. Note that (1.2.87) implies

$$\langle\psi\rangle \ge 1, \qquad \langle\psi\rangle\langle z\rangle^{-1} \ge \frac{1}{2}, \qquad \langle\psi(s,z)\rangle \ge \sqrt{\lambda}|s|. \tag{1.2.88}$$

Now $u \in \mathcal{M}_{\mathbb{R},R;N,+}$ implies

$$|\partial_z^\delta u(s,z)| \le \rho_2(s,z)^{-|\delta|} \tag{1.2.89}$$

by (1.2.81). Now for $|\beta| = 0$ (A.34) is trivially valid, so we assume $|\beta| \geq 1$, By (1.2.16a) and (1.2.16b) there exist constants $c(\beta) > 0$ such that

$$|\partial_z^\beta \omega_+(z;\lambda)| \leq c(\beta) \cdot \rho_2(s,z)^{-1-|\beta|}, \qquad |\partial_z^\beta r_+(z;\lambda)| \leq \begin{cases} c(\beta) & \text{for } |\beta| = 1, \\ c(\beta) \cdot \rho_2(s,z)^{-|\beta|} & \text{for } |\beta| > 1. \end{cases} \tag{1.2.90}$$

Then by (1.2.84a) for $|\delta| = 1$ we have with (1.2.89) and (1.2.90)

$$\begin{aligned} |\partial_z^\delta \psi(s,z)| &= |2\sqrt{\lambda}\partial_z^\delta \omega_+(z;\lambda)s + \partial_z^\delta r_+(z;\lambda) + 2\partial_z^\delta u(s,z)| \\ &\leq 2\sqrt{\lambda}c(\delta)\rho_2(s,z)^{-2}|s| + c(\delta) + 2\rho_2(s,z)^{-1} \end{aligned} \tag{1.2.91}$$

Then with (1.2.88) we have

$$\begin{aligned} |\partial_z^\delta \psi(s,z)| &\leq 2c(\delta)\rho_2(s,z)^{-2}\langle\psi(s,z)\rangle + 2c(\delta)\langle\psi(s,z)\rangle\rho_2^{-1} + 2\rho_2(s,z)^{-1}\langle\psi(s,z)\rangle \\ &\leq c'(\delta) \cdot \rho_2(s,z)^{-1}\langle\psi(s,z)\rangle \end{aligned} \tag{1.2.92}$$

for some $c'(\beta) > 0$. Similarly, (1.2.89) and (1.2.90) yield

$$|\partial_z^\delta \psi(s,z)| \leq c''(\delta) \cdot \rho_2(s,z)^{-|\delta|}\langle\psi(s,z)\rangle \tag{1.2.93}$$

with some $c''(\delta) > 0$ for any δ with $|\delta| > 1$. Thus (A.34) holds and (A.35) yields

$$\left|\partial_z^\beta (\partial_x^\alpha V)(\psi(s,z))\right| \leq C(\alpha,|\beta|) \cdot \rho_2(s,z)^{-|\beta|}\langle\psi(s,z)\rangle^{-\varrho-|\alpha|} \tag{1.2.94}$$

with some constant $C(\alpha, N) > 0$. Together with (1.2.88) we can thus deduce the existence of constants $C_\alpha(N)$ such that (1.2.83) holds. □

Now since

$$\begin{aligned} \|\mathcal{F}_\lambda^+ u\|_{T,\mathbb{H};N,+} &\leq \max_{\substack{(k,\beta)\in\mathbb{N}\times\mathbb{N}^{n-1} \\ |\beta|\leq N}} \sup_{\substack{z\in\mathbb{H} \\ s>T}} \langle z\rangle^{|\beta|} \int_s^\infty \int_t^\infty \left|\partial_z^\beta (\nabla V)(2\sqrt{\lambda}\omega_+(z;\lambda)\tau + r_+(z;\lambda) + 2u(\tau,z))\right| d\tau\, dt \\ &\leq C(N) \max_{\substack{(k,\beta)\in\mathbb{N}\times\mathbb{N}^{n-1} \\ |\beta|\leq N}} \int_T^\infty \int_t^\infty \langle 2\sqrt{\lambda}\omega_-\tau\rangle^{-\varrho-1}\, d\tau\, dt \end{aligned} \tag{1.2.95}$$

where we have applied (1.2.83), we can ensure that $\|\mathcal{F}_\lambda^+ u\|_{T,\mathbb{H};N,+} < 1$. Similarly to the proofs of Lemmas 1.2.17, it is not difficult see that $\mathcal{F}_\lambda^+$ is C^1 and for $T > 0$ large enough a strict contraction on $\mathcal{M}_{T,\mathbb{H};N,+}$. Thus $\boldsymbol{g}_+|_{(T,\infty)\times\mathbb{H}} \in \mathcal{M}_{T,\mathbb{H};N,+}$ and we obtain directly that

$$|\partial_z^\beta \boldsymbol{g}_+(s,z;\lambda)| \leq \langle z\rangle^{-|\beta|}\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{1-\varrho} \qquad \text{for } s > T(N),\ |\beta| \leq N \text{ and } z \in \mathbb{H}. \tag{1.2.96}$$

Repeating the arguments following Lemma 1.2.17 and applying (1.2.83) with the chain rule, we inductively obtain obtain (1.2.16c). □

1.3. The scattering manifold in euclidean phase space

In this section we will show that the union over all $z \in \mathbb{H}$ of the phase trajectories $\mathcal{T}_z$ defined in (1.2.8) is a lagrangian manifold in phase space $T^*\mathbb{R}^n$. It is well-known (cf., e.g., [25]) that the union of the integral curves of a hamiltonian vector field through a non–characteristic $(n-1)$–dimensional surface (given by some initial conditions) gives just such a manifold. Here, the Energy Hypothesis and "initial conditions at $t = -\infty$" will play a major role in the proof of our main result.

1.3.1. Definition *Let (N, σ) be a symplectic manifold, i.e., a manifold N with a smooth, non-degenerate two-form σ. A submanifold $M \subset N$ is called lagrangian if $\sigma|_{TM\times TM} = 0$.*

In our situation it is sufficient to consider cotangent bundles, as we are interested in $T^*\mathbb{R}^n$ (and T^*S^{n-1} in Section 2.1) only. On $T^*\mathbb{R}^n$ we have a canonical one-form $\alpha = \sum \xi_j\, dx_j$. Its exterior derivative, $\sigma = d\xi \wedge dz$ is known as the canonical symplectic form and $(T^*\mathbb{R}^n, \sigma)$ is a symplectic space. In (x,ξ)–coordinates, cf. Notation 1.1.1,

$$\widetilde{\sigma}|_p\colon \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}, \qquad \widetilde{\sigma}_p\big((u,v),(u',v')\big) = \langle u, v'\rangle - \langle u', v\rangle. \tag{1.3.1}$$

1.3.2. Remark For later use, we note that the natural symplectic form on T^*S^{n-1} is $\sigma^\circ = i^*\sigma$, where $i\colon T^*S^{n-1} \to T^*\mathbb{R}^n$ is the natural inclusion map.

1.3.3. Theorem *The map*

$$\iota\colon \mathbb{R}\times\mathbb{R}^{n-1}\to T^*\mathbb{R}^n, \qquad (s,z)\mapsto \big(\boldsymbol{x}_\infty(s,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s,z;\lambda))\big) \tag{1.3.2}$$

is an embedding and

$$\Lambda := \iota(\mathbb{R}\times\mathbb{R}^{n-1}) \tag{1.3.3}$$

is a lagrangian submanifold of $T^*\mathbb{R}^n$.

The proof relies on various preliminary results. We first note that it follows directly from the Energy Hypothesis and the standard theory of ordinary differential equations that any single trajectory (integral curve of H_p) is a 1–dimensional embedded submanifold of $T^*\mathbb{R}^n$.

1.3.4. Lemma *For any* $z\in\mathbb{H}$ *the map* $\tau_z := \iota(\,\cdot\,,z)$ *is an embedding and the trajectories* $\mathcal{T}_z=\tau_z(\mathbb{R})$ *are mutually disjoint smooth 1–dimensional submanifolds of* $T^*\mathbb{R}^n$.

Proof. Recall from Lemma B.3 that a map between manifolds is an embedding if it is an injective immersion that is also proper, i.e., the pre-image of every compact subset is again compact.

The map τ_z is C^∞ by Proposition 1.2.7, and the tangent map $(\tau_z)_*\colon T\mathbb{R}\to T(T^*\mathbb{R}^n)$ is given by

$$\frac{\partial}{\partial t}\Big|_s \mapsto= H_p|_{\tau_z(s)}. \tag{1.3.4}$$

Since the hamiltonian vector field H_p does not vanish on $\tau_z(\mathbb{R})$, cf. Remark 1.1.4, the tangent map is injective and τ_z an immersion. We claim that τ_z is injective. In order to see this, we use the hamiltonian flow g_t (cf. (1.1.9)), which acts on $\tau_z(s)$ through $g_t\tau_z(s)=\tau_z(s+t)$ and has the semi-group property $g_{t_1}g_{t_2}=g_{t_1+t_2}$. Then if there existed two times $s_1, s_2=s_1+\gamma$ such that $\tau_z(s_1)=\tau_z(s_2)=\tau_z(s_1+\gamma)$, the trajectory would be periodic, i.e., $\tau_z(s)=\tau_z(s+\gamma)$ for any $s>s_1$ and therefore bounded as $s\to\infty$. This contradicts the Energy Hypothesis.

It remains to show that τ_z is proper. Let $K\subset\tau_z(\mathbb{R})$ be compact. The continuity of τ_z implies that $\tau^{-1}(K)$ is closed. Furthermore, by the Energy Hypothesis, $\tau_z^{-1}(K)$ is necessarily bounded, therefore compact.

Thus τ_z is an embedding for any $z\in\mathbb{H}$ and $\mathcal{T}_z$ is a smooth submanifold of $T^*\mathbb{R}^n$. The trajectories $\mathcal{T}_z$ are mutually disjoint: for letting $p\in\mathcal{T}_z\cap\mathcal{T}_{z'}$, we have $\mathcal{T}_z=g(\mathbb{R},p)=\mathcal{T}_{z'}$ using the hamiltonian flow (1.2.14). But then $z=z'$ by Proposition 1.2.5. □

Subsequently, we will often make use of "coordinized" maps.

1.3.5. Convention *We will usually denote with a tilde the "coordinized" versions of mappings; if* $A\colon M\to N$ *is a smooth map and* φ_1,φ_2 *are charts on* $M\supset U_1\ni p$ *and* $N\supset U_2\ni A(p)$*, respectively, then* $\widetilde{A}=\varphi_2\circ A\circ\varphi_1^{-1}$ *on* $\varphi_1(U_1)$.

The complementary objects to the trajectories $\mathcal{T}_z$ of (1.2.8) are the "wavefronts" Λ_s, which we now introduce.

1.3.6. Lemma *For any* $s\in\mathbb{R}$*,* $\iota_s=\iota(s,\,\cdot\,)$ *is an embedding and* $\Lambda_s:=\iota_s(\mathbb{R}^{n-1})$ *is a submanifold of* $T^*\mathbb{R}^n$*. Furthermore, for any* $t\in\mathbb{R}$*, the hamiltonian flow* (1.2.14) *considered as a map*

$$g_t^{(s)}\colon \Lambda_s\to\Lambda_{s+t}, \qquad \big(\boldsymbol{x}_\infty(s,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s,z;\lambda))\big)\mapsto\big(\boldsymbol{x}_\infty(s+t,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s+t,z;\lambda))\big) \tag{1.3.5}$$

is a diffeomorphism.

Proof. Step 1: We first show that the map ι_s is an embedding if s is sufficiently small. We will split ι_s into two auxiliary maps. First, for $s\in\mathbb{R}$, we define

$$\kappa_s\colon\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}, \qquad z\mapsto \boldsymbol{x}'_\infty(s,z;\lambda)=(\boldsymbol{x}_1(z;s,\lambda),\dots,\boldsymbol{x}_{n-1}(z;s,\lambda)). \tag{1.3.6}$$

We will show that for some $s_0\ll 0$, any map κ_s with $s<s_0$ is an embedding. Its differential is given by

$$D\kappa_s=\left(\frac{\partial\boldsymbol{x}'_\infty(s,z;\lambda)}{\partial z}\right). \tag{1.3.7}$$

and (1.2.11a) and (1.2.13) imply

$$|D\kappa_s-\mathbb{1}|\le C|s|^{-\varrho} \qquad \text{for all } s<-S \text{ and } z\in\mathbb{H}. \tag{1.3.8}$$

Thus by Lemma B.4, κ_s is an embedding if $s < s_0$ for some $s_0 < -S$. We define

$$\begin{aligned} &\psi_s \colon \kappa_s(\mathbb{R}^{n-1}) \to \mathbb{R}^n \times \mathbb{R}^n, \\ &(x_1, \dots, x_{n-1}) \mapsto (x_1, \dots, x_{n-1}, 2\sqrt{\lambda}s + 2\boldsymbol{g}_n; \partial_s \boldsymbol{g}_1, \dots, \partial_s \boldsymbol{g}_{n-1}, \sqrt{\lambda} + \partial_s \boldsymbol{g}_n). \end{aligned} \tag{1.3.9}$$

where we have written

$$\begin{aligned} (\boldsymbol{g}_1, \dots, \boldsymbol{g}_n) &:= \boldsymbol{g}_-(s, \kappa_s^{-1}(x_1, \dots, x_{n-1}); \lambda), \\ (\partial_s \boldsymbol{g}_1, \dots, \partial_s \boldsymbol{g}_n) &:= (\partial_s \boldsymbol{g}_-)(s, \kappa_s^{-1}(x_1, \dots, x_{n-1}); \lambda) \end{aligned}$$

for short. It is easily seen that ψ_s is an embedding for $s < s_0$, as it maps coordinates on an open domain to the graph of a smooth function of these coordinates. It follows from the representations (1.2.11) that

$$\tilde{\iota}_s = \psi_s \circ \kappa_s, \qquad \text{for } s < s_0, \tag{1.3.10}$$

which is therefore an embedding.

Step 2: Let $g_t \colon T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ denote the hamiltonian flow of (1.2.14). We will show that if $s < s_0$, the restriction $g_t|_{\Lambda_s}$ is an embedding for arbitrary $t \in \mathbb{R}$. By the explicit construction of the trajectories in Proposition 1.2.7, the map $g_t|_{\Lambda_s} \colon \Lambda_s \to T^*\mathbb{R}^n$ exists for all $t \in \mathbb{R}$. Since the hamiltonian vector field $H_p|_{\Lambda_s}$ is non–vanishing for any $s \in \mathbb{R}$ (cf. Remark 1.1.4) $g_t|_{\Lambda_s}$ is an immersion. We next show that $g_t|_{\Lambda_s}$ is also injective for any s and t.

Fix some $s < s_0$. Then $g_t(\Lambda_s)$ simply gives the integral curves of H_p at time t with initial conditions $\Lambda_s = \iota_s(\mathbb{R}^{n-1})$. Any two points $\iota_s(z), \iota_s(z') \in \Lambda_s$ are elements of trajectories $\mathcal{T}_z$ and $\mathcal{T}_{z'}$, respectively, cf. Lemma 1.3.4. Since these trajectories are disjoint, and $g_t(\mathcal{T}_z) = \mathcal{T}_z$, the map $g_t \colon \Lambda_s \to \Lambda_{s+t}$ is clearly injective for any t.

Furthermore, $g_t|_{\Lambda_s}$ is smooth, since the dependence on initial conditions is smooth by the standard theory of ordinary differential equations, cf., e.g., [25]. The inverse map, $g_{-t}|_{\Lambda_{s+t}}$ is continuous by the continuity of ODE solutions with respect to initial conditions.

Thus by Lemma B.3 $g_t|_{\Lambda_s}$ is an embedding for $s < s_0$ and $t \in \mathbb{R}$. It follows immediately that any Λ_s, $s \in \mathbb{R}$ is an $(n-1)$–dimensional submanifold. Since $g_t^{(s)}$ is an injective immersion between submanifolds, it is also a diffeomorphism. Clearly, this also implies that ι_s is an embedding for any $s \in \mathbb{R}$. ☐

1.3.7. Corollary *For any $s \in \mathbb{R}$, the map*

$$\iota_s \colon \mathbb{R}^{n-1} \to \Lambda_s \qquad z \mapsto (\boldsymbol{x}_\infty(s, z; \lambda), X^*(\boldsymbol{\xi}_\infty(s, z; \lambda))) \tag{1.3.11}$$

is a diffeomorphism and for any $z \in \mathbb{R}^{n-1}$ and any $s \in \mathbb{R}$ the manifolds Λ_s are transverse to $\mathcal{T}_z$, i.e., $T_p\mathcal{T}_z \cap T_p\Lambda_s = \{0\}$ for $p = \iota(s, z)$.

Proof. We will verify through direct calculation that for any $z \in \mathbb{H}$ the trajectory $\mathcal{T}_z$ is transverse to Λ_s if $s \ll 0$ is sufficiently small. In fact, $\iota_s(z) = \tau_z(s)$

$$(\tau_z)_* \left.\frac{\partial}{\partial t}\right|_{t=s} = \sum \left.\frac{\partial \boldsymbol{x}_i(t; z, \lambda)}{\partial t}\right|_{t=s} \left.\frac{\partial}{\partial x_i}\right|_{\tau_z(s)} \tag{1.3.12}$$

$$(\iota_s)_* \left.\frac{\partial}{\partial y_j}\right|_{y=z} = \sum \left.\frac{\partial \boldsymbol{x}_i(s; y, \lambda)}{\partial z_j}\right|_{y=z} \left.\frac{\partial}{\partial x_i}\right|_{\iota_s(z)} \tag{1.3.13}$$

and by (1.2.13),

$$\left| (\tau_z)_* \left.\frac{\partial}{\partial t}\right|_{t=s} - 2\sqrt{\lambda} \left.\frac{\partial}{\partial x_n}\right|_{\tau_z(s)} \right| \leq C_1 \cdot \langle s \rangle^{-\varrho}, \tag{1.3.14}$$

$$\left| (\iota_s)_* \left.\frac{\partial}{\partial y_j}\right|_{y=z} - \sum_{i=1}^{n-1} \left.\frac{\partial}{\partial x_i}\right|_{\tau_z(s)} \right| \leq C_2 \cdot \langle s \rangle^{-\varrho}. \tag{1.3.15}$$

This yields the transversality for sufficiently small s. Since g_t is an immersion for any t, the transversality holds for any $s \in \mathbb{R}$. ☐

1.3.8. Lemma *For any $s \in \mathbb{R}$, the restriction of the hamiltonian flow $g|_{\mathbb{R} \times \Lambda_s}$ is an embedding.*

PROOF. We first show that $g|_{\mathbb{R}\times\Lambda_s}$ is injective. Assume that there exist $t, t' \in \mathbb{R}$, $z, z' \in \mathbb{H}$ so that

$$\big(\boldsymbol{x}_\infty(s+t,z;\lambda), X^*(\boldsymbol{\xi}_\infty(s+t,z;\lambda))\big) = \big(\boldsymbol{x}_\infty(s+t';z',\lambda), X^*(\boldsymbol{\xi}_\infty(s+t';z',\lambda))\big). \tag{1.3.16}$$

We argue as in the proof of Lemma 1.3.4; for $z \neq z'$, (1.3.16) contradicts the disjointedness of trajectories $\mathcal{T}_z$ in Lemma 1.3.4, while for $z = z'$, (1.3.16) implies that the trajectory $\mathcal{T}_z$ is periodic, which violates the Non–trapping Condition. Thus $g|_{\mathbb{R}\times\Lambda_s}$ is injective.

Using (x,ξ)–coordinates for $T^*\mathbb{R}^n$ and the global chart $\iota_s^{-1}\colon \Lambda_s \to \mathbb{R}^{n-1}$, we may consider the coordinized map

$$\widetilde{g|_{\mathbb{R}\times\Lambda_s}}\colon \mathbb{R}\times\mathbb{R}^{n-1} \to \mathbb{R}^n\times\mathbb{R}^n, \qquad (t,z) \mapsto \big(\boldsymbol{x}_\infty(s+t,z;\lambda), \boldsymbol{\xi}_\infty(s+t,z;\lambda)\big). \tag{1.3.17}$$

We will show that $\widetilde{g|_{\mathbb{R}\times\Lambda_s}}$ is an immersion. The tangent map at (t_0,z_0) is simply given by

$$\widetilde{g|_{\mathbb{R}\times\Lambda_s}}_*|_{(t_0,z_0)} = \begin{pmatrix} \frac{\boldsymbol{x}_\infty(s+t,z;\lambda)}{\partial t} & \frac{\boldsymbol{x}_\infty(s+t,z;\lambda)}{\partial z} \\ \frac{\boldsymbol{\xi}_\infty(s+t,z;\lambda)}{\partial t} & \frac{\boldsymbol{\xi}_\infty(s+t,z;\lambda)}{\partial z} \end{pmatrix}\Bigg|_{(t_0,z_0)}. \tag{1.3.18}$$

Now comparing with the definitions of $\tau_z = \iota(\,\cdot\,,z)$ and $\iota_s = \iota(s,\,\cdot\,)$, we see that

$$\widetilde{g|_{\mathbb{R}\times\Lambda_s}}_*|_{(t_0,z_0)} = \big((\widetilde{\tau}_{z_0})_*|_{s+t_0} \quad (\widetilde{\iota}_{s+t_0})_*|_{z_0}\big)\,, \tag{1.3.19}$$

where the tilde refers to the coordinized maps as usual. Now the $n-1$ column vectors of the block matrix $(\tilde{\iota}_{s+t_0})_*|_{z_0}$ are simply the coordinate vectors of tangent vectors spanning $T_{\iota_{s+t_0}(z_0)}\Lambda_{s+t_0}$, while the vector $(\widetilde{\tau}_{z_0})_*|_{s+t_0}$ is the coordinate representation of $(\tau_{z_0})_*\,\frac{\partial}{\partial s}\big|_{s+t_0}$. By Corollary 1.3.7, the latter is independent of the $n-1$ former, mutually independent vectors. It follows that the rank of $\widetilde{g|_{\mathbb{R}\times\Lambda_s}}_*|_{(t_0,z_0)}$ is n for any (s_0,t_0), so g is an immersion.

Now by standard theory, the integral curves of (1.1.6) are smooth jointly with respect to initial conditions and time, so g is a smooth injective immersion. The smoothness of the inverse map follows immediately. □

PROOF OF THEOREM 1.3.3. The proof of Theorem 1.3.3 is now an easy consequence of the proof of Lemma 1.3.8. Using (x,ξ)–coordinates on $T^*\mathbb{R}^n$ for ι, we see that from (1.3.17),

$$\widetilde{\iota} = \widetilde{g|_{\mathbb{R}\times\Lambda_0}}, \tag{1.3.20}$$

so ι is an embedding and Λ is a submanifold of $T^*\mathbb{R}^n$. Furthermore, it follows from Corollary 1.3.7 that for sufficiently small $s<0$ the manifold Λ_s is non-characteristic for the hamiltonian vector field H_p. Thus the set of integral curves of H_p through Λ_s (which is just Λ) is a lagrangian manifold, cf. [25]. □

CHAPTER 2

The scattering manifold at infinity; caustics

From the perspective of scattering theory, the "natural" configuration space variable containing information on the scattered state is the scattering angle $\omega_+(\,\cdot\,,\lambda)$. We will show that

$$\boldsymbol{\omega}(s,z) := \frac{\boldsymbol{x}_\infty(s,z;\lambda)}{|\boldsymbol{x}_\infty(s,z;\lambda)|} \to \omega_+(z;\lambda) \qquad \text{as } s\to\infty. \tag{2.0.21}$$

For $x\in\mathbb{R}^n$ we define $\pi_x\colon \mathbb{R}^n\to\mathbb{R}^n$ as the projection $y\mapsto\langle x,y\rangle x$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. Then for $(x,X^*(\xi))\in T^*\mathbb{R}^n$ we define the "radial momentum" by $L_r(x,X^*(\xi))=\pi_{x/|x|}\xi$. [1] Then it follows from (2.0.21) and (1.2.7b) that

$$(\iota^*L_r)(s,z)=\pi_{\omega(s,z)}\boldsymbol{\xi}_\infty(s,z;\lambda)\xrightarrow{s\to\infty}\sqrt{\lambda}\omega_+(z;\lambda). \tag{2.0.22}$$

for all $z\in\mathbb{H}$. Thus the radial momentum does not contain any additional information about the scattering behaviour of a trajectory $\mathcal{T}_z$ to that already contained in $\omega_+(z;\lambda)$.

By contradistinction, we will show that the "angular momentum", which we will define through a certain isometric projection of $(x,X^*(\xi)\in T^*\mathbb{R}^n$ onto $T^*S^{n-1}\subset T^*\mathbb{R}^n$, converges towards an "asymptotic angular momentum" (which we denote by $L_+(z;\omega)$ as $s\to\infty$. For each trajectory we thus obtain a point $(\omega_+(z;\lambda),X^*(L_+(z;\lambda)))\in T^*S^{n-1}$. We will show the convergence (giving explicit estimates) and prove that the union over $z\in\mathbb{H}$ of $(\omega_+(z;\lambda),X^*(L_+(z;\lambda)))$ gives a submanifold $\mathcal{L}_+\subset T^*S^{n-1}$ in Section 2.1.

In Section 2.2 we will prove that the manifolds $\Lambda\subset T^*\mathbb{R}^n$ of Theorem 1.3.3 and $\mathcal{L}_+$ are both lagrangian submanifolds in the cotangent bundles over euclidean space and the sphere, respectively. In Section 2.3 we will recall the role played by lagrangian coordinates and generating functions, and give explicit formulae for generating functions on Λ and $\mathcal{L}_+$.

Finally, in Section 2.4 we will analyse the interplay between lagrangian coordinates on $\mathcal{L}_+$ and Λ, yielding technical prerequisites for the constructions of a useful Maslov operator on Λ, to be used in Chapter 3 for the leading–order term in the semi–classical expansion of the scattering amplitude.

2.1. The asymptotic manifold over the sphere

2.1.1. Convention *We refer to Convention 1.1.1. We denote by g the usual euclidean metric on $T\mathbb{R}^n$, i.e., its fibre-wise action is given by*

$$g_p\colon T_p(\mathbb{R}^n)\times T_p(\mathbb{R}^n)\to\mathbb{R}, \qquad g_p(X_p(v),X_p(v'))=\langle v,v'\rangle, \tag{2.1.1}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidean scalar product on $\mathbb{R}^n$. We introduce the dilated metric h by setting

$$h_p(\,\cdot\,,\,\cdot\,)=|p|^{-2}g_p(\,\cdot\,,\,\cdot\,) \qquad \text{on } T_p\mathbb{R}^n \tag{2.1.2}$$

We will regard the tangent and cotangent bundles on the sphere S^{n-1} as subspaces of $T\mathbb{R}^n$ and $T^*\mathbb{R}^n$, respectively, setting

$$TS^{n-1}=\{(p,X(v))\in T\mathbb{R}^n\colon |p|=1,\ v\perp p\},\quad T^*S^{n-1}=\{(p,X^*(\xi))\in T^*\mathbb{R}^n\colon |p|=1,\ \xi\perp p\}. \tag{2.1.3}$$

Recall that the restriction of h to TS^{n-1} coincides with the same restriction of g and gives a metric on the tangent bundle of the sphere, which we denote by h°. Let

$$\tau_h\colon T\mathbb{R}^n\to T^*\mathbb{R}^n, \qquad (p,X_p(v))\mapsto(p,h_p(X_p(v),\,\cdot\,))=(p,X_p^*(|p|^{-2}v)) \tag{2.1.4}$$

---

[1] Note that for any choice of polar coordinates $(r,\theta)\colon\mathbb{R}^n\to\mathbb{R}_+\times\mathbb{R}^{n-1}$ we then have $dr|_x=X_x^*(x/|x|)$ and $X_x^*(L_r)=\langle x/|x|,\xi\rangle\,dr|_x$.

and

$$\tau_{h^\circ}\colon TS^{n-1}\to T^*S^{n-1},\qquad (q,X_q(u))\mapsto (q,h^\circ_q(X_q(u),\,\cdot\,))=(q,X^*_q(u))\quad (u\perp q) \tag{2.1.5}$$

denote the metric–induced isomorphisms between tangent and cotangent bundles.

We introduce the natural projection

$$\Theta\colon \mathbb{R}^n\to S^{n-1},\qquad \Theta(x):=\hat{x}:=\frac{x}{|x|}. \tag{2.1.6}$$

2.1.2. Remark The action of the push–forward $\Theta_*\colon T\mathbb{R}^n\to TS^{n-1}$ of (2.1.6) is given by

$$\Theta_*\colon (p,X_p(v))\mapsto \big(\hat{p},X_{\hat{p}}(|p|^{-1}\pi^\perp_{\hat{p}}v)\big), \tag{2.1.7}$$

where

$$\pi^\perp_x y=y-\pi_x y=y-\langle x,y\rangle x \qquad \text{for } x,y\in\mathbb{R}^n,\ |x|=1, \tag{2.1.8}$$

is the orthogonal projection onto the orthogonal complement of x.

2.1.3. Definition & Lemma *The map*

$$\boldsymbol{\Theta}\colon T^*\mathbb{R}^n\to T^*S^{n-1},\qquad \boldsymbol{\Theta}:=\tau_{h^\circ}\circ\Theta_*\circ\tau_h^{-1}. \tag{2.1.9}$$

is explicitly given by

$$\boldsymbol{\Theta}(p,X^*_p(\xi))=(\hat{p},X^*_{\hat{p}}(|p|\pi^\perp_{\hat{p}}\xi)). \tag{2.1.10}$$

It is (fibrewise) an isometry on the orthogonal complement of its kernel with respect to the (dual of the) dilated euclidean metric on $T^*\mathbb{R}^n$ and the restricted euclidean metric on T^*S^{n-1}.

Proof. The formula (2.1.10) follows immediately from (2.1.7), (2.1.4) and (2.1.5). It is sufficient to show that $\Theta_*|_p\colon (T_p\mathbb{R}^n,h)\to(T_{\hat{p}}S^{n-1},h^\circ)$ is an isometry when restricted to the orthogonal complement of its kernel. By (2.1.7), the kernel of $\Theta_*|_p$ is given by

$$\ker\Theta_*|_p=\{X_p(v)\in T_p\mathbb{R}^n\colon \pi^\perp_{\hat{p}}v=0\}. \tag{2.1.11}$$

Now for

$$X_p(v),X_p(u)\in(\ker\Theta_*|_p)^\perp=\{X_p(v)\in T_p\mathbb{R}^n\colon \pi^\perp_{\hat{p}}v=v\} \tag{2.1.12}$$

(i.e., $u,v\perp\hat{p}$) we have using (2.1.7)

$$\begin{aligned} h^\circ_{\Theta(p)}(\Theta_*|_pX_p(v),\Theta_*|_pX_p(u)) &= |p|^{-2}h^\circ_{\hat{p}}(X_{\hat{p}}(\pi^\perp_{\hat{p}}v),X_{\hat{p}}(\pi^\perp_{\hat{p}}u))\\ &= |p|^{-2}\langle\pi^\perp_{\hat{p}}v,\pi^\perp_{\hat{p}}u\rangle\\ &= |p|^{-2}\langle v,u\rangle=h_p(X_p(v),X_p(u)). \end{aligned}$$ ☐

Our main interest is in the action of $\boldsymbol{\Theta}$ on Λ, and we will consider $\boldsymbol{\Theta}\circ\iota_s$, where $\iota_s\colon\mathbb{R}^{n-1}\to\Lambda_s\subset\Lambda$ was introduced in Section 1.3 (cf., e.g., (1.3.11)). Effectively, $\boldsymbol{\Theta}\circ\iota_s$ maps points in the impact plane (which parametrize the trajectories) into the angle and angular momentum at time s. We will see that the limit as $s\to\infty$ of these variables exists in T^*S^{n-1} and forms a lagrangian manifold.

In order to keep our notation readable, we set

$$\boldsymbol{\omega}(s,z):=\frac{\boldsymbol{x}_\infty(s,z;\lambda)}{|\boldsymbol{x}_\infty(s,z;\lambda)|},\qquad \boldsymbol{L}(s,z):=|\boldsymbol{x}_\infty(s,z;\lambda)|\pi^\perp_{\boldsymbol{\omega}(s,z)}\boldsymbol{\xi}_\infty(s,z;\lambda), \tag{2.1.13}$$

$$\omega_+(z):=\omega_+(z;\lambda),\qquad L_+(z):=-\sqrt{\lambda}\,\pi^\perp_{\omega_+(z)}r_+(z;\lambda), \tag{2.1.14}$$

cf. Proposition 1.2.10 for the notation employed.

2.1.4. Theorem *For all $z\in\mathbb{H}$ we have*

$$\lim_{s\to+\infty}\boldsymbol{\Theta}\circ\iota_s(z)=(\omega_+(z),X^*(L_+(z)))=:S^+_\lambda(z),\qquad \boldsymbol{\Theta}\circ\iota_s(z)=(\boldsymbol{\omega}(s,z),X^*(\boldsymbol{L}(s,z))). \tag{2.1.15}$$

and this limit is uniform in compact subsets of $\mathbb{R}^{n-1}$*; more precisely, for any* $R, T > 0$*, any* $\alpha \in \mathbb{N}$ *and any multi–index* $\beta \in \mathbb{N}^{n-1}$ *there exist constants* $C_{\alpha,\beta,R,T}, C'_{\alpha,\beta,R,T} > 0$ *so that for* $s > T$*,*

$$\sup_{|z|\le R} |\partial_s^\alpha \partial_z^\beta (\boldsymbol{\omega}(s,z) - \omega_+(z))| \le C_{\alpha,\beta,R,T}\, s^{-1-\alpha}, \tag{2.1.16a}$$

$$\sup_{|z|\le R} |\partial_s^\alpha \partial_z^\beta (\boldsymbol{L}(s,z) - L_+(z))| \le C'_{\alpha,\beta,R,T}\, s^{-1-\alpha}. \tag{2.1.16b}$$

Furthermore, the map $S_\lambda^+ \colon \mathbb{R}^{n-1} \to T^*S^{n-1}$ *given by* (2.1.15) *is an embedding and the image*

$$\mathcal{L}_+ := S_\lambda^+(\mathbb{R}^{n-1}) \tag{2.1.17}$$

is a submanifold of T^*S^{n-1}*.*

2.1.5. Remark The estimates (2.1.16) differ from those of Propositions 1.2.7, 1.2.10 in two important respects: first, they are uniform only for z in compact subsets of $\mathbb{R}^{n-1}$, and second, the speed of convergence as $s \to \infty$ does not depend on $\varrho > 1$. In fact, even if there is no potential ($V \equiv 0$), the estimates cannot be improved. Nevertheless, the convergence expressed in (2.1.16) is the crucial result for our analysis of the behaviour of the scattering amplitude.

2.1.6. Remark Theorem 2.1.4 is our first result where the sharp estimates of Proposition 1.2.10 are crucial. While we have previously used the estimates as a matter of convenience, the limit statements of Definition 1.2.3 together with some uniformity assertion for the impact variable $z \in \mathbb{H}$ would have been sufficient. However, the precise asymptotic behaviour of the trajectories as $s \to +\infty$ is indispensable for the following proof of Theorem 2.1.4.

By contrast, the constructions involving $\boldsymbol{\Theta}$ preceeding Theorem 2.1.4 are not srictly necessary, and serve only as a motivation for the definition of $(\boldsymbol{\omega}(s,z), \mathcal{L}(s,z))$. In fact, we will prove Theorem 2.1.4 without any reference to $\boldsymbol{\Theta}$ at all.

2.1.1. Proof of Theorem 2.1.4.

The proof Theorem 2.1.4 is the main objective of this section. We first give a straightforward technical result on the behaviour of $L_+(z;\lambda)$:

2.1.7. Lemma *For any* $\beta \in \mathbb{N}^{n-1}$ *there exists a constant* $C_{l;\beta} > 0$ *such that*

$$|\partial_z^\beta(\sqrt{\lambda}\check{z} - L_+(z))| \le C_{l;\beta} \cdot \langle z\rangle^{1-\varrho-|\beta|}, \qquad z \in \mathbb{R}^{n-1},\ \check{z} = (z_1, \dots, z_{n-1}, 0). \tag{2.1.18}$$

Proof. Since

$$|\partial_z^\beta(\sqrt{\lambda}\check{z} - L_+(z))| = |\partial_z^\beta(\sqrt{\lambda}\check{z} - \sqrt{\lambda} r_+(z;\lambda) + \sqrt{\lambda}\langle\omega_+(z;\lambda), r_+(z;\lambda)\rangle\omega_+(z;\lambda))| \tag{2.1.19}$$

the result follows from the triangle inequality and the product rule A.1 with the estimates (1.2.16a) and (2.1.20) below. □

2.1.8. Lemma *For any* $\beta \in \mathbb{N}^n$ *there exists a constant* $c(\beta) > 0$ *so that*

$$|\partial_z^\beta \langle\omega_+(z;\lambda), r_+(z;\lambda)\rangle| \le c(\beta) \cdot \langle z\rangle^{1-\varrho-|\beta|} \tag{2.1.20}$$

Proof. The orthogonality of $\check{z}$ and ω_- yields

$$\begin{aligned}\partial_z^\beta\langle\omega_+(z;\lambda), r_+(z;\lambda)\rangle &= \partial_z^\beta\langle\omega_+(z;\lambda) - \omega_-, r_+(z;\lambda)\rangle + \partial_z^\beta\langle\omega_-, r_+(z;\lambda) - \check{z}\rangle \\ &= \partial_z^\beta\langle\omega_+(z;\lambda) - \omega_-, r_+(z;\lambda) - \check{z}\rangle + \partial_z^\beta\langle\omega_+(z;\lambda) - \omega_-, \check{z}\rangle + \partial_z^\beta\langle\omega_-, r_+(z;\lambda) - \check{z}\rangle\end{aligned}$$

Using (1.2.16b) and (1.2.16a), the product rule A.1 yields (2.1.20). □

We will split the proof into distinct parts, starting with the crucial estimates (2.1.16).

Proof of the estimates (2.1.16). It follows that With (1.2.22) and (1.2.23) we can write

$$\boldsymbol{\omega}(s,z) = \frac{\boldsymbol{x}_\infty(s,z;\lambda)}{|\boldsymbol{x}_\infty(s,z;\lambda)|} = \frac{\omega_+(z) + R(s,z)}{\sqrt{1 + 2\langle\omega_+(z), R(s,z)\rangle + |R(s,z)|^2}}. \tag{2.1.21}$$

We assume from now on that $R > 0$ is fixed and $|z| < R$. For $(\alpha,\beta) \in \mathbb{N} \times \mathbb{N}^{n-1}$ we deduce the existence of constants $c_{j;\alpha,\beta,R}$, $j = 1, 2, \dots$ appearing in the formulae below. By (1.2.16c) we have the initial estimates

$$|\partial_s^\alpha \partial_z^\beta R(s,z)| \le c_{1;\alpha,\beta,R}\, s^{-1-\alpha}, \qquad |z| \le R,\ s > T > 1, \tag{2.1.22}$$

where $T > 0$ is given in Proposition 1.2.10. Corollary A.2 then yields

$$|\partial_s^\alpha \partial_z^\beta (2\langle \omega_+(z), R(s,z)\rangle + |R(s,z)|^2)| \le c_{2;\alpha,\beta,R} s^{-1-\alpha}, \qquad |z| \le R,\ s > s_+ > 1. \tag{2.1.23}$$

We choose $S = S(R) > s_+$ so that

$$|(2\langle \omega_+(z), R(s,z)\rangle + |R(s,z)|^2)| \le \frac{1}{2}, \qquad |z| \le R,\ s > S(R), \tag{2.1.24}$$

and note that there exist constants $c_m > 0$ such that

$$\sup_{|x|\le 1/2} \partial_x^m (1+x)^{\frac{1}{2}} \le c_m. \tag{2.1.25}$$

Then from the chain rule (A.32) with (2.1.23), (2.1.24) and (2.1.25) we obtain for $|z| \le R$,

$$|\partial_s^\alpha \partial_z^\beta [(1 + 2\langle \omega_+(z), R(s,z)\rangle + |R(s,z)|^2)^{\pm\frac{1}{2}} - 1]| \le c_{3;\alpha,\beta,R} s^{-1-\alpha}, \quad s > S(\alpha,\beta,R) > s_+. \tag{2.1.26}$$

Setting

$$c_{4;\alpha,\beta,R} := S(\alpha,\beta,R)^{1+\alpha} \sup_{\substack{s_+<t<S(\alpha,\beta,R)\\ |z|<R}} |\partial_s^\alpha \partial_z^\beta [(1 + 2\langle \omega_+(z), R(t,z)\rangle + |R(t,z)|^2)^{\pm\frac{1}{2}} - 1]| + c_{3;\alpha,\beta,R},$$

we can improve (2.1.26) to

$$|\partial_s^\alpha \partial_z^\beta [(1 + 2\langle \omega_+(z), R(s,z)\rangle + |R(s,z)|^2)^{\pm\frac{1}{2}} - 1]| \le c_{4;\alpha,\beta,R} s^{-1-\alpha}, \qquad s > s_+. \tag{2.1.27}$$

The representation (2.1.21) yields

$$\begin{aligned}
&|\partial_s^\alpha \partial_z^\beta [\boldsymbol{\omega}(s,z) - \omega_+(z)]| \\
&\le |\partial_s^\alpha \partial_z^\beta [\omega_+(z)((1 + 2\langle \omega_+(s,z;\lambda), R(s,z)\rangle + |R(s,z)|^2)^{-\frac{1}{2}} - 1)]| + |\partial_s^\alpha \partial_z^\beta R(s,z)| \\
&\quad + |\partial_s^\alpha \partial_z^\beta [R(s,z)((1 + 2\langle \omega_+(s,z;\lambda), R(s,z)\rangle + |R(s,z)|^2)^{-\frac{1}{2}} - 1)]|
\end{aligned}$$

Again, Corollary A.2 with the estimates (2.1.22) and (2.1.27) immediately yields (2.1.16a).

We next consider the estimate (2.1.16b). Note that by (1.2.22)

$$|\boldsymbol{x}_\infty(s,z;\lambda)| = 2\sqrt{\lambda}s + 2\sqrt{\lambda}s\big((1 + 2\langle \omega_+(z), R(s,z)\rangle + |R(s,z)|^2)^{\frac{1}{2}} - 1\big),$$

so we have with the product rule and (2.1.27)

$$|\partial_s^\alpha \partial_z^\beta |\boldsymbol{x}_\infty(s,z;\lambda)|| \le c_{5;\alpha,\beta,R} s^{1-\alpha}, \qquad |z| \le R,\ s > s_+. \tag{2.1.28}$$

By (2.1.13),

$$\boldsymbol{L}(s,z) = |\boldsymbol{x}_\infty(s,z;\lambda)|\boldsymbol{\xi}_\infty(s,z;\lambda) - \langle \boldsymbol{\omega}(s,z), \boldsymbol{\xi}_\infty(s,z;\lambda)\rangle \boldsymbol{x}_\infty(s,z;\lambda) \tag{2.1.29}$$

A straightforward calculation using the representations (1.2.15) yields

$$\begin{aligned}
|\boldsymbol{x}_\infty(s,z;\lambda)|\boldsymbol{\xi}_\infty(s,z;\lambda) &= \sqrt{\lambda}\langle \omega_+(z), \omega_+(z) - \boldsymbol{\omega}_\infty(s,z;\lambda)\rangle |\boldsymbol{x}_\infty(s,z;\lambda)| \cdot \omega_+(z) \\
&\quad + \sqrt{\lambda}\langle \omega_+(z), \boldsymbol{x}_\infty(s,z;\lambda)\rangle \cdot \omega_+(z) + |\boldsymbol{x}_\infty(s,z;\lambda)| \cdot \partial_s \boldsymbol{g}_+(s,z;\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\langle \boldsymbol{\omega}(s,z), \boldsymbol{\xi}_\infty(s,z;\lambda)\rangle \boldsymbol{x}_\infty(s,z;\lambda) \\
&= \langle \boldsymbol{\omega}(s,z), \partial_s \boldsymbol{g}_+(s,z;\lambda)\rangle \boldsymbol{x}_\infty(s,z;\lambda) - \sqrt{\lambda}|\omega_+(z) - \boldsymbol{\omega}(s,z)|^2 \boldsymbol{x}_\infty(s,z;\lambda) \\
&\quad + \sqrt{\lambda}\langle \omega_+(z), \omega_+(z) - \boldsymbol{\omega}(s,z)\rangle \boldsymbol{x}_\infty(s,z;\lambda) + \sqrt{\lambda}\boldsymbol{x}_\infty(s,z;\lambda).
\end{aligned}$$

Then by (2.1.29),

$$\begin{aligned}
\boldsymbol{L}(s,z) &= L_+(z) - 2\sqrt{\lambda}\boldsymbol{g}_+(s,z;\lambda) + 2\sqrt{\lambda}\langle \omega_+(z), \boldsymbol{g}_+(s,z;\lambda)\rangle \cdot \omega_+(z) + \sqrt{\lambda}|\omega_+(z) - \boldsymbol{\omega}(s,z)|^2 \boldsymbol{x}_\infty(s,z;\lambda) \\
&\quad + \sqrt{\lambda}|\boldsymbol{x}_\infty(s,z;\lambda)|\langle \omega_+(z), \omega_+(z) - \boldsymbol{\omega}_\infty(s,z;\lambda)\rangle \cdot (\omega_+(z) - \boldsymbol{\omega}_\infty(s,z;\lambda)) \\
&\quad + |\boldsymbol{x}_\infty(s,z;\lambda)| \cdot \partial_s \boldsymbol{g}_+(s,z;\lambda) - \langle \boldsymbol{\omega}(s,z), \partial_s \boldsymbol{g}_+(s,z;\lambda)\rangle \boldsymbol{x}_\infty(s,z;\lambda)
\end{aligned}$$

where $L_+(z)$ is defined in (2.1.14). Using the estimates (1.2.16c), (2.1.16a), (2.1.28) in conjunction with Corollary A.2, the estimate (2.1.16b) follows immediately. □

Next we show that S_λ^+ is a proper injective immersion, since by Lemma B.3 it then follows that S_λ^+ is a smooth embedding, i.e., the image $\mathcal{L}_+$ is a smooth submanifold of T^*S^{n-1} and $S_\lambda^+\colon \mathbb{R}^{n-1} \to \mathcal{L}_+$ is a diffeomorphism.

2.1.9. Lemma *The map S_λ^+ is injective.*

Proof. Assume that there exist $y, z \in \mathbb{H}$ such that $S_\lambda^+(z) = S_\lambda^+(z)$, i.e., $\omega_+(z;\lambda) = \omega_+(y;\lambda)$ and $L_+(z) = \mathcal{L}_+(y)$. Then by (2.1.14),

$$r_+(z;\lambda) = r_+(y;\lambda) + c\cdot\omega_+(y;\lambda) = r_+(y;\lambda) + c\cdot\omega_+(z;\lambda) \tag{2.1.30}$$

for some $c \in \mathbb{R}$. By (1.2.7) and (2.1.30),

$$\begin{aligned} 0 &= \lim_{s\to+\infty} |\boldsymbol{x}_\infty(s,y;\lambda) - 2\sqrt{\lambda}\omega_+(z;\lambda)(s - c/(2\sqrt{\lambda})) - r_+(z;\lambda)| \\ &= \lim_{s\to+\infty} |\boldsymbol{x}_\infty(s + c/(2\sqrt{\lambda}),y;\lambda) - 2\sqrt{\lambda}\omega_+(z;\lambda)s - r_+(z;\lambda)| \end{aligned} \tag{2.1.31}$$

and

$$0 = \lim_{s\to+\infty} |\boldsymbol{\xi}_\infty(s + c/(2\sqrt{\lambda}),y;\lambda) - \sqrt{\lambda}\omega_+(z;\lambda)|. \tag{2.1.32}$$

But (2.1.31) and (2.1.32) together with the uniqueness statement in Definition 1.2.3 imply $\boldsymbol{\xi}_\infty(s + c/(2\sqrt{\lambda}),y;\lambda) = \boldsymbol{\xi}_\infty(s,z;\lambda)$, $\boldsymbol{x}_\infty(s + c/(2\sqrt{\lambda}),y;\lambda) = \boldsymbol{x}_\infty(s,z;\lambda)$ and by Proposition 1.2.5 we have $y = z$. □

2.1.10. Lemma *The map S_λ^+ is proper.*

Proof. We will show that the pre-image under S_λ^+ of compact sets is compact. We use the canonical coordinates (x,ξ) of $T^*\mathbb{R}^n$ for $\mathcal{L}_+ \subset T^*S^{n-1} \subset T^*\mathbb{R}^n$ and denote by $\widetilde{S}_\lambda^+$ the corresponding coordinate representation of S_λ^+, so that

$$\widetilde{S}_\lambda^+(z) = \begin{pmatrix} \omega_+(z) \\ L_+(z) \end{pmatrix}. \tag{2.1.33}$$

Then estimates (1.2.16b) and (2.1.18) imply the existence of a constant $C > 0$ such that

$$\left| \widetilde{S}_\lambda^+(z) - \begin{pmatrix} \omega_- \\ \sqrt{\lambda}\check{z} \end{pmatrix} \right| \le C\cdot\langle z\rangle^{1-\varrho} \qquad \text{for all } z \in \mathbb{H}. \tag{2.1.34}$$

It follows that if $R > 0$ is sufficiently large,

$$|\widetilde{S}_\lambda^+(z)| > \frac{1}{2}\sqrt{\lambda}|z| \qquad \text{for } z \in \mathbb{H} \text{ with } |z| > R. \tag{2.1.35}$$

Now let $K \subset \mathcal{L}_+$ be some compact set. Then (2.1.35), $(\widetilde{S}_\lambda^+)^{-1}(K)$ is bounded. Since S_λ^+ is continuous, $(\widetilde{S}_\lambda^+)^{-1}(K)$ is closed, hence compact. □

2.1.11. Lemma *The map S_λ^+ is an immersion.*

Proof. We differentiate (2.1.33) and see from the second assertion in (2.1.39) below that the rank of $D\widetilde{S}_\lambda^+|_z$, and therefore of $(S_\lambda^+)_*|_z$ is $n-1$ for all $z \in \mathbb{R}^{n-1}$. □

Up to the proof of Propostion 2.1.13 below, we have shown that S_λ^+ is an embedding and hence $\mathcal{L}_+$ is a submanifold of T^*S^{n-1}. We now verify that $\mathcal{L}_+$ is lagrangian.

2.1.12. Lemma *The manifold $\mathcal{L}_+$ is lagrangian.*

Proof. We need to show that the symplectic form $\sigma = dx \wedge d\xi$ vanishes on $\mathcal{L}_+$. Since $S_\lambda^+\colon \mathbb{R}^{n-1} \to \mathcal{L}_+$ is a diffeomorphism, it is sufficient to check that $(S_\lambda^+)^*\sigma|_{\mathcal{L}_+} = 0$. A straightforward calculation yields

$$\begin{aligned} (S_\lambda^+)^*\sigma|_{\mathcal{L}_+} &= \sum_{i,j=1}^{n-1} \left\langle \frac{\partial\omega_+(z)}{\partial z_i}, \frac{\partial L_+(z)}{\partial z_j} \right\rangle dz_i \wedge dz_j \\ &= \sum_{i,j=1}^{n-1} \left\langle \frac{\partial\omega_+(z)}{\partial z_i}, \frac{\partial r_+(z;\lambda)}{\partial z_j} \right\rangle dz_i \wedge dz_j \end{aligned} \tag{2.1.36}$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the euclidean scalar product in $\mathbb{R}^n$. We recall that the manifold Λ is lagrangian, cf. Theorem 1.3.3, hence $\sigma|_\Lambda = 0$. Now

$$T_{\iota(s,z)}\Lambda = \operatorname{span}\left\{\begin{pmatrix}\frac{\partial \boldsymbol{x}_\infty(z,s;\lambda)}{\partial z_1}\\ \frac{\partial \boldsymbol{\xi}_\infty(z,s;\lambda)}{\partial z_1}\end{pmatrix},\dots,\begin{pmatrix}\frac{\partial \boldsymbol{x}_\infty(z,s;\lambda)}{\partial z_{n-1}}\\ \frac{\partial \boldsymbol{\xi}_\infty(z,s;\lambda)}{\partial z_{n-1}}\end{pmatrix},\begin{pmatrix}\frac{\partial \boldsymbol{x}_\infty(z,s;\lambda)}{\partial s}\\ \frac{\partial \boldsymbol{\xi}_\infty(z,s;\lambda)}{\partial s}\end{pmatrix}\right\}, \tag{2.1.37}$$

and $\sigma|_\Lambda(v,u) = 0$ for $v,u \in T_p\Lambda$. A straightforward calculation using the representation (1.2.7) shows that for all $1 \le i,j \le n-1$

$$\begin{aligned} 0 &= \left\langle \frac{\partial \boldsymbol{x}_\infty(z,s;\lambda)}{\partial z_i}, \frac{\partial \boldsymbol{\xi}_\infty(z,s;\lambda)}{\partial z_j}\right\rangle - \left\langle \frac{\partial \boldsymbol{x}_\infty(z,s;\lambda)}{\partial z_j}, \frac{\partial \boldsymbol{\xi}_\infty(z,s;\lambda)}{\partial z_i}\right\rangle \\ &= 2\lambda\left\langle \frac{\partial \omega_+(z)}{\partial z_j}, \frac{\partial r_+(z;\lambda)}{\partial z_i}\right\rangle - 2\lambda\left\langle \frac{\partial \omega_+(z)}{\partial z_i}, \frac{\partial r_+(z;\lambda)}{\partial z_j}\right\rangle + R(s,z) \end{aligned} \tag{2.1.38}$$

where $R(s,z) \to 0$ uniformly in $z \in \mathbb{H}$ for $s \to \infty$ by (1.2.16c). Since the first terms in (2.1.38) are independent of s they must vanish seperately from the remainder $R(s,z)$, proving (2.1.36). □

This concludes the proof of Theorem 2.1.4 up to the proof of Proposition 2.1.13 below.

2.1.13. PROPOSITION *For any* $z \in \mathbb{R}^{n-1}$,

$$\operatorname{rank}\begin{pmatrix}\frac{\partial r_+(z;\lambda)}{\partial z}\\ \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix} = n-1, \qquad \operatorname{rank}\begin{pmatrix}\frac{\partial L_+(z)}{\partial z}\\ \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix} = n-1 \tag{2.1.39}$$

and

$$\operatorname{rank}\begin{pmatrix}\omega_+(z) & \frac{\partial r_+(z;\lambda)}{\partial z}\\ 0 & \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix} = n, \qquad \operatorname{rank}\begin{pmatrix}\omega_+(z) & \frac{\partial L_+(z)}{\partial z}\\ 0 & \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix} = n \tag{2.1.40}$$

The proof rests on the following technical lemma, which we prove subsequently.

2.1.14. LEMMA *For* $h \in \mathbb{R}^{n-1}$ *we define the directional derivative* $D_h := \sum h_i \frac{\partial}{\partial z_i}$. *Then*

$$|D_h\omega_+|_{z_0}| + |\pi^\perp_{\omega_+} D_h r_+|_{z_0}| > 0 \qquad \textit{for any } h, z_0 \in \mathbb{R}^{n-1}. \tag{2.1.41}$$

PROOF OF PROPOSITION 2.1.13. We consider the first assertion in (2.1.39). First note that (2.1.41) implies that

$$|D_h\omega_+| + |D_h r_+| > 0. \tag{2.1.42}$$

Since any linear combination of columns in the $(n-1)\times 2n$–matrix has the form

$$\sum_{i=1}^{n-1} h_i \frac{\partial}{\partial z_i}\begin{pmatrix} r_+\\ \omega_+\end{pmatrix} = \begin{pmatrix} D_h r_+\\ D_h\omega_+\end{pmatrix} \qquad \text{for some } h = (h_1,\dots,h_{n-1}), \tag{2.1.43}$$

it follows from (2.1.42) that its rank is $n-1$. We now claim that (2.1.41) implies

$$|D_h\omega_+|_{z_0}| + |D_h\pi^\perp_{\omega_+} r_+|_{z_0}| > 0 \qquad \text{for any } z_0 \in \mathbb{H}. \tag{2.1.44}$$

We argue by contradiction and assume that for some $h, z_0 \in \mathbb{R}^{n-1}$

$$D_h\omega_+|_{z_0} = 0 \qquad \text{and} \qquad D_h\pi^\perp_{\omega_+} r_+|_{z_0} = 0. \tag{2.1.45}$$

Now $D_h\omega_+|_{z_0} = 0$ implies

$$D_h\pi^\perp_{\omega_+} r_+|_{z_0} = D_h r_+|_{z_0} - \langle \omega_+(z_0;\lambda), D_h r_+|_{z_0}\rangle \omega_+(z_0;\lambda) = \pi^\perp_{\omega_+}\big(D_h r_+|_{z_0}\big). \tag{2.1.46}$$

But since the left–hand side vanishes by (2.1.45), this contradicts (2.1.41). By (2.1.14),

$$\operatorname{rank}\begin{pmatrix}\frac{\partial L_+(z)}{\partial z}\\ \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix} = \operatorname{rank}\begin{pmatrix}\frac{\partial \pi^\perp_{\omega(z)} r_+(z;\lambda)}{\partial z}\\ \frac{\partial \omega_+(z)}{\partial z}\end{pmatrix}. \tag{2.1.47}$$

Now with the same arguments as those following (2.1.42), the inequality (2.1.44) implies that the ranks in (2.1.47) are equal to $n-1$, so we have shown the second assertion in (2.1.39).

We now turn to the first assertion in (2.1.40). We need to verify that the left column of the matrix is independent of the right block column (which by (2.1.39) contains $n-1$ independent columns). In other word, we will show that there exists no $h = (h_1, \dots, h_{n-1}$ and no $z_0 \in \mathbb{H}$ such that

$$\begin{pmatrix} \omega_+ \\ 0 \end{pmatrix}\Big|_{z_0} = \sum_{i=1}^{n-1} h_i \frac{\partial}{\partial z_i}\begin{pmatrix} r_+ \\ \omega_+ \end{pmatrix}\Big|_{z_0} = \begin{pmatrix} D_h r_+ \\ D_h \omega_+ \end{pmatrix}\Big|_{z_0}. \tag{2.1.48}$$

Clearly, the existence of such an h and z_0 would contradict (2.1.41), so this is impossible. Furthermore, by (2.1.39),

$$\operatorname{rank}\begin{pmatrix} \omega_+(z) & \frac{\partial L_+(z)}{\partial z} \\ 0 & \frac{\partial \omega_+(z)}{\partial z} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \omega_+(z) & -\sqrt{\lambda}\frac{\partial \pi^\perp_{\omega(z)} r_+(z;\lambda)}{\partial z} \\ 0 & \frac{\partial \omega_+(z)}{\partial z} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \omega_+(z) & \frac{\partial \pi^\perp_{\omega(z)} r_+(z;\lambda)}{\partial z} \\ 0 & \frac{\partial \omega_+(z)}{\partial z} \end{pmatrix}, \tag{2.1.49}$$

where we have first divided the upper block row by $-\sqrt{\lambda}$, then multiplied the left column by the same factor. By (2.1.39) we need only show that the left column is independent of the other columns. Assume that for some z_0 the left column is a linear combination of the others, i.e., for some $h = (h_1, \dots, h_{n-1})$ we have

$$D_h \omega_+|_{z_0} = 0 \qquad \text{and} \qquad D_h \pi^\perp_{\omega_+} r_+|_{z_0} = \omega_+(z_0). \tag{2.1.50}$$

But by (2.1.46), $D_h\omega_+|_{z_0} = 0$ implies $D_h \pi^\perp_{\omega_+} r_+|_{z_0} \perp \omega_+(z_0)$, contradicting (2.1.50). It follows that the ranks in (2.1.49) are equal to n, completing the proof. □

Proof of Lemma 2.1.14. We will argue by contradiction and assume that for some $h, z_0 \in \mathbb{R}^{n-1}$ and some $c_r \in \mathbb{R}$ we have

$$D_h \omega_+|_{z_0} = 0 \qquad \text{and} \qquad D_h r_+|_{z_0} = c_r \cdot \omega_+(z_0;\lambda). \tag{2.1.51}$$

Note that by the chain rule, (1.2.15a) and (2.1.51),

$$\begin{aligned} D_h|_{z_0} \frac{\partial V(\boldsymbol{x}_\infty(\tau, z;\lambda))}{\partial x_j} &= \sum_{k=1}^n \frac{\partial^2 V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_k \partial x_j} D_h \boldsymbol{x}_k|_{z_0} \\ &= \sum_{k=1}^n \frac{\partial^2 V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_k \partial x_j}\big(c_r \omega_k(z_0;\lambda) + 2 D_h \boldsymbol{g}_k|_{z_0}\big) \end{aligned} \tag{2.1.52}$$

Using once more the representation (1.2.15a) and the chain rule, we can rewrite (2.1.52) as

$$\begin{aligned} D_h|_{z_0} \frac{\partial V(\boldsymbol{x}_\infty(\tau, z;\lambda))}{\partial x_j} &= \frac{1}{2\sqrt{\lambda}}\frac{\partial}{\partial \tau}\frac{\partial V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_j} \\ &\quad + 2\sum_{k=1}^n \frac{\partial^2 V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_k \partial x_j}\Big(D_h|_{z_0}\boldsymbol{g}_k - \frac{c_r}{2\sqrt{\lambda}}\frac{\partial \boldsymbol{g}_k(\tau; z_0,\lambda)}{\partial s}\Big). \end{aligned} \tag{2.1.53}$$

By (1.2.73),

$$D_h \boldsymbol{g}_j - \frac{c_r}{2\sqrt{\lambda}}\frac{\partial \boldsymbol{g}_j}{\partial s} = \int_s^\infty \int_t^\infty \Big(D_h - \frac{c_r}{2\sqrt{\lambda}}\frac{\partial}{\partial \tau}\Big)\frac{\partial V(\boldsymbol{x}_\infty(\tau; z,\lambda))}{\partial x_j}\, d\tau\, dt. \tag{2.1.54}$$

With (2.1.53), (2.1.54) becomes

$$\Big(D_h \boldsymbol{g}_j - \frac{c_r}{2\sqrt{\lambda}}\frac{\partial \boldsymbol{g}_j}{\partial s}\Big)\Big|_{z=z_0} = 2\int_s^\infty \int_t^\infty \sum_{k=1}^n \frac{\partial^2 V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_k \partial x_j}\Big(D_h|_{z_0}\boldsymbol{g}_k - \frac{c_r}{2\sqrt{\lambda}}\frac{\partial \boldsymbol{g}_k(\tau; z_0,\lambda)}{\partial s}\Big)\, d\tau\, dt. \tag{2.1.55}$$

From the Corollary 1.2.12, estimate (1.2.21), and Lemma B.2 it follows that for sufficiently large $T \gg 0$, the map

$$\mathcal{G}_{\lambda,z_0} \colon \mathcal{M}^+_{T,\mathbb{H}} \to \mathcal{M}^+_{T,\mathbb{H}}, \qquad (\mathcal{G}_{\lambda,z_0} f)_j(s) = 2\int_s^\infty \int_t^\infty \sum_{k=1}^n \frac{\partial^2 V(\boldsymbol{x}_\infty(\tau; z_0,\lambda))}{\partial x_k \partial x_j} f_k(\tau)\, d\tau\, dt. \tag{2.1.56}$$

where $\mathcal{M}^+_{T,\mathbb{H}}$ is defined in (1.2.79), is a strict contraction and has a unique fixed point. By (1.2.16c), $f(s) = D_h \boldsymbol{g}_j(s, z_0; \lambda) - \frac{c_r}{2\sqrt{\lambda}} \frac{\partial \boldsymbol{g}_j}{\partial s}(s, z_0; \lambda) \in \mathcal{M}^+_{T,\mathbb{H}}$ for sufficiently large $T > 0$. Clearly, $f = 0$ is a fixed point of $\mathcal{G}_{\lambda,z_0}$. Therefore, for sufficiently large $T > 0$,

$$D_h \boldsymbol{g}_j\big|_{z=z_0} = \frac{c_r}{2\sqrt{\lambda}} \frac{\partial \boldsymbol{g}_j}{\partial s}\bigg|_{z=z_0} \qquad \text{for } s > T. \tag{2.1.57}$$

Now (1.2.15), (2.1.51) and (2.1.57) give

$$\begin{aligned} D_h \boldsymbol{x}_\infty(s, z; \lambda)|_{z=z_0} &= \frac{c_r}{\sqrt{\lambda}} \frac{\partial}{\partial s} \boldsymbol{x}_\infty(s, z; \lambda)\bigg|_{z=z_0}, \\ D_h \boldsymbol{\xi}_\infty(s, z; \lambda)|_{z=z_0} &= \frac{c_r}{\sqrt{\lambda}} \frac{\partial}{\partial s} \boldsymbol{\xi}_\infty(s, z; \lambda)\bigg|_{z=z_0} \end{aligned} \tag{2.1.58}$$

for $s > T$. But the map ι, cf. (1.3.2), is an immersion, and (2.1.58) implies that

$$\begin{aligned} \iota_*|_{(s,z_0)} \left(D_h - \frac{c_r}{\sqrt{\lambda}} \frac{\partial}{\partial s} \right) &= \sum_{j=1}^n \big(D_h \boldsymbol{x}_j(s, z_0; \lambda) - \partial_s \boldsymbol{x}_j(s, z_0; \lambda) \big) \frac{\partial}{\partial x_j}\bigg|_{\iota(s,z_0)} \\ &\quad + \sum_{j=1}^n \big(D_h \boldsymbol{\xi}_j(s, z_0; \lambda) - \partial_s \boldsymbol{\xi}_j(s, z_0; \lambda) \big) \frac{\partial}{\partial \xi_j}\bigg|_{\iota(s,z_0)} \\ &= 0, \end{aligned} \tag{2.1.59}$$

leading to a contradiction. □

2.2. The Isozaki-Kitada phase functions

A crucial role in the further analysis is played by the Isozaki-Kitada phase functions, which we now introduce.

2.2.1. DEFINITION *For a given triplet (R, d, σ) with $R \gg 1$, $d > 1$, $-1 < \sigma < 1$, we introduce the notation* [2]

$$\Gamma_\pm(R, d, \sigma) := \big\{ (x, X^*(\xi)) \in T^*\mathbb{R}^n : |x| > R,\ d^{-1} < |\xi| < d,\ \langle \hat{x}, \hat{\xi} \rangle \gtrless \sigma \big\} \tag{2.2.1}$$

(where $\hat{x} = x/|x|$, $\hat{\xi} = \xi/|\xi|$) and for non–vanishing momentum

$$\Sigma_\pm(R, \sigma, \xi) := \big\{ x \in \mathbb{R}^n : |x| > R,\ \langle \widetilde{x}, \widetilde{\xi} \rangle \gtrless \sigma \big\}, \qquad \xi \neq 0 \textit{ fixed.} \tag{2.2.2}$$

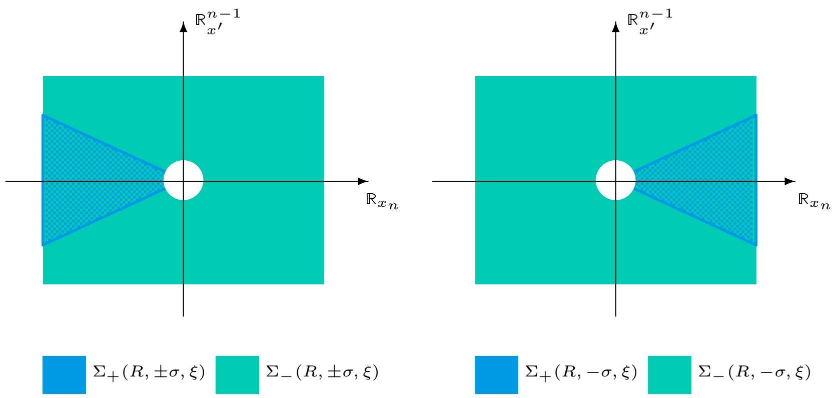


FIGURE 1. The sets $\Sigma_\pm(R, \pm\sigma, \xi) \subset \mathbb{R}^n_x$ for $\xi = (0, \ldots, 0, 1)$. Here σ is taken close to -1.

[2] Note that this definition differs from the convention used by Isozaki and Kitada [14], who set

$$\widetilde{\Gamma}_\pm(R, d, \sigma) := \{ (x, X^*(\xi)) \in T^*\mathbb{R}^n : |x| > R,\ d^{-1} < |\xi| < d,\ \langle \hat{x}, \hat{\xi} \rangle > \pm\sigma \}.$$

2.2.2. PROPOSITION [14, Proposition 2.4] *Let V obey the Potential Hypothesis and fix $d_0 > 1$ and $\sigma_0 \in (-1,1)$. Then there exist real functions $\varphi_\pm \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ (Isozaki-Kitada phase functions) with the following properties:*

i) *The functions $\varphi_\pm(x,\xi)$ solve the eikonal equation*

$$|\nabla_x \varphi_\pm(x,\xi)|^2 + V(x) = |\xi|^2 \tag{2.2.3}$$

in $\Gamma_\pm(R_0, d_0, \pm\sigma_0)$ for some $R_0 \gg 1$,

ii) *For any $L > 0$ and $\alpha, \beta \in \mathbb{N}^n$ there exist constants $C_{\alpha\beta L}$ such that for all $(x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$,*

$$|\partial_x^\alpha \partial_\xi^\beta (\varphi_\pm(x,\xi) - \langle x,\xi\rangle)| \le C_{\alpha\beta L} \langle x\rangle^{-|\alpha|} \langle \xi\rangle^{-L} \tag{2.2.4}$$

iii) *for all $(x,\xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$,*

$$\left| \frac{\partial^2}{\partial x_j \partial \xi_k} (\varphi_\pm(x,\xi) - \langle x,\xi\rangle) \right| \le \varepsilon(R_0) < \frac{1}{2},$$

where $\varepsilon(R_0)$ can be made arbitrarily small by taking R_0 large enough.

2.2.3. LEMMA *Let $\Lambda_s \subset T^*\mathbb{R}^n$, $s \in \mathbb{R}$, be the manifold defined in Lemma 1.3.6 and $\boldsymbol{x}_\infty$ the configuration space trajectory of Definition 1.2.3. For any $R > 0$, $\sigma \in (-1,0)$ and $d > 1$ with $d^{-1} < \sqrt{\lambda} < d$ there exist times $s_\pm(R,d,\sigma)$ such that*

i) *$\Lambda_s \subset \Gamma_-(R,d,-\sigma)$ for all $s < s_-(R,d,\sigma)$,*
ii) *$\boldsymbol{x}_\infty(s,z;\lambda) \in \Sigma_-(R,-\sigma,\sqrt{\lambda}\omega_-)$ for all $z \in \mathbb{H}$ and $s < s_-(R,d,\sigma)$,*
iii) *$\Lambda_s \subset \Gamma_+(R,d,\sigma)$ for all $s > s_+(R,d,\sigma)$,*
iv) *$\boldsymbol{x}_\infty(s,z;\lambda) \in \Sigma_+(R,\sigma,\sqrt{\lambda}\omega_+(z;\lambda))$ for all $z \in \mathbb{H}$ and $s > s_+(R,d,\sigma)$.*

PROOF. We fix some (R,d,σ) as in the hypothesis and start with assertion i). By the triangle inequality and (1.2.11a), (1.2.13) we have a constants $c > 0$ and $S < 0$ such that

$$\sup_{z\in\mathbb{H}} \big| |\boldsymbol{\xi}_\infty(s,z;\lambda)| - \sqrt{\lambda} \big| \le \sup_{z\in\mathbb{H}} \big| \boldsymbol{\xi}_\infty(s,z;\lambda) - \sqrt{\lambda} \big| < c|s|^{-\varrho} \qquad \text{for } s < S. \tag{2.2.5}$$

It follows that

$$d^{-1} < \sup_{z\in\mathbb{H}} |\boldsymbol{\xi}_\infty(s,z;\lambda)| < d \qquad \text{for } s < 0 \text{ sufficiently small.} \tag{2.2.6}$$

Again we can use the triangle inequality and (1.2.11a) to see that

$$\begin{aligned} |\boldsymbol{x}_\infty(s,z;\lambda)| &= |2\sqrt{\lambda}\omega_- s + \check{z} + 2\boldsymbol{g}_-(s,z;\lambda)| \\ &\ge |2\sqrt{\lambda}\omega_- s + \check{z}| - 2|\boldsymbol{g}_-(s,z;\lambda)| \\ &= \sqrt{\lambda}|s| + (\sqrt{\lambda}|s| - 2|\boldsymbol{g}_-(s,z;\lambda|), \end{aligned} \tag{2.2.7}$$

so with (1.2.13),

$$\sup_{z\in\mathbb{H}} |\boldsymbol{x}_\infty(s,z;\lambda)| > \sqrt{\lambda}|s| > R \qquad \text{for } s < 0 \text{ sufficiently small.} \tag{2.2.8}$$

It follows from (1.2.11) with (1.2.13) that for some $c' > 0$,

$$|\langle \boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda)\rangle - 2\lambda s| \le c' \cdot |s|^{1-\varrho} \qquad \text{for all } z \in \mathbb{R}^{n-1}. \tag{2.2.9}$$

Using (2.2.5), (2.2.8), we have

$$|\boldsymbol{x}_\infty(s,z;\lambda)||\boldsymbol{\xi}_\infty(s,z;\lambda)| > \sqrt{\lambda}|s|(\sqrt{\lambda} - c|s|^{-\varrho})) > \lambda|s| - c\sqrt{\lambda}|s|^{1-\varrho} \tag{2.2.10}$$

for some $c > 0$ if $s \ll 0$ is sufficiently small. It follows from (2.2.9) and (2.2.10) that

$$\frac{\langle \boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda)\rangle}{|\boldsymbol{x}_\infty(s,z;\lambda)| \cdot |\boldsymbol{\xi}_\infty(s,z;\lambda)|} < \frac{2\lambda s - c' \cdot |s|^{1-\varrho}}{2(\lambda|s| - c''\sqrt{\lambda}|s|^{1-\varrho})} = \frac{-1 - c'\lambda^{-1} \cdot |s|^{-\varrho}}{1 - c''\lambda^{-\frac{1}{2}}|s|^{-\varrho}}, \tag{2.2.11}$$

which implies

$$\frac{\langle \boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda)\rangle}{|\boldsymbol{x}_\infty(s,z;\lambda)| \cdot |\boldsymbol{\xi}_\infty(s,z;\lambda)|} < 0 < -\sigma \qquad \text{for } s < 0 \text{ sufficiently small.} \tag{2.2.12}$$

Now by (2.2.1), the assertion i) follows from (2.2.6), (2.2.8) and (2.2.12). By calculations analogous to (2.2.10) and (2.2.11) we easily see that

$$\frac{\langle \boldsymbol{x}_\infty(s,z;\lambda),\omega_-\rangle}{|\boldsymbol{x}_\infty(s,z;\lambda)|} < 0 < -\sigma \qquad \text{for } s<0 \text{ sufficiently small.} \tag{2.2.13}$$

Then by (2.2.2), assertion ii) follows from (2.2.8) and (2.2.13).

Similarly to (2.2.6) above, we obtain from (1.2.15a) and (1.2.16c) that

$$d^{-1} < \sup_{z\in\mathbb{H}}|\boldsymbol{\xi}_\infty(s,z;\lambda)| < d \qquad \text{for } s>0 \text{ sufficiently large.} \tag{2.2.14}$$

Furthermore, using the representation (1.2.15) with (1.2.16c) and noting $\langle \check{z},\omega_-\rangle = 0$, we have

$$\begin{aligned} |\langle \boldsymbol{x}_\infty(s,z;\lambda),\boldsymbol{\xi}_\infty(s,z;\lambda)\rangle - 2\lambda s| &\le c\cdot\langle z\rangle^{1-\varrho}s^{1-\varrho} + \langle\omega_+(z;\lambda),r_+(z;\lambda)\rangle \\ &= c\cdot\langle z\rangle^{1-\varrho}s^{1-\varrho} + \langle\omega_+(z;\lambda),r_+(z;\lambda)-\check{z}\rangle + \langle\omega_+(z;\lambda)-\omega_-,\check{z}\rangle. \end{aligned} \tag{2.2.15}$$

for some $c>0$ and $s>0$ sufficiently large. Now applying (1.2.16a) and (1.2.16b), we see that

$$|\langle \boldsymbol{x}_\infty(s,z;\lambda),\boldsymbol{\xi}_\infty(s,z;\lambda)\rangle - 2\lambda s| \le C \qquad \text{for some } C>0 \text{ and } s>0 \text{ sufficiently large.} \tag{2.2.16}$$

it follows that

$$\langle \boldsymbol{x}_\infty(s,z;\lambda),\boldsymbol{\xi}_\infty(s,z;\lambda)\rangle > 0 \qquad \text{for all } z\in\mathbb{H} \text{ nd } s>0 \text{ sufficiently large.} \tag{2.2.17}$$

Checking (2.2.1), we see that the assertion iii) follows from (1.2.19), (2.2.14) and (2.2.17). We omit the proof of iv), which is verified in the same way. ☐

The next lemma is just a result of Robert and Tamura [24, (4.2)-(4.5), Lemma 4.1], which we have reformulated in terms of the notation of Lemma 2.2.3.

2.2.4. LEMMA *Let (R,d,σ) be some triple with $\sigma\in(-1,0)$, $d>1$ with $d^{-1}<\sqrt{\lambda}<d$ and $R>0$ sufficiently large such that we can define Isozaki-Kitada functions $\varphi_\pm$ as in Proposition 2.2.2. Set $s_\pm = s_\pm(R,d,\sigma)\in\mathbb{R}$ as in Lemma 2.2.3. Then, using the notation of Definition 1.2.3,*

$$\boldsymbol{\xi}_\infty(s,z;\lambda) = (\nabla_x\varphi_-)(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-) \qquad \textit{for all } z\in\mathbb{H} \textit{ and } s<s_-. \tag{2.2.18}$$

Moreover,

$$(\nabla_x\varphi_+)(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z;\lambda)) = \boldsymbol{\xi}_\infty(s,z;\lambda), \tag{2.2.19}$$

$$(\nabla_\xi\varphi_+)(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z;\lambda)) = 2\sqrt{\lambda}\omega_+(z;\lambda)s + r_+(z;\lambda), \tag{2.2.20}$$

for all $z\in\mathbb{R}^{n-1}$ and $s>s_+$. Additionally, we have the representation

$$\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z;\lambda)) \;=\; 2s\lambda - \langle r_+(z;\lambda),\sqrt{\lambda}\omega_+(z,\lambda)\rangle - 2\int_s^\infty (|\boldsymbol{\xi}_\infty(\tau,z;\lambda)|^2-\lambda)\,d\tau. \tag{2.2.21}$$

In fact, we can sharpen (2.2.19) to a uniqueness result:

2.2.5. LEMMA *Let (R,d,σ) be some triple with $\sigma\in(-1,0)$, $d>1$ with $d^{-1}<\sqrt{\lambda}<d$ and $R>0$ sufficiently large such that we can define an Isozaki-Kitada function φ_+ as in Proposition 2.2.2. Choose $s_+ = s_+(R,d,\sigma)\in\mathbb{R}$ as in Lemma 2.2.3. Then there exists a time $S>s_+(R,d,\sigma)$ such that for any $z\in\mathbb{H}$ and any $s>S$*

$$\boldsymbol{\xi}_\infty(s,z;\lambda) = \nabla\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega) \qquad \textit{implies} \qquad \omega = \omega_+(z;\lambda). \tag{2.2.22}$$

PROOF. By (2.2.4) we have

$$|\nabla_x\varphi_+(x,\sqrt{\lambda}\omega) - \sqrt{\lambda}\omega| \le C_{100}\langle x\rangle^{-1}. \tag{2.2.23}$$

Inserting the hypothesis of (2.2.22) together with (1.2.19) into (2.2.23) we obtain

$$|\boldsymbol{\xi}_\infty(s,z;\lambda) - \sqrt{\lambda}\omega| \le c_1\cdot|s|^{-1}, \tag{2.2.24}$$

for all $z \in \mathbb{H}$, $s > s_+$ sufficiently large and some constant $c_1 > 0$. By Lemma 2.2.3 iii), we can assume that $(\boldsymbol{x}_\infty(s,z;\lambda), \boldsymbol{\xi}_\infty(s,z;\lambda)) \in \Gamma_+(R,d,\sigma)$ for all $z \in \mathbb{H}$ and $s > s_+$. Then from the definition (2.2.1) of $\Gamma_+(R,d,\sigma)$ and (2.2.24) it follows that

$$\frac{1}{|\boldsymbol{x}_\infty(s,z;\lambda)|} \langle \boldsymbol{x}_\infty(s,z;\lambda), \omega\rangle > \sigma. \tag{2.2.25}$$

for all $z \in \mathbb{H}$ and $s > s_+$ sufficiently large. Hence $(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega) \in \Gamma_+(R,d,\sigma)$. Under this condition it was shown by Robert and Tamura [24, Beginning of Section 4.1] that (using the notation of Definition 1.1.2)

$$\lim_{t\to+\infty} \frac{\widetilde{\boldsymbol{x}}(t,s,z;\omega)}{|\widetilde{\boldsymbol{x}}(t,s,z;\omega)|} = \omega, \tag{2.2.26}$$

where $\widetilde{\boldsymbol{x}}(t,s,z;\omega) := \boldsymbol{x}(t;\boldsymbol{x}_\infty(s,z;\lambda), \nabla_x\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega))$. However, by the hypothesis of (2.2.22), $\widetilde{\boldsymbol{x}}(t,s,z;\omega) = \boldsymbol{x}_\infty(s+t,z;\lambda)$ and by (2.1.13), (2.1.16a) the limit in (2.2.26) is $\omega_+(z;\lambda)$. □

2.2.6. Definition *We fix a triple (R_0,d_0,σ_0) by first choosing $\sigma_0 \in (-1,0)$ and $d_0 > 1$ so that $d_0^{-1} < \sqrt{\lambda} < d_0$. We then choose $R_0 > 0$ sufficiently large to ensure that Proposition 2.2.2 i) holds and fix a choice of corresponding Isozaki-Kitada functions φ_- and φ_+. In the notation of Lemmas 2.2.3 and 2.2.5, we define*

$$s_- := s_-(R_0,d_0,\sigma_0) \qquad \textit{and} \qquad s_+ := \max(s_+(R_0,d_0,\sigma_0), S) \tag{2.2.27}$$

and set

$$\Lambda_- := \bigcup_{s<s_-} \Lambda_s \qquad \textit{and} \qquad \Lambda_+ := \bigcup_{s>s_+} \Lambda_s. \tag{2.2.28}$$

2.3. Generating functions

This section discusses the concepts of lagrangian coordinates and local generating functions, which are both central to the construction of the canonical Maslov operator, the classification of caustics and the asymptotic formula for the scattering amplitude. We first need to introduce some notation

2.3.1. Convention *Let $\mathcal{N}$ denote the index set $\{1,\dots,n\}$. For $I \subset \mathcal{N}$ we define $|I| := \#I$ (the number of indices in I) and $\overline{I} := \mathcal{N}\setminus I$. Further, we set $\mathcal{N}_i := \mathcal{N}\setminus\{i\}$ for short. We use the index "I" to denote an ordered $|I|$–tuple, e.g., with $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ we write $x_I = (x_{i_1},\dots,x_{i_{|I|}})$ with $i_m < i_{m+1}$.*

2.3.2. Definition *Let M be a n–dimensional manifold, $U \subset T^*M$ an open set and $(x,\xi)\colon U \to \mathbb{R}^n\times\mathbb{R}^n$ a local coordinate chart, canonically induced by a chart $x\colon \pi(U)\to\mathbb{R}^n$, where $\pi\colon T^*M \to M$ denotes the canonical projection. Let $\Lambda \subset T^*M$ be a lagrangian manifold and $\Omega \subset U$ an open set in Λ.*

i) *For any $I \subset \mathcal{N}$ we define*

$$\pi_{\Omega,I}\colon \Omega \to \mathbb{R}^{|I|}_{x_I}\times\mathbb{R}^{|\overline{I}|}_{\xi_{\overline{I}}}, \qquad p \mapsto (x_I(p), \xi_{\overline{I}}(p)) \tag{2.3.1}$$

If (2.3.1) is a diffeomorphism on its image, we say that $(\Omega,\pi_{\Omega,I})$ is a lagrangian chart of order $|I|$ and $(x_I,\xi_{\overline{I}})$ are lagrangian coordinates. If $|I| = \min_{p\in\Omega}\operatorname{rank} d\pi_{\Lambda,\mathcal{N}}|_p$, we call the lagrangian chart canonical. A (canonical) lagrangian atlas on Λ is a locally finite open covering of Λ by (canonical) lagrangian charts. We say that Ω is well-projected if x can be chosen as lagrangian coordinates on Ω.

ii) *a function $S \in C^\infty(\Lambda)$ is called a global generating function for Λ if*

$$dS = \sum \xi_j\, dx_j|_\Lambda. \tag{2.3.2}$$

If such a function exists and $(\Omega,\pi_{\Omega,I})$ is some lagrangian chart on Λ, we define a local generating function $S_{\Omega,I} \in C^\infty(\Omega)$ via

$$S_{\Omega,I} := S - \langle x_{\overline{I}}, \xi_{\overline{I}}\rangle \qquad \textit{on } \Omega. \tag{2.3.3}$$

It then follows that on Ω

$$\xi_I = \frac{\partial S_{\Omega,I}}{\partial x_I} \qquad \textit{and} \qquad x_{\overline{I}} = -\frac{\partial S_{\Omega,I}}{\partial \xi_{\overline{I}}}. \tag{2.3.4}$$

The concept of lagrangian coordinates and their role in the construction of the canonical Maslov operator on lagrangian manifolds in $T^*\mathbb{R}^n$ has been extensively explored in [18], [27] and elsewhere. For completeness we repeat the proof of the following basic result.

2.3.3. LEMMA ([18]) *Any lagrangian manifold admits a canonical lagrangian atlas.*

PROOF. We keep the notation of Definition 2.3.2. We can without loss consider the coordinized manifold $\widetilde{\Lambda} := (x,\xi)\Lambda \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$; indeed, the canonical symplectic form is given by

$$\sigma = \sum_{j=1}^{n} dx_j \wedge d\xi_j$$

and $\operatorname{rank} d\pi|_\Lambda = \operatorname{rank} d\pi_x|_{\widetilde{\Lambda}}$ in any set of induced coordinates. We will henceforth drop the tilde and consider the case of $\Lambda \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$.

Let $p \in \Lambda$. By the implicit function theorem, a neighbourhood of p can be diffeomorphically mapped into the tangent plane of Λ at p, i.e., we obtain a coordinate chart in a neighbourhood Ω of p through the coordinates of the tangent plane at p, which we denote by L. By definition, the tangent plane is lagrangian (i.e., the symplectic form σ vanishes on L.) In order to prove that there exists an index set $I \subset \mathcal{N}$ such that $\pi_{\Omega,I}$ is a diffeomorphism on its image it is then sufficient to show that the map $\pi_{L,I}$ is a diffeomorphism. We also claim that if the neighbourhood Ω is sufficiently small, then $\min_\Omega \operatorname{rank} d\pi_{\Omega,\mathcal{N}} = \operatorname{rank} d\pi_{\Omega,\mathcal{N}}|_p$. This can be seen from the fact that the rank of a matrix does not change under small perturbations.

In conclusion, it suffices to show that for any lagrangian plane there exists a set of canonical coordinates. A covering of Λ with appropriate neighbourhoods mapped onto tangent planes then yields a canonical lagrangian atlas for Λ.

Let $L \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ be a lagrangian plane. We will prove that for some $I \subset \mathcal{N}$ with $|I| = \operatorname{rank} d\pi_x|_L$ the canonical projection

$$\pi_I \colon \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{R}^I_{x_I} \times \mathbb{R}^{\overline{I}}_{\xi_{\overline{I}}} \tag{2.3.5}$$

is a diffeomorphism when restricted to L. We need only show that $\ker \pi_I = \{0\}$. Choose $I \subset \mathcal{N}$ with $|I| = \operatorname{rank} d\pi_x|_L$ such that $\{x_I\}$ is a basis of $\pi_x L$. Then $\pi_I u = 0$ implies $\pi_x u = 0$ for $u \in L$. Since $\pi_{\xi_{\overline{I}}} u = 0$ by assumption, it remains to show that $\pi_{\xi_I} u = 0$. However, the symplectic form has the form

$$\sigma(u,v) = \langle \pi_\xi v, \pi_x u\rangle - \langle \pi_\xi u, \pi_x v\rangle \qquad \text{for } u,v \in L$$

so $\pi_I u = 0$ implies

$$\sigma(u,v) = -\sum_{i\in I} \pi_{\xi_i} u \cdot \pi_{x_i} v = 0 \qquad \text{for all } v \in L.$$

This completes the proof. ∎

The proof of Lemma 2.3.3 is actually much more insightful than the statement of the result, In fact, the proof yields an important corollary.

2.3.4. COROLLARY *Let Λ be an n–dimensional lagrangian manifold in the notation of Definition 2.3.2 and let $p \in \Lambda \cap U$ for some chart domain U.*

i) *We can find an index set $I \subset \mathcal{N}$ with $\operatorname{rank} d\pi_{\Lambda\cap U,\mathcal{N}}|_p = |I|$ such that $\operatorname{rank} d\pi_{\Lambda\cap U,I}|_p = n$.*
ii) *For any $I \subset \mathcal{N}$, $\operatorname{rank} d\pi_{\Lambda,I}|_p = n$ if and only if $(x_I, \xi_{\overline{I}})$ are lagrangian coordinates in some neighbourhood Ω of p. They are canonical lagrangian coordinates if and only if $\min_{q\in\Omega} \operatorname{rank} d(\pi|_\Lambda)|_q = \operatorname{rank} d(\pi|_\Lambda)|_p$.*

2.3.5. REMARK For an arbitrary cotangent bundle we use the notation $\pi \colon T^*M \to M$ for the canonical projection onto the base, employing π_x for the special case $M = \mathbb{R}^n_x$ and also denoting the corresponding map $\mathbb{R}^n_\xi \times \mathbb{R}^n_x \to \mathbb{R}^n_x$ by π_x. Note that $\pi_{\Lambda\cap U,\mathcal{N}} = x \circ \pi|_\Lambda$, where $x\colon \pi(U) \to \mathbb{R}^n$ is a chart on the base manifold. Clearly the ranks of the differentials of $\pi_{\Lambda\cap U,\mathcal{N}}$ and $\pi|_\Lambda$ are equal, and we will make use of both maps. In $T^*\mathbb{R}^n$, the mapa are identical.

We will first use the Isozaki-Kitada phase functions to construct local generating functions on the scattering manifold $\Lambda \subset T^*\mathbb{R}^n$.

2.3.6. LEMMA *Let $\varphi_-(\,\cdot\,,\sqrt{\lambda}\omega_-) \in C^\infty(\mathbb{R}^n_x)$ be the Isozaki-Kitada phase function of Lemma 2.2.4. Then*

$$\varphi_-(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega^-) = \langle \boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-\rangle + \int_{\mathcal{T}^-_{z,s}} \langle \xi - \sqrt{\lambda}\omega_-, dx\rangle \tag{2.3.6}$$

$$= 2s\lambda + 2\int_{-\infty}^{s} (|\boldsymbol{\xi}_\infty(\tau,z;\lambda)|^2 - \lambda)\,d\tau. \tag{2.3.7}$$

PROOF. We now show (2.3.6). Let $s' < s < s_-$. Then

$$\begin{aligned}
&\varphi_-(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-)\\
&= \varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) + \int_{s'}^{s} \frac{\partial}{\partial\tau}\varphi_-(\boldsymbol{x}_\infty(\tau,z;\lambda),\sqrt{\lambda}\omega_-)\,d\tau\\
&= \varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) + \int_{s'}^{s} \langle \nabla_x\varphi_-(\boldsymbol{x}_\infty(\tau,z;\lambda),\sqrt{\lambda}\omega_-), \partial_\tau\boldsymbol{x}_\infty(\tau,z;\lambda)\rangle\,d\tau
\end{aligned} \tag{2.3.8}$$

Inserting (2.2.18) into (2.3.8) yields

$$\begin{aligned}
\varphi_-(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-) &= \varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) - \langle\sqrt{\lambda}\omega_-,\boldsymbol{x}_\infty(s',z;\lambda)\rangle + \langle\sqrt{\lambda}\omega_-,\boldsymbol{x}_\infty(s,z;\lambda)\rangle\\
&\quad + \int_{s'}^{s} \langle \boldsymbol{\xi}_\infty(\tau,z;\lambda) - \sqrt{\lambda}\omega_-, \partial_\tau\boldsymbol{x}_\infty(\tau,z;\lambda)\rangle\,d\tau.
\end{aligned} \tag{2.3.9}$$

By (2.2.4),

$$\big|\varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) - \langle \boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-\rangle\big| \to 0 \qquad \text{as } s' \to -\infty. \tag{2.3.10}$$

so letting $s' \to -\infty$, we obtain

$$\varphi_-(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-) = \langle\sqrt{\lambda}\omega_-,\boldsymbol{x}_\infty(s,z;\lambda)\rangle + \int_{-\infty}^{s} \langle \boldsymbol{\xi}_\infty(\tau,z;\lambda) - \sqrt{\lambda}\omega_-, \partial_\tau\boldsymbol{x}_\infty(\tau,z;\lambda)\rangle\,d\tau. \tag{2.3.11}$$

But the integral in (2.3.11) is by definition just the line integral of $\xi - \sqrt{\lambda}\omega_-$ along $\mathcal{T}^-_{z,s}$.

In order to show (2.3.7), we start from (2.3.8). Using (1.1.6) and (2.2.18),

$$\begin{aligned}
\varphi_-(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-) &= \varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) - 2\lambda s' + 2\lambda s\\
&\quad + 2\int_{s'}^{s} (|\nabla_x\varphi_-(\boldsymbol{x}_\infty(\tau,z;\lambda),\sqrt{\lambda}\omega_-)|^2 - \lambda)\,d\tau.
\end{aligned} \tag{2.3.12}$$

By (1.2.6) and (1.2.13) there exists a constant $c > 0$ such that

$$\big|\langle \boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-\rangle - 2\lambda s'\big| \le c|s'|^{1-\varrho}. \tag{2.3.13}$$

Now (2.3.13) and (2.3.10) imply that $|\varphi_-(\boldsymbol{x}_\infty(s',z;\lambda),\sqrt{\lambda}\omega_-) - 2\lambda s'| \to 0$ as $s' \to -\infty$ and therefore (2.3.7) follows from (2.3.12) by letting $s' \to -\infty$. □

2.3.7. LEMMA *The function $S\colon \Lambda \to \mathbb{R}$ given by*

$$S \circ \iota(s,z;\lambda) = \langle \boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_-\rangle + \int_{\mathcal{T}^-_{z,s}} \langle \xi - \sqrt{\lambda}\omega_-, dx\rangle \tag{2.3.14}$$

$$= 2s\lambda + 2\int_{-\infty}^{s} (|\boldsymbol{\xi}_\infty(\tau,z;\lambda)|^2 - \lambda)\,d\tau. \tag{2.3.15}$$

is a global generating function for Λ.

PROOF. By (2.3.14),

$$S(x,\xi) = \langle x,\sqrt{\lambda}\omega_-\rangle + \int_{\mathcal{T}^-_{z,s}} \langle \xi - \sqrt{\lambda}\omega_-, dx\rangle \qquad \text{on } \Lambda,\ (s,z) = \iota^{-1}(x,\xi). \tag{2.3.16}$$

It follows from (2.3.16) that $dS = \sum \xi_j\,dx_j$ on Λ, which proves the lemma. □

Note that on Λ_-, the functions S and $\varphi_-(\,\cdot\,,\sqrt{\lambda}\omega_-)\circ\pi_x$ coincide.

The construction of local generating functions for $\mathcal{L}_+ \subset T^*S^{n-1}$ is slightly more complicated, involving the “second” Isozaki-Kitada phase function $\varphi_+(x,\xi)$.

2.3.8. LEMMA *The function*

$$F\colon \Lambda \to \mathbb{R}, \qquad (F\circ\iota)(s,z;\lambda) = S\circ\iota(s,z;\lambda) - \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z,\lambda)) \tag{2.3.17}$$

does not depend on s; explicitly,

$$(F\circ\iota)(s,z;\lambda) = 2\int_{-\infty}^{\infty}(|\boldsymbol{\xi}_\infty(\tau,z;\lambda)|^2-\lambda)\,d\tau - \langle r_+(z;\lambda), \sqrt{\lambda}\omega_+(z,\lambda)\rangle. \tag{2.3.18}$$

The function $F_+ \in C^\infty(\mathcal{L}_+)$ *defined through*

$$F_+\circ S^+_\lambda(z;\lambda) := F\circ\iota(s,z;\lambda) \tag{2.3.19}$$

is a generating function for $\mathcal{L}_+$*, i.e.,*

$$dF_+ = \langle \xi, dx\rangle|_{\mathcal{L}_+}. \tag{2.3.20}$$

PROOF. The first part of the Lemma, equation (2.3.18), follows immediately from (2.3.7) and (2.2.21). We will show that

$$(S^+_\lambda)^* dF_+|_z = (S^+_\lambda)^*\langle\xi,dx\rangle|_{\mathcal{L}_+} = -\sqrt{\lambda}\langle \pi^\perp_{\omega_+(\,\cdot\,;\lambda)} r_+(\,\cdot\,;\lambda), (S^+_\lambda)^* dx\rangle|_z. \tag{2.3.21}$$

The first equality is equivalent to (2.3.20), while the second follows by combining (2.1.17), (2.1.15) and (2.1.14). By (2.3.17) and (2.3.19), for any $s > s_+$,

$$\begin{aligned}(S^+_\lambda)^* dF_+|_z = d(S\circ\iota)(s,\,\cdot\,;\lambda)|_z &- \sum_{i=1}^{n-1}\langle \nabla_x\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z,\lambda)), \partial_{z_i}\boldsymbol{x}_\infty(s,z;\lambda)\rangle\,dz_i|_z\\ &- \sqrt{\lambda}\sum_{i=1}^{n-1}\langle\nabla_\xi\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z,\lambda)), \partial_{z_i}\omega_+(z,\lambda)\rangle\,dz_i|_z\end{aligned} \tag{2.3.22}$$

By (2.2.19),

$$\begin{aligned}&\sum_{i=1}^{n-1}\langle\nabla_x\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda),\sqrt{\lambda}\omega_+(z,\lambda)), \partial_{z_i}\boldsymbol{x}_\infty(s,z;\lambda)\rangle\,dz_i|_z\\ &= \sum_{i=1}^{n-1}\langle\boldsymbol{\xi}_\infty(s,z;\lambda), \partial_{z_i}\boldsymbol{x}_\infty(s,z;\lambda)\rangle\,dz_i|_z\\ &= \sum_{j=1}^{n}\boldsymbol{\xi}_j(s,z;\lambda)\Big(\sum_{i=1}^{n-1}\frac{\partial\boldsymbol{x}_j(s,z;\lambda)}{\partial z_i}\,dz_i|_z\Big)\\ &= \sum_{j=1}^{n}(\xi_j\circ\iota)\,\iota_s^* dx_j|_z = \iota_s^*(\langle\xi,dx\rangle)|_z = d(S\circ\iota)(s,\,\cdot\,;\lambda)|_z\end{aligned} \tag{2.3.23}$$

Inserting (2.3.23) and (2.2.20) into (2.3.22) and noting that $\langle\omega_+(z;\lambda),\partial_{z_i}\omega_+(z;\lambda)\rangle = 0$, we obtain

$$\begin{aligned}(S^+_\lambda)^* dF_+|_z &= -\sqrt{\lambda}\sum_{i=1}^{n-1}\langle r_+(z;\lambda), \partial_{z_i}\omega_+(z,\lambda)\rangle\,dz_i|_z\\ &= -\sqrt{\lambda}\sum_{i=1}^{n-1}\langle\pi^\perp_{\omega_+(z;\lambda)} r_+(z;\lambda), \partial_{z_i}\omega_+(z;\lambda)\rangle\,dz_i|_z\\ &- \sqrt{\lambda}\langle\pi^\perp_{\omega_+(\,\cdot\,;\lambda)} r_+(\,\cdot\,;\lambda), (S^+_\lambda)^* dx\rangle|_z\end{aligned}$$

completing the proof. □

2.4. Lagrangian coordinates

We will now further explore the relationship between lagrangian coordinates on $\mathcal{L}_+$ and on Λ. In order to formulate the result as explicitly as possible, we will need to introduce coordinates on $\mathcal{L}_+$.

2.4.1. Remark Let (Σ, χ) be a chart on the unit sphere which we write as

$$\chi\colon \Sigma \to \mathbb{R}^{n-1}, \qquad \omega \mapsto \theta = (\theta_k)_{k\in\mathcal{N}_i} \tag{2.4.1}$$

for some $i \in \mathcal{N} := \{1, \ldots, n\}$ and $\mathcal{N}_i := \mathcal{N} \setminus \{i\}$. We obtain an induced chart

$$\widetilde{\chi}\colon T^*\Sigma \to \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \qquad (\omega, X^*(L)) \mapsto (\theta, l) \tag{2.4.2}$$

where $l = (l_k)_{k\in\mathcal{N}_i}$,

$$l_k = \Big\langle L, \frac{\partial \chi^{-1}(\theta)}{\partial \theta_k}\Big\rangle, \qquad k \in \mathcal{N}_i. \tag{2.4.3}$$

Writing $\theta_+(z) = \chi(\omega_+(z))$, For $(\omega_+, X^*(L_+)) \in \mathcal{L}_+$ defined in Theorem 2.1.4 we define $\theta_+(z) := \chi(\omega_+(z))$ and $l_+(z) := (l_k(z))_{k\in\mathcal{N}_i}$ with

$$l_k(z) := \Big\langle L_+(z), \frac{\partial \chi^{-1}(\theta)}{\partial \theta_k}\Big|_{\theta_+(z)}\Big\rangle, \qquad k \in \mathcal{N}_i. \tag{2.4.4}$$

Noting that $\langle \frac{\partial \chi^{-1}(\theta)}{\partial \theta_k}, \chi^{-1}(\theta)\rangle = 0$ for $k \in \mathcal{N}_i$, we obtain from (2.1.14) and (2.2.20) that for any $s > s_+$ (see Definition 2.2.6) we have

$$l_k(z) = -\sqrt{\lambda}\Big\langle r_+(z;\lambda), \frac{\partial \chi^{-1}(\theta)}{\partial \theta_k}\Big|_{\theta_+(z)}\Big\rangle = -\frac{\partial}{\partial \theta_k}\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\chi^{-1}(\theta))\Big|_{\theta_+(z)}, \qquad k \in \mathcal{N}_i. \tag{2.4.5}$$

We now choose an atlas on S^{n-1} that is well–suited to our situation, with charts simply consisting of the orthogonal projection of hemispheres along their poles.

2.4.2. Definition *On the sphere $S^{n-1} \subset \mathbb{R}^n$ we define charts $\{(\Sigma_i^\pm(\delta), \chi_i^\pm)\colon i = 1, \ldots, n\}$ for $\delta \in [0, 1/4)$ by*

$$\begin{aligned} \Sigma_i^\pm(\delta) &:= \{x = (x_1, \ldots, x_n) \in S^{n-1}\colon x_i \gtrless \pm\delta\} \\ \chi_i^\pm\colon \Sigma_i^\pm(\delta) &\to B^{n-1} \subset \mathbb{R}^{n-1} \qquad \chi_i^\pm(x_1, \ldots, x_n) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = x_{\mathcal{N}_j} \end{aligned} \tag{2.4.6}$$

where $B^{n-1} := \{x \in \mathbb{R}^{n-1}\colon |x| < 1\}$, $\mathcal{N}_j := \mathcal{N} \setminus \{j\}$, $\mathcal{N} = \{1, \ldots, n\}$. Note that we have not included δ in our notation for the maps $\chi_i^\pm\colon \Sigma_i^\pm(\delta) \to B^{n-1} \subset \mathbb{R}^{n-1}$. Furthermore, we simply write $\Sigma_i^\pm$ for $\Sigma_i^\pm(0)$.

2.4.3. Lemma *On T^*S^{n-1}, the maps $\chi_i^\pm$ induce charts $\{(T^*\Sigma_i^\pm(\delta), \widetilde{\chi}_i^\pm)\colon i = 1, \ldots, n\}$ given by*

$$\widetilde{\chi}_i^\pm\colon T^*\Sigma_i^\pm(\delta) \to B^{n-1} \times \mathbb{R}^{n-1}, \qquad (\omega, X^*(L)) \mapsto \Big(\omega_{\mathcal{N}_j}, L_{\mathcal{N}_i} - \frac{L_i}{\omega_i}\omega_{\mathcal{N}_i}\Big). \tag{2.4.7}$$

Here we have used the notation of Convention 2.3.1 for L_+ and ω_+, suppressing the subscript when referring to the components.

Proof. Writing $|y|^2 = \sum_{j=1}^{n-1} y_j^2$ we have

$$(\chi_i^\pm)^{-1}(y_1, \ldots, y_{n-1}) = (y_1, \ldots, y_{i-1}, \pm\sqrt{1-|y|^2}, y_i, \ldots, y_{n-1}). \tag{2.4.8}$$

Then (2.4.7) follows directly from (2.4.3). □

2.4.4. Convention i) *For any $i \in \mathcal{N}$ we use $\Sigma_i(\delta)$ to denote either $\Sigma_i^+(\delta)$ or $\Sigma_i^-(\delta)$ and χ_i to denote either χ_i^+ or χ_i^- of (2.4.6).*

ii) *In the context of Definition 2.3.2, let $M = S^{n-1}$, $U = T^*\Sigma_i$ for some $i \in \mathcal{N}$ and $(x, \xi) = \widetilde{\chi}_i$. Let $\Lambda = \mathcal{L}_+$ be the lagrangian manifold (2.1.17). Denote by $\Omega = \Gamma$ an open set in $\mathcal{L}_+ \cap T^*\Sigma_i$ and $I = J \subset \mathcal{N}_i := \mathcal{N} \setminus \{i\}$. In this situation, we denote the map $\pi_{\Omega,I}$ in (2.3.1) by $\pi_{\Gamma,J}^{(i)}$. Conversely, $\pi_{\Gamma,J}^{(i)}$ shall always refer to Γ, J, i as above.*

iii) *The notation $\pi_{\Omega,I}$ shall henceforth refer to a lagrangian chart as in (2.3.1) with $M = \mathbb{R}^n$, (x,ξ)–coordinates on $U = T^*\mathbb{R}^n$, the lagrangian manifold Λ of (1.3.3) and $\Omega \subset \Lambda$, $I \subset \mathcal{N}$.*

2.4.5. Remark i) Setting

$$l^{(i)} := L - \frac{L_i}{\omega_i}\omega, \tag{2.4.9}$$

the lagrangian charts $\pi^{(i)}_{J,\Gamma}$ are given by

$$\pi^{(i)}_{J,\Gamma}(\omega, X^*(L)) = \big(\omega_J(z), l^{(i)}_{\mathcal{N}_i\setminus J}\big). \tag{2.4.10}$$

Here we have used the notation of Convention 2.4.4 ii) for $l^{(i)}_+$. For $\omega_+(z)$ and $L_+(z)$ of (2.1.14), we define

$$l^{(i)}_+(z) := L_+(z) - \frac{L_i(z)}{\omega_i(z)}\omega_+(z). \tag{2.4.11}$$

Noting that with $\theta_+(z) := \chi_i(\omega_+(z))$, $l^{(i)}_+(z) = (l^{(i)}_k(z))_{i\in\mathcal{N}}$ and Remark 2.4.1,

$$l^{(i)}_k(z) = \Big\langle L_+(z), \frac{\partial\chi_i^{-1}(\theta)}{\partial\theta_k}\Big|_{\theta_+(z)}\Big\rangle = \frac{\partial}{\partial\theta_k}\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \chi^{-1}(\theta))\Big|_{\theta_+(z)}, \qquad k \in \mathcal{N}_i. \tag{2.4.12}$$

ii) We have established global coordinates for $\Lambda \subset T^*\mathbb{R}^n$ and $\mathcal{L}_+ \subset T^*S^{n-1}$ in Sections 1.3 and 2.1 through the mappings

$$\iota\colon \mathbb{R}\times\mathbb{R}^{n-1} \to \Lambda \qquad \text{and} \qquad S^+_\lambda\colon \mathbb{R}^{n-1} \to \mathcal{L}_+, \tag{2.4.13}$$

given by (1.3.2) and (2.1.15), respectively. Then by Corollary 2.3.4 $(x_I, \xi_{\overline{I}})$, $\overline{I} = \mathcal{N}\setminus I$, are lagrangian coordinates in some neighbourhood of $p \in \Lambda$ if and only if

$$n = \operatorname{rank} d\pi_{\Lambda,I}|_p = \operatorname{rank}\left(\frac{\partial(\boldsymbol{x}_I(s,z;\lambda), \boldsymbol{\xi}_{\mathcal{N}\setminus I}(s,z;\lambda))}{\partial(s,z)}\right). \tag{2.4.14}$$

Similarly, for $I \subset \mathcal{N}$, $(\omega_J, l^{(i)}_{\mathcal{N}_i\setminus J})$ are lagrangian coordinates in a neighbourhood of $q \in \mathcal{L}_+ \cap T^*\Sigma_i$ if and only if

$$n-1 = \operatorname{rank} d\pi^{(i)}_{\mathcal{L}_+,J}|_p = \operatorname{rank}\left(\frac{\partial(\omega_J(z;\lambda), l_{\mathcal{N}_i\setminus J}(z;\lambda))}{\partial z}\right). \tag{2.4.15}$$

The main result of this section is the relationship between lagrangian coordinates on $\mathcal{L}_+$ and coordinates on Λ. Again, the sharp estimates on the behaviour of the trajectories in Proposition 1.2.10 are crucial for the proof of Proposition 2.4.7, on which Proposition 2.4.6 is based.

2.4.6. Proposition *Let $K \subset \mathbb{R}^{n-1}$ be a compact set and $T \gg 0$ sufficiently large. Fix $\delta \in (0, 1/4)$. Then for any $T_1 > T_0 > T$ there exists an open covering $\{Z_k\}$ of K, numbers $i_k \in \mathcal{N}$ and index sets $J_k \subset \mathcal{N}_{i_k}$ such that*

(i) *$\Gamma_k := S^+_\lambda(Z_k) \subset \mathcal{L}_+ \cap T^*\Sigma_{i_k}(\delta)$ and $(\Gamma_k, \pi^{(i_k)}_{\Gamma_k,J_k})$ are lagrangian charts on $\mathcal{L}_+$ and*
(ii) *$(\Omega_k, \pi_{\Omega_k,\mathcal{N}\setminus J_k})$, $\Omega_k := \iota((T_0,T_1), Z_k)$ are lagrangian charts on Λ.*

Proposition 2.4.6 relies on various preliminary results, which we give below before completing the proof at the end of this section.

2.4.7. Proposition *For $z_0 \in \mathbb{R}^{n-1}$, let $i \in \mathcal{N}$ such that $S^+_\lambda(z_0) \in \mathcal{L}_+ \cap T^*\Sigma_i$. Let $\pi_\omega\colon T^*S^{n-1} \to S^{n-1}$ denote the canonical projection onto the base and let $\operatorname{rank} d(\pi_\omega|_{\mathcal{L}_+})|_{S^+_\lambda(z_0)} = m$. Let $J \subset \mathcal{N}_i$ be an index set such that $|J| = m$ and $\operatorname{rank} d\pi^{(i)}_{\mathcal{L}_+,J}|_{S^+_\lambda(z_0)} = n-1$ (see Corollary 2.3.4). Then there exists some $T = T(z_0) > 0$ and some $\varepsilon = \varepsilon(z_0) > 0$ so that*

$$\left|\det\left(\frac{\partial(\boldsymbol{x}_{\mathcal{N}\setminus J}(s,z;\lambda), \boldsymbol{\xi}_J(s,z;\lambda))}{\partial(s,z)}\right)\right|_{z=z_0} > \varepsilon \qquad \textit{for all } s > T. \tag{2.4.16}$$

Proof. We first claim that for any $k \in \mathcal{N}$,

$$\frac{\partial\omega_k}{\partial z}\Big|_{z_0} \in \operatorname{span}\left\{\frac{\partial\omega_j}{\partial z}\Big|_{z_0}\right\}_{j\in J}. \tag{2.4.17}$$

By (2.4.15)

$$\operatorname{rank}\left(\frac{\partial\omega_J(z;\lambda)}{\partial z}\right)\Big|_{z=z_0} = |J| = m. \tag{2.4.18}$$

Furthermore,

$$m = \operatorname{rank} d(\pi_\omega|_{\mathcal{L}_+})|_{S^+_\lambda(z_0)} = \operatorname{rank} d\pi^{(i)}_{\mathcal{L}_+,\mathcal{N}_i}|_{S^+_\lambda(z_0)} = \operatorname{rank}\left(\frac{\partial(\omega_{\mathcal{N}_i}(z;\lambda)}{\partial z}\right)\Big|_{z=z_0}, \tag{2.4.19}$$

using (2.4.15) in the last step. Then (2.4.18) and (2.4.19) imply (2.4.17) for $k \in \mathcal{N}_i$. Differentiating $\omega_i^2 = \sum_{j\in\mathcal{N}_i}\omega_j^2$ gives

$$\frac{\partial\omega_i}{\partial z}\Big|_{z_0} = -\frac{1}{\omega_i}\sum_{j\in\mathcal{N}_i}\omega_j\,\frac{\partial\omega_j}{\partial z}\Big|_{z_0}. \tag{2.4.20}$$

and hence (2.4.17) holds for all $k \in \mathcal{N}$.

Now by (1.2.15),

$$\begin{aligned} M &:= \left(\frac{\partial(\boldsymbol{x}_{\mathcal{N}\setminus J}(s,z;\lambda),\boldsymbol{\xi}_J(s,z;\lambda))}{\partial(s,z)}\right)\Big|_{z=z_0} \\ &= \begin{pmatrix} 2\sqrt{\lambda}\omega_{\mathcal{N}\setminus J}(z;\lambda) + 2\frac{\partial g_{\mathcal{N}\setminus J}(s,z;\lambda)}{\partial s} & 2\sqrt{\lambda}s\frac{\partial\omega_{\mathcal{N}\setminus J}(z;\lambda)}{\partial z} + \frac{\partial r_{\mathcal{N}\setminus J}(z;\lambda)}{\partial z} + 2\frac{\partial g_{\mathcal{N}\setminus J}(s,z;\lambda)}{\partial z} \\ \frac{\partial^2 g_J(s,z;\lambda)}{\partial s^2} & \sqrt{\lambda}\frac{\partial\omega_J(z;\lambda)}{\partial z} + \frac{\partial^2 g_J(s,z;\lambda)}{\partial z\partial s}\end{pmatrix}\Bigg|_{z=z_0} \end{aligned} \tag{2.4.21}$$

By (2.4.17), for any $k \in \mathcal{N}$ we can write the vector $\partial_z\omega_k$ as a linear combination of vectors $\partial_z\omega_j$, $j \in J$. We will abbreviate this linear combination as

$$\frac{\partial\omega_k}{\partial z}\Big|_{z_0} = \operatorname{lc}_k\left(\frac{\partial\omega_J}{\partial z}\Big|_{z_0}\right), \qquad \text{where } \operatorname{lc}_k(v_J) := \sum_{j\in J}\lambda_{jk}v_j, \qquad k\in\mathcal{N}, \text{ for } v_j \in \mathbb{R}^m,\ m\in\mathbb{N}. \tag{2.4.22}$$

By adding suitable linear combinations of the lower rows of D to the upper rows, we obtain

$$\det M = \det(A + A'), \tag{2.4.23}$$

where

$$A := \begin{pmatrix} 2\sqrt{\lambda}\omega_{\mathcal{N}\setminus J} & \frac{\partial r_{\mathcal{N}\setminus J}}{\partial z} \\ 0 & \sqrt{\lambda}\frac{\partial\omega_J}{\partial z}\end{pmatrix}\Bigg|_{z=z_0}. \tag{2.4.24}$$

and

$$A' := \begin{pmatrix} 2\frac{\partial g_{\mathcal{N}\setminus J}(s,z;\lambda)}{\partial s} + 2s\operatorname{lc}_{\mathcal{N}\setminus J}\left(\frac{\partial^2 g_J(s,z;\lambda)}{\partial s^2}\right) & 2\frac{\partial g_{\mathcal{N}\setminus J}(s,z;\lambda)}{\partial z} + 2s\operatorname{lc}_{\mathcal{N}\setminus J}\left(\frac{\partial^2 g_J(s,z;\lambda)}{\partial z\partial s}\right) \\ \frac{\partial^2 g_J(s,z;\lambda)}{\partial s^2} & \frac{\partial^2 g_J(s,z;\lambda)}{\partial z\partial s}\end{pmatrix}\Bigg|_{z=z_0} \tag{2.4.25}$$

Applying estimate (1.2.16c), we see that for sufficiently large T there exists a constant $C_T > 0$ such that

$$\|A'\|_{z=z_0} \le C_T\cdot s^{1-\varrho} \qquad \text{for all } s > T. \tag{2.4.26}$$

Note that

$$|\det A| = 2\lambda^{1+|J|-\frac{n}{2}}|\det B|, \qquad B = \begin{pmatrix}\omega_{\mathcal{N}\setminus J} & -\sqrt{\lambda}\frac{\partial r_{\mathcal{N}\setminus J}}{\partial z} \\ 0 & \frac{\partial\omega_J}{\partial z}\end{pmatrix}\Bigg|_{z=z_0} \tag{2.4.27}$$

We will show that $\operatorname{rank} B = n$, i.e.,

$$0 < |\det A| =: 2\varepsilon. \tag{2.4.28}$$

Then by (2.4.23), (2.4.26) and (2.4.28) the continuity of the determinant yields the existence of some $T' > T$ such that

$$|\det D| > \frac{1}{2}|\det A| = \varepsilon \qquad \text{for } s > T'. \tag{2.4.29}$$

The proof of (2.4.16) is thus completed. We now show $\operatorname{rank} B = n$. By (2.4.11) and (2.1.14) we have

$$\begin{aligned} \frac{\partial l_j}{\partial z_k} &= \frac{\partial L_j}{\partial z_k} - \frac{L_i}{\omega_i}\frac{\partial\omega_j}{\partial z_k} - \frac{\partial}{\partial z_k}\left(\frac{L_i}{\omega_i}\right)\omega_j & (2.4.30) \\ &= -\sqrt{\lambda}\frac{\partial r_j}{\partial z_k} + \beta\frac{\partial\omega_j}{\partial z_k} + \alpha_k\omega_j & (2.4.31) \end{aligned}$$

with

$$\beta(z) = \sqrt{\lambda}\langle\omega_+(z), r_+(z;\lambda)\rangle - \frac{L_i(z)}{\omega_i(z)}, \qquad \alpha_k(z) = \frac{\partial}{\partial z_k}\beta(z). \tag{2.4.32}$$

For $k = 1, \ldots, n-1$ we add the first column multiplied by $\alpha_k(z)$ to the $(k+1)$st column of B. Furthermore, for $k = 1, \ldots, n - |J|$, we add the linear combination $\beta(z) \cdot \mathrm{lc}_k((0, \frac{\partial\omega_J}{\partial z})|_{z_0})$ (see (2.4.22)) of the lower $|J|$ rows to the kth row of B. Then by (2.4.31) and (2.4.27),

$$\operatorname{rank} B = \operatorname{rank} C, \qquad C = \left.\begin{pmatrix} \omega_i & \frac{\partial l_i}{\partial z} \\ \omega_{\mathcal{N}\setminus J} & \frac{\partial l_{\mathcal{N}\setminus J}}{\partial z} \\ 0 & \frac{\partial \omega_J}{\partial z} \end{pmatrix}\right|_{z=z_0}. \tag{2.4.33}$$

By (2.4.15), the rank of the lower $n-$ rows of C is $n-1$. We will now show that the uppermost row is independent of the lowers rows, hence $\operatorname{rank} C = n - 1$.

Similarly to the arguments leading to (2.4.33), it follows from (2.4.30) that

$$\operatorname{rank}\begin{pmatrix} \omega_{\mathcal{N}_i} & \frac{\partial l_{\mathcal{N}_i}}{\partial z} \\ 0 & \frac{\partial \omega_{\mathcal{N}_i}}{\partial z} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \omega_{\mathcal{N}_i} & \frac{\partial L_{\mathcal{N}_i}}{\partial z} \\ 0 & \frac{\partial \omega_{\mathcal{N}_i}}{\partial z} \end{pmatrix} = n - 1 \tag{2.4.34}$$

(where we have used (2.4.36) below) and

$$\operatorname{rank} D := \operatorname{rank}\left.\begin{pmatrix} \omega_+ & \frac{\partial l_+}{\partial z} \\ 0 & \frac{\partial \omega_+}{\partial z} \end{pmatrix}\right|_{z=z_0} = \operatorname{rank}\left.\begin{pmatrix} \omega_+ & \frac{\partial L_+}{\partial z} \\ 0 & \frac{\partial \omega_+}{\partial z} \end{pmatrix}\right|_{z=z_0} = n, \tag{2.4.35}$$

where we have used (2.1.40). It follows from (2.4.20), (2.4.34) and (2.4.35) that in D the row $(\omega_i, \frac{\partial l_i}{\partial z})$ is independent of all other rows. The same is then true in the matrix C. □

2.4.8. Lemma *For $z \in \mathbb{R}^{n-1}$ choose $i \in \mathcal{N}$ such that $S_\lambda^+(z) \in \mathcal{L}_+ \cap T^*\Sigma_i$. Then*

$$\operatorname{rank}\begin{pmatrix} \frac{\partial L_{\mathcal{N}_i}(z)}{\partial z} \\ \frac{\partial \omega_{\mathcal{N}_i}(z)}{\partial z} \end{pmatrix} = n - 1 \qquad \textit{and} \qquad \operatorname{rank}\begin{pmatrix} \omega_{\mathcal{N}_i}(z) & \frac{\partial L_{\mathcal{N}_i}(z)}{\partial z} \\ 0 & \frac{\partial \omega_{\mathcal{N}_i}(z)}{\partial z} \end{pmatrix} = n - 1. \tag{2.4.36}$$

Proof. We start with the first assertion. By Proposition 2.1.13,

$$\operatorname{rank}\begin{pmatrix} \frac{\partial L_+(z)}{\partial z} \\ \frac{\partial \omega_+(z;\lambda)}{\partial z} \end{pmatrix} = n - 1. \tag{2.4.37}$$

By (2.4.20) the $(n+i)$th row of the matrix is a linear combination of the other rows, so it suffices to show that $\frac{\partial L_i(z)}{\partial z}$ is also a linear combination of the other rows. Now $\langle L_+, \omega_+\rangle = 0$, so

$$\omega_i L_i = -\sum_{j\in\mathcal{N}_i} L_j \omega_j \tag{2.4.38}$$

and hence

$$\frac{\partial L_i}{\partial z} = -\frac{1}{\omega_i}\sum_{j\in\mathcal{N}} L_j \frac{\partial \omega_j}{\partial z} - \frac{1}{\omega_i}\sum_{j\in\mathcal{N}_i} \omega_j \frac{\partial L_j}{\partial z}. \tag{2.4.39}$$

This proves the first assertion. Once more Proposition 2.1.13 yields

$$\operatorname{rank}\begin{pmatrix} \omega_+ & \frac{\partial L_+}{\partial z} \\ 0 & \frac{\partial \omega_+}{\partial z} \end{pmatrix} = n, \tag{2.4.40}$$

and by (2.4.20) the $n + i$th row is a linear combination of the others. We will show that the row $(\omega_i, \frac{\partial L_i(z;\lambda)}{\partial z})$ is independent of the others, thereby completing the proof. Now (2.4.39) expresses $\frac{\partial L_i}{\partial z}$ as a linear combination of $\frac{\partial L_j(z;\lambda)}{\partial z}$, $j \in \mathcal{N}_i$, and $\frac{\partial \omega_j(z;\lambda)}{\partial z}$, $j \in \mathcal{N}$. Hence if $(\omega_i, \frac{\partial L_i(z;\lambda)}{\partial z})$ were a combination of the rows of (2.4.40), we would get

$$\omega_i = -\frac{1}{\omega_i}\sum_{j\in\mathcal{N}_i} \omega_j^2, \tag{2.4.41}$$

contradicting $|\omega| = 1$. Thus $(\omega_i, \frac{\partial L_i(z;\lambda)}{\partial z})$ is independent of the rows of (2.4.40), completing the proof. □

2.4.9. Lemma *Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^n$ be a smooth map. Let $K \subset U$ be some compact set, $f|_K$ be injective and $df|_K$ be invertible. Then there exists some open set $V \supset K$ so that f is injective on V.*

Proof. The inverse map $(f|_K)^{-1}\colon f(K) \to K$ exists (by injectivity of $f|_K$) and is C^1 by the invertibility of $df|_K$. The compactness of K and $f(K)$ together with the continuity of the derivatives of $f|_K$ and $(f|_K)^{-1}$ give the two–sided estimate

$$C_1|x - x'| \le |f(x) - f(x')| \le C_2|x - x'|, \qquad x, x' \in K, \tag{2.4.42}$$

with uniform constants $C_1, C_2 > 0$. By continuity of f, for any $\varepsilon > 0$ we can find some open $U_\delta \supset K$ such that

$$|f(x) - f(x')| \ge C_1|x - x'| - \varepsilon \qquad \text{for all } x, x' \in U_\delta. \tag{2.4.43}$$

Assume that there is no open set $V \supset K$ such that $f|_V$ is injective. By (2.4.43), $|x - x'| \le \varepsilon/C_1$ for all points $x, x' \in U_\delta$ for which $f(x) = f(x')$. But by the inverse function theorem, for all $x \in K$ there exist neighbourhoods $B_r(x)$, where $r = r(x) > 0$ can be chopsen to continuously depend on x, so that $f|_{B_r(x)}$ is injective. By compactness of K and continuity of r the minimum $r_0 := \min_{x\in K} r(x)$ exists and is strictly positive, so we need only choose $\varepsilon < C_1 r_0$ to arrive at a contradiction. □

2.4.10. Lemma *For $z \in \mathbb{R}^{n-1}$, let $i \in \mathcal{N}$ such that $S^+_\lambda(z) \in T^*\Sigma_i$. Let $\pi_\omega\colon T^*S^{n-1} \to S^{n-1}$ denote the canonical projection onto the base and let $\operatorname{rank} d(\pi_\omega|_{\mathcal{L}_+})|_{S^+_\lambda(z)} = m$. Let $J \subset \mathcal{N}_i$ be an index set such that $|J| = m$ and $\operatorname{rank} d\pi^{(i)}_{\mathcal{L}_+,J}|_{S^+_\lambda(z)} = n - 1$ (see Corollary 2.3.4). Choose $T(z)$ so that (2.4.16) holds for $z_0 = z$. Then for any $T_1 > T_0 > T(z)$ there exists some $\delta > 0$ so that for*

$$U_\delta := (T_0 - \delta, T_1 + \delta) \times B_\delta(z) \subset \mathbb{R} \times \mathbb{R}^{n-1}, \qquad B_\delta(z) = \{y \in \mathbb{R}^{n-1}\colon |z - y| \le \delta\}, \tag{2.4.44}$$

the map

$$\pi_{\iota(U_\delta),\mathcal{N}\setminus J}\colon (x, X^*(\xi)) \mapsto (x_{\mathcal{N}\setminus J}, \xi_J). \tag{2.4.45}$$

is a diffeomorphism on its image.

Proof. Since ι is a diffeomorphism on its image, it suffices to show that

$$\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota\colon (s, z) \mapsto (\boldsymbol{x}_{\mathcal{N}\setminus J}(s, z; \lambda), \boldsymbol{\xi}_J(s, z; \lambda)). \tag{2.4.46}$$

is a diffeomorphism on its image. We choose $T(z)$ so that Proposition 2.4.7 and (1.2.20) (with $\beta = 0$ and $\varepsilon < 1/2$) both hold. By (1.2.20) the map $\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota$ is injective on $K := [T_0, T_1] \times \{z\}$; in fact

$$\begin{aligned}|\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota(s, z) - \pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota(s', z)| &> C_1|\boldsymbol{x}_i(s, z; \lambda) - \boldsymbol{x}_i(s', z; \lambda)|\\ &> C_2|s - s'|\end{aligned}$$

for some $C_1, C_2 > 0$. Furthermore, by (2.4.16), the differential of $\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota$ is invertible on K. Thus we can apply Lemma 2.4.9 to obtain the existence of some open set V_1 containing K so that $\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota|_{V_1}$ is injective. Furthermore, since the differential $\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota$ is invertible on K, we can find some open set $V_2 \supset K$ where the differential is invertible, too. Taking $\delta > 0$ small enough that $U_\delta \subset V_1 \cap V_2$, it follows that $\pi_{\iota(U_\delta),\mathcal{N}\setminus J} \circ \iota|_{U_\delta}$ is an injective immersion. Since U_δ is bounded, the map is trivially proper, so it is also an embedding, □

Proof of Proposition 2.4.6. For any $z \in K$, choose a continuous function $T(z)$ so that Lemma 2.4.10 holds for that z. Set $T := \max_{z\in K} T(z)$. Then for any z, there exists an open set $U_z \supset z$ such that the assertions (i) (by Corollary 2.3.4) and (ii) (with Z_k replaced by U_z; applying Lemma 2.4.10; $U_z \subset B_\delta(z)$) hold. By compactness, we can cover K with a finite number of these U_z; denoting this covering by $\{Z_k\}$, we are finished. □

CHAPTER 3

The scattering amplitude

Using the results of Chapter 2, we can now build on Robert and Tamura's basic representation formula for the scattering amplitude. In Section 3.1 we review the relevant results from [24] and establish some basic definitions and choices of constants. The aim of Section 3.2 is to establish hard estimates which will allow us to approximate the action of $e^{\frac{i}{h}Pt}$ using Maslov theory; the main results here are Proposition 3.2.16 and Corollary 3.2.17.

Having obtained an integral formula for the scattering amplitude based on a Maslov operator on Λ, we apply the results of Chapter 2, in particular Lemma 2.3.8 and Proposition 2.4.6, to recast this formula in terms of a Maslov operator on $\mathcal{L}_+$. This leads directly to a proof of Theorem 1. A discussion of the main result and some aspects of previous results concerning caustics follows in Section 3.4, while Section 3.5 concludes with a review of the caustics encountered in scattering in $\mathbb{R}^2$ and $\mathbb{R}^3$.

3.1. The representation formula

The aim of this section is to formulate the representation formula for the scattering amplitude that was obtained by Robert and Tamura [24]. We will therefore first give a summary of their constructions.

We start with a few essential definitions.

3.1.1. Definition *For $\Omega \subset \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ we denote by $A_m(\Omega)$ the set of all $a \in C^\infty(\Omega)$, such that for any $\alpha, \beta \in \mathbb{N}^n$ and any $L > 1$ there exist constants $C_{\alpha,\beta,L} > 0$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta,L} \langle x\rangle^{m-|\alpha|} \langle \xi\rangle^{-L}. \tag{3.1.1}$$

If, in particular, $\Omega = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, we write A_m for $A_m(\Omega)$.

3.1.2. Proposition *Let (R_0, d_0, σ_0) be fixed by Definition 2.2.6 and denote by $\varphi_\pm$ the Isozaki-Kitada phase functions of Proposition 2.2.2. Then for any $\sigma, \sigma' \in (\sigma_0, 0)$, $\sigma' > \sigma$, $d, d' \in (1, d_0)$, $d' < d$, and $R' > R > R_0$ there exist functions $c_{\pm j}$, $j \in \mathbb{N}$, such that*

$$c_{\pm j} \in A_{-j}, \qquad \operatorname{supp} c_{\pm j} \subset \Gamma_\pm(R, d, \pm\sigma), \tag{3.1.2a}$$

$$2\langle \nabla_x \varphi_\pm, \nabla_x c_{\pm j}\rangle + (\Delta_x \varphi_\pm) c_{\pm j} = \begin{cases} 0 & j = 0 \\ i\Delta_x c_{\pm j-1} & j \ge 1 \end{cases} \qquad \text{on } \Gamma_\pm(R', d, \pm\sigma'), \tag{3.1.2b}$$

$$c_{\pm j} \xrightarrow{|x|\to\infty} \begin{cases} 1 & j = 0 \\ 0 & j \ge 1 \end{cases} \qquad \text{on } \Gamma_\pm(R', d', \pm\sigma'), \tag{3.1.2c}$$

where $\Gamma_\pm(R, d, \sigma)$ were defined in (2.2.1).

3.1.3. Definition & Lemma *Denote by $R(\zeta, P) = (P - \zeta)^{-1}$, $\operatorname{Im}\zeta \ne 0$, the resolvent of P, cf. (1). Then $R(\zeta, P)$ is well defined as an operator $L^2_\gamma \to L^2_{-\gamma}$ for any $\gamma > 1/2$ and we can use the principle of limiting absorption to define $R(\lambda + i0, P) \colon L^2_\gamma \to L^2_{-\gamma}$, $\gamma > 1/2$, by*

$$R(\lambda + i0, P) := \operatorname*{s-lim}_{\kappa \searrow 0} R(\lambda + i\kappa, P) \qquad \text{in } L^2_{-\gamma}, \quad \lambda \in \mathbb{R} \tag{3.1.3}$$

3.1.4. Theorem [24, Corollary of Lemma 2.1] *Choose $N > 100n$. Fix $1 < d_4 < d_3 < d_2 < d_1 < d_0$ such that $d_4^{-1} < \sqrt{\lambda} < d_4$, $\sigma_0 < \sigma_1 < \sigma_2 < \sigma_3 < \sigma_4 < 0$, $R_0 < R_1 < R_2 < R_3 < R_4$. Define*

$$a_\pm(x,\xi;h) := \sum_{j=0}^{N} a_{\pm j}(x,\xi) h^j, \qquad b_\pm(x,\xi;h) := \sum_{j=0}^{N} b_{\pm j}(x,\xi) h^j, \tag{3.1.4}$$

with $a_{\pm j}$ *defined to have the properties of* $c_{\pm j}$ *in Proposition 3.1.2 for* $R = R_1$, $R' = R_2$, $d = d_1$, $d' = d_2$, $\sigma = \sigma_1$, $\sigma' = \sigma_2$. *In the same way,* $b_{\pm j}$ *is defined with* $R = R_3$, $R' = R_4$, $d = d_3$, $d' = d_4$, $\sigma = \sigma_3$, $\sigma' = \sigma_4$.

Let $R_a \gg R_b \gg R_4$ *(we will choose suitable* R_a *and* R_b *below). Let* $\chi(\,\cdot\,, \rho) \in C^\infty(\mathbb{R})$, $0 < \chi < 1$, *be a function such that* $\chi(x, \rho) = 1$ *if* $|x| < \rho$ *and* $\chi(x, \rho) = 0$ *if* $|x| > \rho + 1$. *Set*

$$\chi_a(x) := \chi(x, R_a), \qquad \chi_b(x) := \chi(x, R_b). \tag{3.1.5}$$

Let $[A\,, B] = AB - BA$ *denote the usual commutator. Define*

$$\begin{aligned} g_{-b}(x; h, \omega_-) &:= e^{-\frac{i}{h}\varphi_-(x,\sqrt{\lambda}\omega_-)}[\chi_b(x), P_0]b_-(x, \sqrt{\lambda}\omega_-; h)e^{\frac{i}{h}\varphi_-(x,2\sqrt{\lambda}\omega_-)}, \\ g_{+a}(x; h, \omega_+) &:= e^{-\frac{i}{h}\varphi_+(x,2\sqrt{\lambda}\omega_+)}[\chi_a(x), P_0]a_+(x, \sqrt{\lambda}\omega_+; h)e^{\frac{i}{h}\varphi_+(x,\sqrt{\lambda}\omega_+)} \end{aligned} \tag{3.1.6}$$

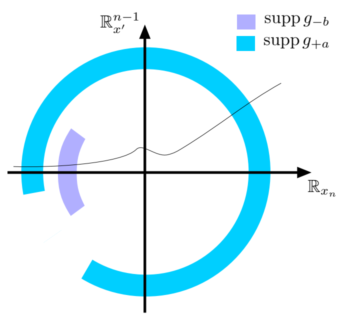
and

$$G_0(\omega_-, \omega_+; \lambda, h) := \big(R(\lambda + i0; P)g_{-b}(\,\cdot\,; h, \omega_-)e^{\frac{i}{h}\varphi_-(\,\cdot\,,2\sqrt{\lambda}\omega_-)} \,\big|\, g_{+a}(\,\cdot\,; h, \omega_+)e^{\frac{i}{h}\varphi_+(\,\cdot\,,\sqrt{\lambda}\omega_+)}\big)_{L^2}. \tag{3.1.7}$$

Then

$$f(\omega_-, \omega_+; \lambda, h) = c_1(\lambda, h)G_0(\omega_-, \omega_+; \lambda, h) + O(h^{N/3}), \tag{3.1.8}$$

where $c_1(\lambda, h) = 2\pi\lambda^{(n-3)/4}(2\pi h)^{-(n+1)/2}e^{-(n-3)i\frac{\pi}{4}}$.[1] *Fixing* $\omega_- \in S^{n-1}$, $O(h^n)$ *denotes a function of* ω_+ *whose supremum over the sphere is bounded by a constant multiplied by* h^n.



Defining

$$\widetilde{\Sigma}_\pm(R, \sigma, \xi) := \Sigma_\pm(R, \sigma, \xi) \cap \{x \in \mathbb{R}^n \colon R < |x| < R + 1\} \tag{3.1.9}$$

we see that

$$\begin{aligned} \operatorname{supp} g_{-b}(\,\cdot\,; h, \omega_-) &\subseteq \widetilde{\Sigma}_-(R_b, \sigma_4, \sqrt{\lambda}\omega_-), \\ \operatorname{supp} g_{+a}(\,\cdot\,; h, \omega_+) &\subseteq \widetilde{\Sigma}_+(R_a, \sigma_1, \sqrt{\lambda}\omega_+). \end{aligned} \tag{3.1.10}$$

For short we will write

$$\begin{aligned} \Sigma_{-b} &:= \widetilde{\Sigma}_-(R_b, \sigma_4, \sqrt{\lambda}\omega_-), \\ \Sigma'_{-b} &:= \widetilde{\Sigma}_-(R_b, \sigma_3, \sqrt{\lambda}\omega_-). \end{aligned} \tag{3.1.11}$$

Having thus reviewed the basic setting of Robert and Tamura, we now adapt the construction for our purposes. We now fix R_a, R_b and various other objects in a suitable way, which will be essential for our analysis of G_0.

3.1.5. Definition & Lemma *Let* σ_3 *and* σ_4 *be fixed as in Theorem 3.1.4 and* σ_0, R_0, s_- *and* Λ_- *be fixed as in Definition 2.2.6. For some sufficiently small* $\varepsilon > 0$ *we define the compact set*

$$Z_\varepsilon := \{z \in \mathbb{R}^{n-1} \colon |\omega_+(z; \lambda) - \omega_-| > \varepsilon\} \tag{3.1.12}$$

We choose $R_b > R_0$ *so large that (see (3.1.11))*

i) $\Sigma_{-b} \subset \pi_x\Lambda_-$,
ii) $\Sigma'_{-b} \cap \pi_x\mathcal{T}^-_{z,s_-} \neq \emptyset$ *for all* $z \in Z_\varepsilon$ *and*
iii) $\pi_x\mathcal{T}^-_{Z_\varepsilon,s_-} \cap (\{x \colon R_b < |x| < R_b + 1\} \setminus \Sigma'_{-b}) = \emptyset$.

We set

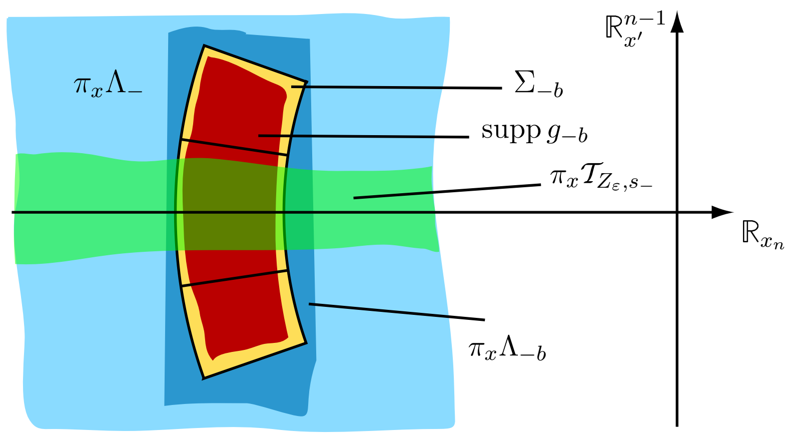
$$\Omega_0 := \operatorname{int}\operatorname{supp} g_{-b}(\,\cdot\,; h, \omega_-) \circ \pi_x|_\Lambda, \qquad Z_0 := \operatorname{int}\operatorname{supp} g_{-b}(\boldsymbol{x}_\infty(s, \,\cdot\,; \lambda); h, \omega_-), \tag{3.1.13}$$

(where "$\operatorname{int} A$*" denotes the interior of the set* $A \subset \mathbb{R}^n$*) and remark that by the above constructions we have* $Z_\varepsilon \subset Z_0$. *We further choose a bounded open set* $\mathcal{Z} \subset \mathbb{R}^{n-1}$ *and* $S_0 < S_1 < s_-$ *such that*

$$\Sigma_{-b} \subset \pi_x\Lambda_{-b}, \qquad \Lambda_{-b} := \iota((S_0, S_1) \times \mathcal{Z}) \tag{3.1.14}$$

---

[1] The constant c_1 contains a factor $\lambda^{(n-3)/4}$ instead of $(2\lambda)^{(n-3)/4}$ in [24] due to a differing factor of 2 in the hamiltonian system (1.1.6).



Proof. The compactness of Z_ε follows immediately from (1.2.16b).

i) We will show that it is possible to ensure $\Sigma_{-b} \subset \pi_x\Lambda_-$ by choosing R_b large enough. Note that $x \in \widetilde{\Sigma}_-(R_b, \sigma_4, \sqrt{\lambda}\omega_-)$ implies

$$x_n = \langle x, \omega_- \rangle < |x|\sigma_4, \qquad \text{and} \qquad |x| > R_b, \qquad \text{hence } x_n < R_b \cdot \sigma_4. \tag{3.1.15}$$

Now for $s_0 < s_-$ sufficiently small, by (1.2.11a) with (1.2.13),

$$\begin{aligned} \mathbb{R}^{n-1} \times (-\infty, 2\sqrt{\lambda}(s_0 - 1)) &\subset \pi_x\iota((-\infty, s_0) \times \mathbb{H}) \\ &\subset \mathbb{R}^{n-1} \times (-\infty, 2\sqrt{\lambda}(s_0 + 1)) \subset \pi_x\Lambda_-. \end{aligned} \tag{3.1.16}$$

We now fix such an s_0. By (3.1.15), if $x \in \widetilde{\Sigma}_-(R_b, \sigma, \sqrt{\lambda}\omega_-)$ and R_b is chosen sufficiently large then $x_n < 2\sqrt{\lambda}(s_0 - 1)$ which by (3.1.16) implies $x \in \pi_x\iota((-\infty, s_0) \times \mathbb{H}) \subset \pi_x\Lambda_-$. Thus we can choose R_b as stated.

ii) A simple geometrical argument shows that

$$\Sigma'_{-b} \supset \{x = (x', x_n) \in \mathbb{R}^n \colon R_b < |x| < R_b + 1,\ x_n < 0,\ |x'| \le R_b\sqrt{1 - \sigma_3^2}\}. \tag{3.1.17}$$

Thus any line $\pi_x\mathcal{T}_z^0$ (defined in (1.2.5)) with $|z| < R_b\sqrt{1 - \sigma_4^2}$ intersects Σ'_{-b} transversely. Since the distance between $\pi_x\mathcal{T}_z \cap \{x \colon x_n < 0,\ |x| > R_b\}$ and $\pi_x\mathcal{T}_z^0 \cap \{x \colon x_n < 0,\ |x| > R_b\}$ decreases as R_b increases, we can choose R_b large enough to ensure ii).

iii) This is seen by using the same argument as for ii) above.

Finally, since Σ_{-b} is bounded, we can find suitable S_0, S_1 and bounded $\mathcal{Z}$. □

3.1.6. Definition & Lemma *Let $T > 0$ be the time T of Proposition 2.4.6 for $K = Z_\varepsilon$, s_-, s_+ fixed in Definition 2.2.6 and R_b, $S_0, S_1 < s_-$ and $\mathcal{Z} \supset Z_\varepsilon$ fixed in Definition 3.1.5. We set*

$$T_1 := T + s_+ - S_0 + 1 \tag{3.1.18}$$

and choose $R_a > R_b$ such that

$$\bigcup_{0 \le t \le T_1} g_t\Lambda_{-b} \subset \{x \colon |x| < R_a - 1\}. \tag{3.1.19}$$

We then choose $T_0 > T_1 + S_1 - S_0$ such that

$$\iota((S_0 + T_0, S_1 + T_0) \times \mathcal{Z}) \subset \Gamma_+(R_a + 2, d_4, 0), \tag{3.1.20}$$

where Γ_+ is defined in (2.2.1).

Proof. It is possible to choose R_a as stated since $\pi_x \circ \iota$ is continuous on the compact set $[S_0 + T_1, S_1 + T_1] \times \mathcal{Z}$. Furthermore, by Lemma 2.2.3 iii) we can achieve (3.1.20) by choosing T_0 large enough. □

Using the above specific choice of T_0 we can follow Robert and Tamura's use of the Egorov theorem to obtain a formula for G_0.

3.1.7. LEMMA [24, page 173] *Let $T_0 > 0$ be given as in Definition 3.1.6. Then*

$$G_0(\omega_-, \omega_+; \lambda, h) = \frac{1}{ih} \int_0^{T_0} e^{\frac{i}{h}t\lambda} \big(e^{-\frac{i}{h}tP} g_{-b} e^{\frac{i}{h}\varphi_-(\,\cdot\,, \sqrt{\lambda}\omega^-)} \mid g_{+a} e^{\frac{i}{h}\varphi_+(\,\cdot\,,\sqrt{\lambda}\omega^+)}\big)_{L^2}\, dt + O(h^\infty). \quad (3.1.21)$$

3.2. The Maslov Operator in extended phase space

In this section we will construct an approximation to the function $e^{-\frac{i}{h}tP} g_{-b} e^{\frac{i}{h}\varphi_-}$ in (3.1.21). This will involve the construction of a Maslov operator on Λ, the manifold of integral curves of the hamiltonian system (1.1.6) defined in Theorem 1.3.3. The definition of a Maslov operator is summarised in Appendix D, and the procedure relies on the main results of Sections 2.2 and 2.3, as well as crucial results from [18]. The final result is then stated in Corollary 3.2.17.

We consider the Cauchy problem

$$Q\psi(t, x; h) = 0, \quad (3.2.1a)$$

$$\psi(0, x; h) = u_0(x) e^{\frac{i}{h}S_0(x)}, \qquad u_0 \in C_0^\infty(\mathbb{R}^n),\ S_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}). \quad (3.2.1b)$$

where

$$Q := ih\frac{\partial}{\partial t} - P \quad (3.2.2)$$

and $P = P(h)$ is the Hamiltonian defined in (1).

We henceforth consider u_0 and S_0 to be given by

$$u_0(x) := g_{-b}(x; h, \omega_-), \qquad \text{and} \qquad S_0(x) := \varphi_-(x, \sqrt{\lambda}\omega_-), \quad (3.2.3)$$

and set $U_0 := \operatorname{int}\operatorname{supp} u_0$.

3.2.1. REMARK The function

$$\psi(x, t; h) = e^{-\frac{i}{h}tP}(g_{-b}(\,\cdot\,; h, \omega_-) e^{\frac{i}{h}\varphi_-(\,\cdot\,,\sqrt{\lambda}\omega_-)})|_x. \quad (3.2.4)$$

solves the Cauchy problem (3.2.1) with (3.2.3). This solution of the Cauchy problem is unique and smooth in x and t.

We will construct an "approximate" solution of (3.2.1) by considering the associated Hamiltonian

$$q(x, t, \xi, E) := p(x, \xi) + E = |\xi|^2 + V(x) + E \quad (3.2.5)$$

and studying integral curves of the hamiltonian vector field given by

$$\frac{\partial \widetilde{\boldsymbol{x}}}{\partial \tau} = \widetilde{\boldsymbol{\xi}}, \qquad \frac{\partial \widetilde{\boldsymbol{t}}}{\partial \tau} = 1, \qquad \frac{\partial \widetilde{\boldsymbol{\xi}}}{\partial \tau} = -\nabla_x V(\widetilde{\boldsymbol{x}}), \qquad \frac{\partial \widetilde{\boldsymbol{E}}}{\partial \tau} = 0. \quad (3.2.6a)$$

with suitable initial conditions. Deviating slightly from standard notation, for $y \in U_0$ we will denote by $\{(\widetilde{\boldsymbol{x}}(\tau, y; \lambda), \widetilde{\boldsymbol{t}}(\tau, y; \lambda); \widetilde{\boldsymbol{\xi}}(\tau, y; \lambda), \widetilde{\boldsymbol{E}}(\tau, y; \lambda)) \colon \tau \in \mathbb{R}\} \subset \mathbb{R}^{2(n+1)}$ solutions of (3.2.6a) with

$$\widetilde{\boldsymbol{x}}(0, y; \lambda) = y, \qquad \widetilde{\boldsymbol{t}}(0, y; \lambda) = 0, \qquad \widetilde{\boldsymbol{\xi}}(0, y; \lambda) = \nabla_x S_0(y), \qquad \widetilde{\boldsymbol{E}}(0, y; \lambda) = -\lambda. \quad (3.2.6b)$$

3.2.2. REMARK In this section we use $(x, t; \xi, E)$–coordinates on $T^*\mathbb{R}^{n+1}$, treating t as x_{n+1} and E as ξ_{n+1}. We will sometimes identify $T^*\mathbb{R}^{n+1}$ with $T^*\mathbb{R}^n \times T^*\mathbb{R}$ by identifying $((x, t), X^*(\xi, E))$ with $((x, X^*(\xi)), (t, X^*(E))$. We denote by

$$\pi_{(x,\xi)} \colon T^*\mathbb{R}^{n+1} \to T^*\mathbb{R}^n, \qquad ((x, t), X^*(\xi, E)) \mapsto (x, X^*(\xi)) \quad (3.2.7)$$

the projection onto the standard phase space induced by the (x, t)–coordinates.

3.2.3. LEMMA *The "initial value set"*

$$\widehat{\Omega}_0 := \{((\widetilde{\boldsymbol{x}}(0, y; \lambda), \widetilde{\boldsymbol{t}}(0, y; \lambda)), X^*(\widetilde{\boldsymbol{\xi}}(0, y; \lambda), \widetilde{\boldsymbol{E}}(0, y; \lambda))) \colon y \in U_0\} \quad (3.2.8)$$

$$\simeq \Omega_0 \times \{(0, X^*(-\lambda))\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}, \quad (3.2.9)$$

is an n–dimensional isotropic submanifold. We denote the flow of the hamiltonian vector field H_q (restricted to $\widehat{\Omega}_0$) by $\widetilde{g}_\tau \colon \widehat{\Omega}_0 \to T^*\mathbb{R}^{n+1}$. The integral curves of H_q through $\widehat{\Omega}_0$,

$$\widetilde{\Lambda} := \bigcup_{\tau\in\mathbb{R}} \widetilde{\Lambda}_\tau, \qquad\qquad \widetilde{\Lambda}_\tau := \widetilde{g}_\tau \widehat{\Omega}_0, \tag{3.2.10}$$

form a lagrangian manifold and the coordinate map

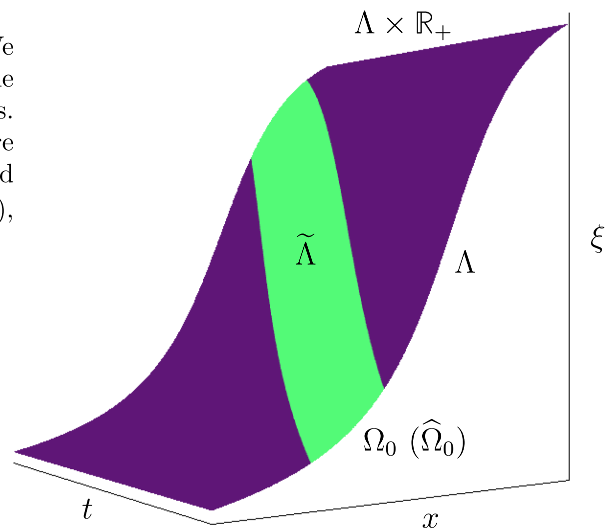
$$\widetilde{\iota}_1 \colon \mathbb{R}\times U_0 \to \widetilde{\Lambda}, \qquad (\tau, y) \mapsto ((\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \widetilde{\boldsymbol{t}}(\tau,y;\lambda)), X^*(\widetilde{\boldsymbol{\xi}}(\tau,y;\lambda), \widetilde{\boldsymbol{E}}(\tau,y;\lambda))) \tag{3.2.11}$$

is a diffeomorphism. Identifying $T^*\mathbb{R}^{n+1}$ with $T^*\mathbb{R}^n \times T^*\mathbb{R}$, $\widetilde{\iota}_1$ is

$$\widetilde{\iota}_1(\tau, y) = (\iota(s+\tau, z), (\tau, X^*(-\lambda))), \qquad\qquad (s,z) = \iota^{-1}(y, X^*(\nabla S_0(y))), \tag{3.2.12}$$

where ι was defined in (1.3.2).

The surface represents $\Lambda \times \{(t, X^*(-\lambda)) \colon t\in\mathbb{R}\}$. We have used $(x,t;\xi,E)$–coordinates, but omitted the E-axis and drawn only the positive part of the t–axis. The right edge of the surface then represents Λ (more precisely, $\Lambda\times\{(0,X^*(-\lambda))\}$). If the lightly shaded part of the right edge shows Ω_0 (more precisely, $\widehat{\Omega}_0$), the lightly shaded part of the surface represents $\widetilde{\Lambda}$.



3.2.4. Remark We define $\widetilde{\iota}_2 := \widetilde{\iota}_1 \circ (\mathrm{id} \otimes \pi_x \circ \iota)$, i.e.,

$$\begin{aligned}&\widetilde{\iota}_2 \colon \mathbb{R}\times \iota^{-1}(\Omega_0) \to \widetilde{\Lambda},\\ &\qquad (\tau, s, z) \mapsto \widetilde{\iota}_1(\tau, \pi_x\iota(s,z)) = ((\boldsymbol{x}_\infty(s+\tau, z;\lambda), X^*(\boldsymbol{\xi}_\infty(s+\tau,z;\lambda))), (\tau, X^*(-\lambda))).\end{aligned} \tag{3.2.13}$$

It follows from Lemma 3.2.3 that $\widetilde{\iota}_2$ is a diffeomorphism. Furthermore, we see directly from (3.2.11) and (3.2.12) that

$$(\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \widetilde{\boldsymbol{\xi}}(\tau,y;\lambda) = (\boldsymbol{x}_\infty(s+\tau,z;\lambda), \boldsymbol{\xi}_\infty(s+\tau,z;\lambda)) \tag{3.2.14}$$

for $(s,z) = \iota^{-1}(y, \nabla S_0(y)) = \iota^{-1}(\pi_x|_{\Omega_0})^{-1} y$.

Proof of Lemma 3.2.3. Throughout this proof we will consider $T^*\mathbb{R}^{n+1}$ to be identified with $\mathbb{R}^{2n+2}$ using $(x,t;\xi,E)$–coordinates, and similarly identify $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ using (x,ξ)–coordinates. We start by showing that the map $\widetilde{\iota}_1$ is given by (3.2.12). A crucial observation is the fact that the initial value problem (3.2.6) splits into two independent initial value problems for $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{\xi}})$ and $(\widetilde{\boldsymbol{E}}, \widetilde{\boldsymbol{t}})$. The latter can be solved immediately, yielding $\widetilde{\boldsymbol{E}}(\tau,y;\lambda) = -\lambda$ and $\widetilde{\boldsymbol{t}}(\tau,y;\lambda) = \tau$. Thus

$$\widetilde{\iota}_1(\tau,y) = (\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \widetilde{\boldsymbol{\xi}}(\tau,y;\lambda); \tau, -\lambda), \qquad\qquad y\in U_0. \tag{3.2.15}$$

On the other hand, $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{\xi}})$ are integral curves of the system

$$\frac{\partial\widetilde{\boldsymbol{x}}}{\partial\tau} = \widetilde{\boldsymbol{\xi}}, \qquad\qquad \frac{\partial\widetilde{\boldsymbol{\xi}}}{\partial\tau} = -\nabla_x V(\widetilde{\boldsymbol{x}}) \tag{3.2.16}$$

with initial conditions $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{\xi}})|_{\tau=0} = (y, \nabla S_0(y)) \in \Omega_0$, $y \in U_0$. But any point in Ω_0 is in fact a point on an integral curve of (1.1.6), cf. Theorem 1.3.3. Thus for any $y \in U_0$,

$$(y, \nabla_x S_0(y)) = (\boldsymbol{x}_\infty(s, z; \lambda), \boldsymbol{\xi}_\infty(s, z; \lambda)), \qquad (s, z) = \iota^{-1}(y, \nabla S_0(y)), \tag{3.2.17}$$

Now (1.1.6) is identical to (3.2.16), so $\pi_{(x,\xi)} \widetilde{g}_\tau|_{\widehat{\Omega}_0} = g_\tau|_{\Omega_0}$ and hence (3.2.14) holds. Together with (3.2.15) this implies (3.2.12) and $\widetilde{\Lambda} = \widetilde{\iota}_1(\mathbb{R} \times \iota^{-1}(\Omega_0))$.

We first consider $\tilde{\iota}_1$ as a map $\mathbb{R} \times \iota^{-1}(\Omega_0) \to T^*\mathbb{R}^{n+1}$ and show that it is an embedding, so $\widetilde{\Lambda}$ is a submanifold. As usual we will verify that $\tilde{\iota}_1$ is a smooth, proper injective immersion. By (3.2.12) any bounded subset in $\widetilde{\Lambda}$ is bounded in $t = \widetilde{t}(t, y; \lambda)$. Since $\widetilde{t}(t, y; \lambda) = \tau$ by (3.2.12) and U_0 is bounded, it follows that the pre-image of any compact subset in $\widetilde{\Lambda}$ is bounded. By (3.2.14) $\tilde{\iota}_1$ is smooth, so the pre-image will also be closed, hence compact. Thus $\tilde{\iota}_1$ is proper. Now $(y, \tau) \mapsto \tilde{\iota}_1(y, \tau)$ can be expressed as the map $y \mapsto (y, 0; \nabla S_0(y), -\lambda)$ composed with $\widetilde{g}_\tau$. The former map is clearly an injective immersion. Since $\widetilde{g}_\tau = (g_\tau, \tau, -\lambda)$, where g_τ is the (injective and immersive) flow of H_p, $\widetilde{g}_\tau$ and hence $\tilde{\iota}_1$ are also injective immersions. Thus $\tilde{\iota}_1 \colon \mathbb{R} \times \iota^{-1}(\Omega_0) \to \widetilde{\Lambda}$ is a diffeomorphism

Now Ω_0 is a subset of the lagrangian manifold $\Lambda \subset \mathbb{R}^{2n}$, so we immediately see that $\widehat{\Omega}_0$ defined by (3.2.8) is an n–dimensional isotropic manifold. Furthermore, $\widetilde{\Lambda}$ is just the union of integral curves of a hamiltonian vector field transverse to $\widehat{\Omega}_0$, and thus is lagrangian by standard arguments in the theory of ordinary differential equations (cf., e.g., [25]). ☐

Defining a Maslov operator $K_{\widetilde{\Lambda}}$ on $\widetilde{\Lambda}$ we can approximate the solution ψ of the Cauchy problem (see Remark 3.2.1) arbitrarily closely. Explicitly, we have the following result of Maslov:

3.2.5. Proposition [18, Theorem 12.4] *for any $T_0 > 0$ there exist functions $\phi_k \in C_0^\infty(\widetilde{\Lambda})$, $k \in \mathbb{N}$, such that for any N there exists a function $R_{N+1}(\,\cdot\,, \cdot\,; h) \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}_t)$ such that*

$$\psi(x, t; h) = K_{\widetilde{\Lambda}}\Big[\sum_{k=0}^{N} \phi_k h^k\Big] + R_{N+1}(x, t; h), \qquad 0 \le t \le T_0, \tag{3.2.18}$$

where

$$\max_{t \in [0, T_0]} \|R_{N+1}(\,\cdot\,, t; h)\|_{L^2} < C_{T_0, N} h^{N+1}. \tag{3.2.19}$$

3.2.6. Remark The functions R_{N+1}, $N \in \mathbb{N}$, are smooth due to the smoothness of the functions ϕ_k and the smoothness of ψ. It follows from the estimate (3.2.18) that the functions φ_k are unique and that

$$\psi(x, t; h) = K_{\widetilde{\Lambda}}\Big[\varphi_0 + \varphi_R\Big], \qquad 0 \le t \le T_0, \tag{3.2.20}$$

where $\varphi_R \in C_0^\infty(\widetilde{\Lambda})$ and $\|\varphi_R\|_\infty < C_R h$ for some $C_R > 0$.

For an open set $\widetilde{\Omega} \subset \widetilde{\Lambda}$ we define, similarly to (2.3.1), the map

$$\widetilde{\pi}_{\widetilde{\Omega}, I} \colon \widetilde{\Omega} \to \mathbb{R}^{|I|}_{x_I} \times \mathbb{R}_t \times \mathbb{R}^{|\overline{I}|}_{\xi_{\overline{I}}}, \qquad p \mapsto (x_I(p), t(p); \xi_{\overline{I}}(p)) \tag{3.2.21}$$

where $I \subset \mathcal{N} = \{1, \ldots, n\}$ and $\overline{I} = \mathcal{N} \setminus I$. If $\widetilde{\pi}_{\widetilde{\Omega}, I}$ is a diffeomorphism on its image, $(x_I, t, \xi_{\overline{I}})$ are lagrangian coordinates on $\widetilde{\Omega}$ and $(\widetilde{\Omega}, \widetilde{\pi}_{\widetilde{\Omega}, I})$ is a lagrangian chart in the sense of Definition 2.3.2.

Denote by

$$\pi_{(x,\xi)} \colon T^*\mathbb{R}^{n+1} \to T^*\mathbb{R}^n, \qquad ((x, t), X^*(\xi, E)) \mapsto (x, X^*(\xi)) \tag{3.2.22}$$

the canonical projection onto the standard phase space.

By (3.1.13), (3.2.7), (3.2.14) and (1.2.8),

$$\pi_{(x,\xi)}(\widetilde{\Lambda}) = \mathcal{T}_{Z_0} \tag{3.2.23}$$

and for any open set $\Omega \subset \mathcal{T}_{Z_0}$ the pre-image

$$\widetilde{\Omega} := (\pi_{(x,\xi)}|_{\widetilde{\Lambda}})^{-1}(\Omega) \tag{3.2.24}$$

is open by continuity. Later we shall see that in this way we can induce Maslov data (in the sense of Definition D.1) on $\widetilde{\Lambda}$ from Maslov data given on Λ.

3.2.7. LEMMA *Let $S \in C^\infty(\Lambda)$ denote a global generating function on Λ. Then*

$$\widetilde{S} = S \circ \pi_{(x,\xi)} - \sqrt{\lambda} t, \qquad \widetilde{S} \in C^\infty(\widetilde{\Lambda}), \tag{3.2.25}$$

where t is regarded as the usual coordinate function on $\widetilde{\Lambda}$, is a global generating function on $\widetilde{\Lambda}$.

PROOF. Let $p \in \widetilde{\Lambda}$. Noting $\xi(\pi_{(x,\xi)}(p)) = \xi(p)$, $dx|_{\pi_{(x,\xi)}(p)} = dx|_p$ and that S is a generating function, we see from (3.2.25)

$$\begin{aligned} d\widetilde{S}|_p &= dS|_{\pi_{(x,\xi)}(p)} \circ d\pi_{(x,\xi)}|_p - \sqrt{\lambda}\, dt|_p \\ &= \xi\, dx|_{\pi_{(x,\xi)}(p)} \circ d\pi_{(x,\xi)}|_p - \sqrt{\lambda}\, dt|_p \\ &= \xi\, dx|_p + E\, dt|_p. \end{aligned}$$

□

We first fix some basic data for $K_{\widetilde{\Lambda}}$.

3.2.8. DEFINITION *In this section $K_{\widetilde{\Lambda}}$ denotes a Maslov operator constructed on $\widetilde{\Lambda}$ using the following Maslov data (cf. Definitions D.1 and D.5):*

i) *Some lagrangian atlas $\{(\widetilde{\Omega}_m, \widetilde{\pi}_{\widetilde{\Omega}_m, I_m})\}_{m\geq 0}$, $I_m \subset \mathcal{N} := \{1,\dots,n\}$, where $\widetilde{\Omega}_0 = (\pi_{(x,\xi)}|_{\widetilde{\Lambda}})^{-1}\Omega_0$, Ω_0 defined in (3.1.13), and $I_0 = \mathcal{N}$.*
ii) *The global coordinate map $\widetilde{\iota}_1 \colon \mathbb{R}^n \to \Lambda$ of (3.2.11).*
iii) *The global generating function $\widetilde{S}$ given by (3.2.25).*
iv) *Some partition of unity $\{\widetilde{e}_m\}$ subordinate to the covering $\{\widetilde{\Omega}_m\}$ (i.e., $\widetilde{e}_m \in C_0^\infty(\widetilde{\Omega}_m)$, $\sum \widetilde{e}_m = 1$)*
v) *Some set of functions $\widetilde{g}_m \in C^\infty(\mathbb{R}^{n+1})$ such that $\widetilde{g}_m(x) = 0$ for $\operatorname{dist}(x, \pi_{(x,t)}\widetilde{\Omega}_m) > 1$ and $\widetilde{g}_m = 1$ on $\pi_{(x,t)}\widetilde{\Omega}_m$. (Here $\pi_{(x,t)} \colon T^*\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ denotes the canonical projection onto the base.)*

3.2.9. LEMMA *Define $\widetilde{u}_0 \in C^\infty(\widetilde{\Lambda})$ through*

$$\widetilde{u}_0 \circ \widetilde{\iota}_1(\tau, y) := u_0(y), \tag{3.2.26}$$

let $K_{\widetilde{\Lambda}}$ denote a Maslov operator as in Definition 3.2.8 and set

$$v(x,t;h) := K_{\widetilde{\Lambda}}[\widetilde{u}_0](x,t). \tag{3.2.27}$$

Then

$$v(\,\cdot\,, 0; h) = \psi(\,\cdot\,, 0; h), \qquad \text{on } U_0, \tag{3.2.28}$$

where ψ is defined in (3.2.4).

We can now apply a crucial result from Maslov theory [18] concerning the construction of approximate solutions to the Cauchy problem (3.2.1) using a Maslov operator defined on $\widetilde{\Lambda}$ and satisfying the initial condition (3.2.1b).

3.2.10. THEOREM [18, Theorem 10.1] *The function v defined in (3.2.27) is an approximate solution to the Cauchy problem (3.2.1), i.e., it satisfies the initial conditions (3.2.1b) and for any $N, T \geq 0$ there exist constants $C_{N,T} \geq 0$ such that*

$$Qv(t,x;h) = r(t,x;h)\cdot h^2, \qquad \text{for } 0 \leq t \leq T, \text{ where } |r(t,x;h)| \leq C_{N,T}\langle x\rangle^{-N}. \tag{3.2.29}$$

We quote the a direct consequence of the Duhamel principle, cf., e.g., [18, Proposition 10.6].

3.2.11. LEMMA *Let $f(t,\,\cdot\,;h) \in C^\infty(\mathbb{R}^n)$ be a smooth function for any $0 \leq t \leq T$ for some $T > 0$. Then the (unique) solution of the Cauchy problem*

$$Q\phi(t,x;h) = f(t,x;h), \qquad 0 \leq t \leq T, \qquad \phi(0,\,\cdot\,;h) \in C_0^\infty(\mathbb{R}^n) \tag{3.2.30}$$

satisfies the estimate

$$\|\phi(t,\,\cdot\,;h)\|_\infty \leq \|\phi(0,\,\cdot\,;h)\|_\infty + \int_0^t \|f(\tau,\,\cdot\,;h)\|_\infty\, d\tau \tag{3.2.31}$$

We hence obtain the basis for the approximate representation of ψ.

3.2.12. Corollary *Let ψ of (3.2.4) and v of (3.2.27) be the exact and an approximate solution, respectively, to the Cauchy problem (3.2.1). Then for any $N, T \geq 0$ there exist constants $C_T \geq 0$ such that*

$$\sup_{t\in[0,T]} \|\psi(t,\,\cdot\,;h) - v(t,\,\cdot\,;h)\|_\infty \leq C_T \cdot h^2. \tag{3.2.32}$$

3.2.13. Remark It follows from Proposition 3.2.5, (3.2.20) and Corollary 3.2.12 that $\varphi_0 = \widetilde{u}_0$. hence for any $T_0 > 0$ we have

$$\psi(x,t;h) = K_{\widetilde{\Lambda}}\Big[\widetilde{u}_0 + \varphi_R\Big], \qquad 0 \leq t \leq T_0, \tag{3.2.33}$$

where $\varphi_R \in C_0^\infty(\widetilde{\Lambda})$ and $\|\varphi_R\|_\infty < C_R h$ for some $C_R > 0$.

We will now explicitly construct a Maslov operator $K_{\widetilde{\Lambda}}$ with Maslov data as in Definition 3.2.8 from any given Maslov operator K_Λ with the following Maslov data:

3.2.14. Definition *We define K_Λ as a Maslov operator constructed on Λ using the following Maslov data (cf. Definitions D.1 and D.5):*

i) *Some lagrangian atlas $\{(\Omega_m, \pi_{\Omega_m,I_m})\}_{m\geq 0}$, $I_m \subset \mathcal{N}$, where Ω_0 is either given by (3.1.13) or some superset in Λ, $I_0 = \mathcal{N}$ and the maps π_{Ω_m,I_m} are those of Convention 2.4.4 ii) with $\Omega = \Omega_m$ and $I = I_m$.*
ii) *The global coordinate map $\iota\colon \mathbb{R}^n \to \Lambda$ of (1.3.2).*
iii) *The global generating function S given by (2.3.14).*
iv) *Some partition of unity $\{e_m\}$ subordinate to the covering $\{\Omega_m\}$ (i.e., $e_m \in C_0^\infty(\Omega_m)$, $\sum e_m = 1$)*
v) *Some set of functions $g_m \in C^\infty(\mathbb{R}^n)$ such that $g_m = 1$ on $\pi_x\Omega_m$ and $g_m(x) = 0$ for $\operatorname{dist}(x, \pi_x\Omega_m)$.*

3.2.15. Lemma *Let K_Λ be a Maslov operator constructed on Λ as in Definition 3.2.14, Setting*

$$\widetilde{e}_m := e_m \circ \pi_{(x,\xi)}|_{\widetilde{\Lambda}}, \qquad \widetilde{g}_m := g_m \circ \pi_{(x,\xi)}|_{\widetilde{\Lambda}} \tag{3.2.34}$$

and defining $\{\widetilde{\Omega}_m, \widetilde{\pi}_{\widetilde{\Omega}_m,I_m}\}$ by (3.2.24) and (3.2.21), the operator K_Λ induces a Maslov operator $K_{\widetilde{\Lambda}}$ on $\widetilde{\Lambda}$ as in Definition 3.2.8.

Proof. We first show that $\{\widetilde{\Omega}_m, \widetilde{\pi}_{\widetilde{\Omega}_m,I_m}\}$ is a lagrangian atlas on $\widetilde{\Lambda}$. By the continuity of $\pi_{(x,\xi)}$, $\{\widetilde{\Omega}_m\}$ is an open covering of $\widetilde{\Lambda}$. We shall see that $\widetilde{\pi}_{\widetilde{\Omega},I}$ is a diffeomorphism on its image. Eeach $p \in \widetilde{\Omega}$ has the representation $p = \widetilde{\iota}_1(\tau,y) = (\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \tau, \widetilde{\boldsymbol{\xi}}(\tau,y;\lambda), -\lambda)$ for some unique $(\tau, y) \in \mathbb{R}\times U_0$, so $\widetilde{\pi}_{\widetilde{\Omega}_m,I}$ acts via

$$\widetilde{\pi}_{\widetilde{\Omega}_m,I}\colon (\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \tau, \widetilde{\boldsymbol{\xi}}(\tau,y;\lambda), -\lambda) \mapsto (\widetilde{\boldsymbol{x}}_I(\tau,y;\lambda), \tau, \widetilde{\boldsymbol{\xi}}_{\overline{I}}(\tau,y;\lambda)). \tag{3.2.35}$$

By working through the usual criteria (injectivity, immersiveness, properness, smoothness), we easily see that this map is an embedding if and only if the map

$$\widetilde{\pi}_{\widetilde{\Omega}_m,I}|_{\pi_{(x,\xi)}\Lambda_\tau}\colon (\widetilde{\boldsymbol{x}}(\tau,y;\lambda), \widetilde{\boldsymbol{\xi}}(\tau,y;\lambda)) \mapsto (\widetilde{\boldsymbol{x}}_I(\tau,y;\lambda), \widetilde{\boldsymbol{\xi}}_{\overline{I}}(\tau,y;\lambda)), \qquad (y,\tau) \in \widetilde{\iota}_1^{-1}(\widetilde{\Omega}) \tag{3.2.36}$$

is an embedding for fixed τ. In other words, we need to check that $\widetilde{\pi}_{\widetilde{\Omega}_m,I} \circ \pi_{(x,\xi)} \circ \widetilde{\iota}_1(\tau,\,\cdot\,)$ is a diffeomorphism. But by (3.2.13),

$$\widetilde{\pi}_{\widetilde{\Omega}_m,I} \circ \pi_{(x,\xi)} \circ \widetilde{\iota}_1(\tau,\,\cdot\,) = \pi_{\Omega_m,I} \circ g_\tau \circ (\pi_x|_\Lambda)^{-1} \qquad \text{on } U_0 \subset \mathbb{R}^n, \tag{3.2.37}$$

where $\pi_{\Omega_m,I}$ is defined in (2.3.1) and $(\Omega_m, \pi_{\Omega_m,I})$ is a lagrangian chart. Since $(\pi_x|_\Lambda)^{-1}$ is a diffeomorphism onto Ω_0 and $g_\tau(\Omega_0) \subset \Omega_m$, we have shown that $\widetilde{\pi}_{\widetilde{\Omega},I}|_{\pi_{(x,\xi)}\Lambda_\tau}$ and hence $\widetilde{\pi}_{\widetilde{\Omega}_m,I}$ is an embedding. Thus $\{\widetilde{\Omega}_m, \widetilde{\pi}_{\widetilde{\Omega}_m,I_m}\}$ is a lagrangian atlas on $\widetilde{\Lambda}$. It is easily verified that the functions $\widetilde{e}_m$ and $\widetilde{g}_m$ have the required properties of Definition 3.2.8 iv) and v), respectively. □

3.2.16. Proposition *Let K_Λ be a Maslov operator on Λ as in Definition 3.2.14 and $K_{\widetilde{\Lambda}}$ an induced Maslov operator on $\widetilde{\Lambda}$ in the sense of Lemma 3.2.15. Then for any function $\widetilde{\phi} \in C^\infty(\widetilde{\Lambda})$ there exists a function $\phi \in C^\infty(\Lambda\times\mathbb{R})$ such that*

$$K_{\widetilde{\Lambda}}[\widetilde{\phi}](x,t) = e^{-\frac{i}{h}\lambda t} K_\Lambda[\phi(\,\cdot\,,t)](x) \qquad \textit{and} \qquad \operatorname{supp}\phi(\,\cdot\,,t) \subset g_t\Omega_0. \tag{3.2.38}$$

In particular, if $\widetilde{\varphi} \in C^\infty(\widetilde{\Lambda})$ *such that* $\partial_\tau \widetilde{\varphi} \circ \widetilde{\iota}_1(\tau, y) = 0$, *i.e.,* $\widetilde{\varphi} \circ \widetilde{\iota}_1(\tau, y)$ *does not depend on* τ, *and* $\varphi \in C^\infty(\Omega_0)$ *with* $\varphi \circ \iota(s, z) := \widetilde{\varphi} \circ \widetilde{\iota}_2(0, s, z)$, *then*

$$K_{\widetilde{\Lambda}}[\widetilde{\varphi}](x,t) = e^{-\frac{i}{h}\lambda t} K_\Lambda[(\mathsf{D}^{\frac{1}{2}}_{\Omega_0,\mathcal{N}}\varphi) \circ g_{-t}](x), \tag{3.2.39}$$

where $\mathsf{D}_{\Omega_0,\mathcal{N}}$ *is defined in* (D.103).

With (3.2.3), (3.2.4) and Remark 3.2.13, Proposition 3.2.16 yields immediately

3.2.17. Corollary *Let* K_Λ *be a Maslov operator on* Λ *as in Definition* 3.2.14. *Then*

$$\begin{aligned} & e^{-\frac{i}{h}tP}[g_{-b}(\,\cdot\,;h,\omega_-)e^{\frac{i}{h}\varphi_-(\,\cdot\,,\sqrt{\lambda}\omega_-)}] \\ &\qquad = e^{-\frac{i}{h}\lambda t} K_\Lambda[(\mathsf{D}^{\frac{1}{2}}_{\Omega_0,\mathcal{N}} g_{-b}(\,\cdot\,;h,\omega_-) \circ \pi_x|_{\Omega_0}) \circ g_{-t} + R_1(\,\cdot\,,t;h)](x) \end{aligned} \tag{3.2.40}$$

where for some $C_{T_0} > 0$ *we have*

$$\sup_{t\in[0,T]} \|R_1(\,\cdot\,,t;h)\|_\infty \le C_{T_0}\cdot h \qquad \textit{and} \qquad \operatorname{supp} R_1(\,\cdot\,,t;h) \subset g_t\Omega_0. \tag{3.2.41}$$

Before proving Proposition 3.2.16, we establish a useful result.

3.2.18. Lemma *Let* $\Omega \subset \Lambda$ *be an open set with lagrangian coordinates* $(x_I, \xi_{\overline{I}})$, $I \subset \mathcal{N}$. *Then using Convention* 2.4.4 *ii) and the definitions* (1.3.2), (3.2.11), (3.2.24) *and* (3.2.21), *we have for all* $(\tau, y) \in \widetilde{\iota}_1^{-1}(\widetilde{\Omega})$

$$\begin{aligned} \left|\det d(\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_1)\right| \circ \widetilde{\iota}_1^{-1} &= \left|\det d(\pi_{\Omega,I} \circ \iota)\right| \circ \iota^{-1} \circ \pi_{(x,\xi)} \\ &\qquad \times \left|\det d(\pi_x \circ \iota)\right|^{-1} \circ \iota^{-1} \circ g_{-t} \circ \pi_{(x,\xi)}, \qquad \textit{on } \widetilde{\Omega} \cap \widetilde{\Lambda}_t \end{aligned} \tag{3.2.42}$$

Proof. We will prove

$$\left|\det d(\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_1)\right|_{(\tau,y)} = \left|\det d(\pi_{\Omega,I} \circ \iota)\right|_{(s+\tau,z)} \cdot \left|\det d(\pi_x \circ \iota)\right|^{-1}_{(s,z)}, \qquad (s,z) = (\pi_x|_\Lambda \iota)^{-1}(y). \tag{3.2.43}$$

by calculating the modulus of the determinant of the Jacobian of $\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_2$. By (3.2.13) and the chain rule we have

$$\left|\det d(\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_2)\right|_{(\tau,s,z)} = \left|\det d(\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_1)\right|_{(\tau,\pi_x\iota(s,z))} \cdot \left|\det d\pi_x\iota(s,z)\right|_{(s,z)}. \tag{3.2.44}$$

On the other hand, direct calculation from (3.2.13) yields

$$\begin{aligned} \left|\det d(\widetilde{\pi}_{\widetilde{\Omega},I} \circ \widetilde{\iota}_2)|_{(\tau,s,z)}\right| &= \left|\det \begin{pmatrix} \frac{\partial \boldsymbol{x}_I(s+\tau,z;\lambda)}{\partial\tau} & \frac{\partial \boldsymbol{x}_I(s+\tau,z;\lambda)}{\partial s} & \frac{\partial \boldsymbol{x}_I(s+\tau,z;\lambda)}{\partial z} \\ 1 & 0 & 0 \\ \frac{\partial \boldsymbol{\xi}_{\overline{I}}(s+\tau,z;\lambda)}{\partial\tau} & \frac{\partial \boldsymbol{\xi}_{\overline{I}}(s+\tau,z;\lambda)}{\partial s} & \frac{\partial \boldsymbol{\xi}_{\overline{I}}(s+\tau,z;\lambda)}{\partial z} \end{pmatrix}\right| \\ &= \left|\det \begin{pmatrix} \frac{\partial \boldsymbol{x}_I(s+\tau,z;\lambda)}{\partial s} & \frac{\partial \boldsymbol{x}_I(s+\tau,z;\lambda)}{\partial z} \\ \frac{\partial \boldsymbol{\xi}_{\overline{I}}(s+\tau,z;\lambda)}{\partial s} & \frac{\partial \boldsymbol{\xi}_{\overline{I}}(s+\tau,z;\lambda)}{\partial z} \end{pmatrix}\right| \\ &= \left|\det d(\pi_{\Omega,I} \circ \iota)|_{(s+\tau,z)}\right| \end{aligned} \tag{3.2.45}$$

Now (3.2.44) and (3.2.45) together yield (3.2.43). □

Proof of Proposition 3.2.16. We will prove (3.2.39) only; the assertion (3.2.38) will follow immediately from these considerations. An induced Maslov operator on $\widetilde{\Lambda}$ in the sense of Lemma 3.2.15 is given by

$$K_{\widetilde{\Lambda}}[\widetilde{\varphi}] := \sum_m e^{i\frac{\pi}{2}\widetilde{\gamma}_m} K_{\widetilde{\Omega}_m,I_m}[\widetilde{e}_m\widetilde{\varphi}], \qquad \widetilde{\varphi} \in \widetilde{\Lambda}, \tag{3.2.46}$$

where $\widetilde{\gamma}_m$ is the index of the chain of charts joining $\{\widetilde{\Omega}_0, \widetilde{\pi}_{\widetilde{\Omega}_0,\mathcal{N}}\}$ to $\{\widetilde{\Omega}_m, \widetilde{\pi}_{\widetilde{\Omega}_m,I_m}\}$ and

$$K_{\widetilde{\Omega}_m,I_m}[\widetilde{e}_m\widetilde{\varphi}](x,t) = \widetilde{g}_m(x,t)\mathcal{F}_h^{-1}\big[e^{\frac{i}{h}\widetilde{S}_{\widetilde{\Omega}_m,I_m}\circ\widetilde{\pi}^{-1}_{\widetilde{\Omega}_m,I_m}(x_{I_m},t,\,\cdot\,)}(\widetilde{\mathsf{D}}^{-\frac{1}{2}}_{\widetilde{\Omega}_m,I_m}\cdot\widetilde{e}_m\widetilde{\varphi}) \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m,I_m}(x_{I_m},t,\,\cdot\,)\big]\big|_{x_{\overline{I}_m}}. \tag{3.2.47}$$

By a slight abuse of notation we use the index set $I_m \subset \mathcal{N}$ to denote the index set of the "space variables" (x_{I_m}, t) of extended phase space and write $\overline{I}_m = \mathcal{N} \setminus I_m$. We thus use $K_{\widetilde{\Omega}_m, I_m}$ to denote the Maslov operator on $\widetilde{\Omega}_m$ constructed with lagrangian coordinates $(x_{I_m}, t, \xi_{\overline{I}_m})$, define

$$\widetilde{S}_{\widetilde{\Omega}_m, I_m} = \widetilde{S} - \langle x_{\overline{I}_m}, \xi_{\overline{I}_m} \rangle \qquad \text{on } \widetilde{\Omega}_m \tag{3.2.48}$$

as in (2.3.3) with (3.2.25) and set

$$\widetilde{\mathsf{D}}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\iota}_1(\tau, y) := \left|\det d(\widetilde{\pi}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\iota}_1)\right|_{(\tau,y)} \qquad \text{for } (\tau, y) \in \widetilde{\iota}_1^{-1}(\widetilde{\Omega}_m). \tag{3.2.49}$$

Since the index set $\overline{I} = \mathcal{N} \setminus I$ of the "fibre variables" $\xi_{\overline{I}}$ is identical in Ω and $\widetilde{\Omega}$ it follows from (D.100) that $\gamma(\widetilde{\Omega}_m, \widetilde{\Omega}_k) = \gamma(\Omega_m, \Omega_k)$ and hence $\widetilde{\gamma}_m = \gamma_m$, the index of the chain of charts joining $\{\Omega_0, \pi_{\widetilde{\Omega}_0, \mathcal{N}}\}$ to $\{\Omega_m, \pi_{\widetilde{\Omega}_m, I_m}\}$. It is therefore sufficient to show that for $\widetilde{\varphi}$ given as in the proposition,

$$K_{\widetilde{\Omega}_m, I_m}[\widetilde{e}_m \widetilde{\varphi}](x, t) = e^{-\frac{i}{h}\lambda t} K_{\Omega_m, I_m}[(\mathsf{D}^{\frac{1}{2}}_{\Omega_0, \mathcal{N}} \varphi) \circ g_{-t}](x), \tag{3.2.50}$$

where K_{Ω_m, I_m} is the local Maslov operator on Ω_m constructed using the data of Definition 3.2.14. First, note that

$$\pi_{(x,\xi)} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(t, x_I, \xi_{\overline{I}}) = \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) \qquad \text{for } (t, x_I, \xi_{\overline{I}}) \in \widetilde{\pi}_{\widetilde{\Omega}_m, I_m} \widetilde{\Omega}_m. \tag{3.2.51}$$

By (3.2.48), (3.2.25) and (3.2.51) we obtain

$$\widetilde{S}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(t, x_I, \xi_{\overline{I}}) = S_{\Omega_m, I_m} \circ \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) - \sqrt{\lambda} t, \tag{3.2.52}$$

where S_{Ω_m, I_m} is the local generating function on Ω_m constructed from the global generating function S in (2.3.14). Similarly, (3.2.34) implies that $\widetilde{g}_m(x, t) = g_m(x)$ and, using (3.2.51), $\widetilde{e}_m \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(t, x_I, \xi_{\overline{I}}) = e_m \circ \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}})$. Hence (3.2.47) becomes

$$\begin{aligned} &K_{\widetilde{\Omega}_m, I_m}[\widetilde{e}_m \widetilde{\varphi}](x, t) \\ &= g_m(x) e^{-\frac{i}{h}\lambda t} \mathcal{F}_h^{-1}\left[e^{\frac{i}{h} S_{\Omega_m, I_m} \circ \pi^{-1}_{\Omega_m, I_m}(x_{I_m}, \,\cdot\,)} (\mathsf{D}^{-\frac{1}{2}}_{\widetilde{\Omega}_m, I_m} \cdot \widetilde{\varphi}) \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(x_{I_m}, t, \,\cdot\,) \cdot e_m \circ \pi^{-1}_{\Omega, I}(x_{I_m}, \,\cdot\,)\right]\Big|_{x_{\overline{I}}}. \end{aligned} \tag{3.2.53}$$

By (3.2.49),

$$\begin{aligned} \widetilde{\mathsf{D}}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m} &= \widetilde{\mathsf{D}}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\iota}_1 \circ \widetilde{\iota}_1^{-1} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m} \\ &= s\left|\det d(\widetilde{\pi}_{\widetilde{\Omega}, I} \circ \widetilde{\iota}_1)\right| \circ \widetilde{\iota}_1^{-1} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}. \end{aligned} \tag{3.2.54}$$

Applying (3.2.42) with (3.2.51) we obtain

$$\begin{aligned} \widetilde{\mathsf{D}}_{\widetilde{\Omega}_m, I_m} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(t, x_I, \xi_{\overline{I}}) &= \left|\det d(\pi_{\Omega, I} \circ \iota)\right| \circ \iota^{-1} \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) \\ &\quad \times \left|\det d(\pi_x \circ \iota)\right|^{-1} \circ \iota^{-1} \circ g_{-t} \cdot \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) \\ &= \mathsf{D}_{\Omega_m, I_m} \circ \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) \cdot \mathsf{D}_{\Omega_0, \mathcal{N}} \circ g_{-t} \circ \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}) \end{aligned} \tag{3.2.55}$$

From the fact that $\widetilde{\varphi} \circ \widetilde{\iota}_1(\tau, y)$ is independent of τ, we see that

$$\widetilde{\varphi} \circ \widetilde{\pi}^{-1}_{\widetilde{\Omega}_m, I_m}(x_I, t, \xi_{\overline{I}}) = \varphi \circ g_{-t} \circ \pi^{-1}_{\Omega_m, I_m}(x_I, \xi_{\overline{I}}), \tag{3.2.56}$$

where $\varphi \in C^\infty(\Omega_0)$ is defined by $\varphi \circ \iota(s, z) := \widetilde{\varphi} \circ \widetilde{\iota}_2(0, s, z)$. With (3.2.55) and (3.2.56) (3.2.53) becomes

$$\begin{aligned} K_{\widetilde{\Omega}_m, I_m}[\widetilde{e}_m \widetilde{\varphi}](x, t) &= g_m(x) e^{-\frac{i}{h}\lambda t} \mathcal{F}_h^{-1}\left[e^{\frac{i}{h} S_{\Omega_m, I_m} \circ \pi^{-1}_{\Omega_m, I_m}(x_{I_m}, \,\cdot\,)} (e_m \cdot \mathsf{D}^{-\frac{1}{2}}_{\Omega_m, I_m}) \circ \pi^{-1}_{\Omega_m, I_m}(x_{I_m}, \,\cdot\,)\right. \\ &\qquad \left. \times (\varphi \cdot \mathsf{D}^{\frac{1}{2}}_{\Omega_0, \mathcal{N}}) \circ g_{-t} \circ \pi^{-1}_{\Omega, I}(x_{I_m}, \,\cdot\,)\right]\Big|_{x_{\overline{I}}} \\ &= e^{-\frac{i}{h}\lambda t} K_{\Omega_m, I_m}[e_m \cdot (\varphi \mathsf{D}^{\frac{1}{2}}_{\Omega_0, \mathcal{N}}) \circ g_{-t}]. \end{aligned}$$

□

3.3. The scattering amplitude as a Maslov operator on the asymptotic manifold

In this section, we will finally give the leading term in the asymptotic expansion of the scattering amplitude as $h \to 0$, proving Theorem 1, which is the main objective of the present work.

From Lemma 3.1.7 and Corollary 3.2.17 we obtain

$$G_0(\omega_-, \omega_+; \lambda, h) = \frac{1}{ih} \int_0^{T_0} \left(K_\Lambda[\psi_0] \mid g_{+a} e^{\frac{i}{h}\varphi_+}\right)_{L^2} dt + O(h^\infty), \tag{3.3.1}$$

where we have write φ_+ for $\varphi_+(\,\cdot\,, \sqrt{\lambda}\omega^+)$, g_{+a} for $g_{+a}(\,\cdot\,; h, \omega_+)$ and written ψ_0 for

$$\psi_0(\,\cdot\,, t; h) := (\mathsf{D}^{\frac{1}{2}}_{\Omega_0,\mathcal{N}} g_{-b}(\,\cdot\,; h, \omega_-) \circ \pi_x|_{\Omega_0}) \circ g_{-t} + R_1(\,\cdot\,, t; h) \in C_0^\infty(g_t\Omega_0), \qquad 0 \le t \le T_0. \tag{3.3.2}$$

Recall from Proposition 2.4.6 that we can find open sets on $\mathbb{H}$ such that their images under S^+_λ and $\iota(I, \cdot\,)$ ($I \subset \mathbb{R}$ is open and bounded) are well-projected onto certain tuples of lagrangian coordinates. We hence construct a suitable covering of Z_ε.

3.3.1. Definition *We fix some $\delta \in (0, 1/4)$ and choose an open covering $\{Z_k\}_{k=1}^M$ of Z_0 (defined in (3.1.13)) such that*

i) $\overline{\bigcup Z_k} \subset \mathcal{Z}$.

ii) *there exist $i_k \in \mathcal{N}$ and index sets $J_k \subset \mathcal{N}_{i_k}$ such that $\Gamma_k := S^+_\lambda(Z_k) \subset T^*\Sigma_{i_k}(\delta)$ and $(\Gamma_k, \pi^{(i_k)}_{\Gamma_k, J_k})$ are lagrangian charts on $\mathcal{L}_+$ (see Convention 2.4.4 for the notation used) and*

iii) $\{(\Omega_k, \pi_{\Omega_k, \mathcal{N}\setminus J_k})\}_{k=1}^M$, $\Omega_k := \iota((S_1 + T_1, S_0 + T_0) \times Z_k)$, *(see Definitions 3.1.5, 3.1.6) are lagrangian charts on Λ_+ (defined in the Scattering Angle Hypothesis). We will write $I_k := \mathcal{N} \setminus J_k$ and $\overline{I} = J_k$ for short.*

Recall that we have not yet specified the lagrangian atlas on Λ used in the construction of K_Λ in (3.2.40) (with the exception of the chart $(\Omega_0, \pi_{\Omega_0,\mathcal{N}})$. We first construct an open covering of $\mathcal{T}_{Z_0}$ contained in $\mathcal{T}_{\mathcal{Z}}$ (note that $\mathcal{Z} \supset Z_0$, cf. (3.1.14)) and then add a covering of the complement.

3.3.2. Definition *We choose an open set Λ_{+a} such that $\overline{\Lambda_{+a}} \subset \bigcup_{k=1}^M \Omega_k$ and*

$$\pi_x(\iota((S_1 + T_1, S_0 + T_0), Z_0) \setminus \Lambda_{+a}) \cap \{x \colon R_a < |x| < R_a + 1\} = \emptyset.$$

Then there exist charts Ω_k, $k > M$ such that

i) $\Omega_k \cap \Lambda_{+a} = \emptyset$,

ii) *there exist $I_k \subset \mathcal{N}$ such that $(\Omega_k, \pi_{\Omega_k, I_k})$ are lagrangian charts on Λ,*

iii) *with Λ_{-b} defined in (3.1.14) and Ω_k for $1 \le k \le M$ defined in Definition 3.3.1 we have*

$$\mathcal{T}_{Z_0} \subset \mathcal{C} \subset \mathcal{T}_{\mathcal{Z}}, \qquad \mathcal{C} := \Lambda_{-b} \cup \bigcup_{k \ge 1} \Omega_k. \tag{3.3.3}$$

We next choose open sets $\{\Omega'_k\}_{k\ge1}$ such that $(\Omega'_k, \pi_{\Omega'_k, I'_k})$ are lagrangian charts for some $I'_k \subset \mathcal{N}$,

$$\Lambda \setminus \mathcal{T}_{\mathcal{Z}} \subset \mathcal{C}' \subset \Lambda \setminus \mathcal{T}_{Z_0} \qquad \text{for } \mathcal{C}' = \bigcup_{k\ge1} \Omega'_k, \tag{3.3.4}$$

and $\Lambda = \mathcal{C} \cup \mathcal{C}'$. We hence obtain a lagrangian atlas

$$\mathcal{A} := \{(\Lambda_{-b}, \pi_{\Lambda_{-b},\mathcal{N}})\} \cup \bigcup_{k\ge1} \{(\Omega_k, \pi_{\Omega_k, I_k})\} \cup \bigcup_{k\ge1} \{(\Omega'_k, \pi_{\Omega'_k, I'_k})\} \tag{3.3.5}$$

on Λ.

3.3.3. Definition *We fix some function $e_b \in C_0^\infty(\Lambda_{-b})$ so that $e_b = 1$ on Ω_0 and choose some $g_b \in C_0^\infty(\mathbb{R}^n)$ equal to unity on $\pi_x\Lambda_{-b}$ and supported within $\{x \in \mathbb{R}^n \colon |x| < R_a\}$.*

We define functions $\zeta_k \in C_0^\infty(Z_k)$, $k = 1, \dots, M$, and $\tau \in C_0^\infty((S_1 + T_1, S_1 + T_1))$ such that for

$$e_k \in C_0^\infty(\Omega_k) \qquad \text{defined via} \qquad e_k \circ \iota(s, z) := \tau(s) \cdot \zeta_k(z). \tag{3.3.6}$$

we have $\sum e_k = 1$ on Λ_{+a}. We denote by $g_k \in C^\infty(\mathbb{R}^n)$, $k = 1, \dots, M$ functions supported in a small neighbourhood of $\pi_x\Omega_k$, vanishing outside $\pi_x\mathcal{T}_{\mathcal{Z}}$ and equal to unity on $\pi_x\Omega_k$.

We define functions $e_k \in C^\infty(\Omega_k)$, $k > M$, such that $e_b + \sum_{k\ge1} e_k = 1$ on $\mathcal{T}_{Z_0}$. We further introduce functions $\{g_k\}_{k>M}$ equal to unity on $\pi_x\Omega_k$, $k > M$, and supported in a small neighbourhood of $\pi_x\Omega_k$, so *that* $\operatorname{supp} g_k \cap \pi_x\Lambda_{+a} = \emptyset$ *and* $g_k(x) = 0$ *if* $\operatorname{dist}(x, \Omega_k) \geq 1$. *In particular, for* $\Omega_k \cap \mathcal{T}^+_{Z_0,S_0} \neq \emptyset$ *and* $k > M$ *we require that* $\operatorname{supp} g_k \cap \{x\colon R_a < |x| < R_a + 1\} = \emptyset$.

We further cover $\mathcal{T}_{Z_0}$ *with charts* $(\Omega_k, \pi_{\Omega_k,I_k})$, $k > M$, *such that* $\Omega_k \subset \mathcal{T}_Z$ *and a subordinate partition of unity* $\{e_k\}$ *such that* $e_b + \sum_{k\geq 1} e_k = 1$ *on* $\mathcal{T}_{Z_0}$. *We define functions* $\{g_k\}_{k>M}$ *equal to unity on* $\pi_x\Omega_k$, $k > M$, *and supported in a small neighbourhood of* $\pi_x\Omega_k$, *so that* $\operatorname{supp} g_k \cap \pi_x\Lambda_{+a} = \emptyset$ *and* $g_k(x) = 0$ *if* $\operatorname{dist}(x, \Omega_k) \geq 1$.

Lastly, we define functions $e'_k \in C_0^\infty(\Omega'_k)$, $k \geq 1$, *so that* $e_b + \sum_{k\geq 1} e_k + e'_k = 1$. *(Hence* $\{e_b, e_k, e'_k\}_{k\geq 1}$ *is a partition of unity subordinate to* $\mathcal{A}$ *of Definition 3.3.2.) We define functions* g'_k *equal to unity on* $\pi_x\Omega'_k$ *and vanishing outside a small neighbourhood of* $\pi_x\Omega'_k$.

3.3.4. Definition *We define* K_Λ *to be the Maslov operator constructed on* Λ *using*

i) *The lagrangian atlas* $\mathcal{A}$ *given in Definition 3.3.2. (The Maslov index of any given chart is that of a chain of charts joining it to* Λ_{-b}*, defined in (3.1.14).)*
ii) *The global coordinate map* $\iota\colon \mathbb{R}^n \to \Lambda$ *of (1.3.2).*
iii) *The global generating function* S *given by (2.3.14).*
iv) *The partition of unity* $\{e_b, e_k, e'_k\}_{k\geq 1}$ *of Definition 3.3.3.*
v) *The set of functions* $\{g_b, g_k, g'_k\}_{k\geq 1}$ *of Definition 3.3.3.*

3.3.5. Lemma *Let* K_Λ *be the Maslov operator constructed in Definition 3.3.4. Then*

$$G_0(\omega_-, \omega_+; \lambda, h) = \frac{1}{ih} \sum_{k=1}^{M} e^{i\frac{\pi}{2}\gamma_k} \int_{T_1}^{T_0} G_k(t; \omega_+, h)\, dt + O(h^\infty) \tag{3.3.7}$$

with $G_k(t; \omega_+, h) = \big(K_{\Omega_k,I_k}[e_k\psi_0(\,\cdot\,, t; h)] \;\big|\; g_{+a} e^{\frac{i}{h}\varphi_+}\big)_{L^2}$.

Proof. Since $\psi_0(\,\cdot\,, t; h) \subset g_t\Omega_0$ and $\Omega_0 \subset \iota((S_0, S_1) \times Z_0)$, we have

$$\operatorname{supp} \psi_0(\,\cdot\,, t; h) \subset \iota((S_0 + t, S_1 + t) \times Z_0) \subset \mathcal{T}^-_{Z_0,S_1+t} \cap \mathcal{T}^+_{Z_0,S_0+t}. \tag{3.3.8}$$

Hence $\operatorname{supp} \psi_0(\,\cdot\,, t; h) \cap \operatorname{supp} e'_k = \emptyset$ for all $k \in \mathbb{N}$ and $t \in [0, T_0]$. It follows that

$$K_\Lambda[\psi_0(\,\cdot\,, t; h)] = K_{\Lambda_{-b},\mathcal{N}}[e_b\psi_0(\,\cdot\,, t; h)] + \sum_{\substack{k\geq 1 \\ \Omega_k \cap \mathcal{T}^+_{Z_0,S_0+t} \neq \emptyset}} e^{i\frac{\pi}{2}\gamma_k} K_{\Omega_k,I_k}[e_k\psi_0(\,\cdot\,, t; h)].$$

We now note that $\operatorname{supp} g_{+a} \subset \{x\colon R_a < |x| < R_a + 1\}$ and

$$\operatorname{supp} K_{\Lambda_{-b},\mathcal{N}}[e_b\psi_0(\,\cdot\,, t; h)] \subset \operatorname{supp} g_b \subset \{x \in \mathbb{R}^n\colon |x| < R_a\}. \tag{3.3.9}$$

Using (3.3.9) and the definition of g_k, $k > M$ and $\Omega_k \cap \mathcal{T}^+_{Z_0,S_0+t} \neq \emptyset$, we obtain

$$\int_0^{T_0} \big(K_\Lambda[\psi_0(\,\cdot\,, t; h)] \;\big|\; g_{+a} e^{\frac{i}{h}\varphi_+}\big)_{L^2}\, dt = \sum_{k=1}^{M} e^{i\frac{\pi}{2}\gamma_k} \int_0^{T_0} \big(K_{\Omega_k,I_k}[e_k\psi_0(\,\cdot\,, t; h)] \;\big|\; g_{+a} e^{\frac{i}{h}\varphi_+}\big)_{L^2}. \tag{3.3.10}$$

Furthermore, for $t < T_1$, $\operatorname{supp} e_k \cap \operatorname{supp} \psi_0(\,\cdot\,, t; h) = \emptyset$ by (3.1.19), and we obtain (3.3.7). □

Writing out the terms G_k using Definitions 3.3.4 and (D.5) for the local Maslov operator K_{Ω_k,I_k}, we have

$$G_k(t; \omega_+, h) = \int g_k(x) \big(\mathsf{D}^{-\frac{1}{2}}_{\Omega_k,I_k} e_k \cdot \psi_0\big) \circ \pi^{-1}_{\Omega_k,I_k}(x_{I_k}, \xi_{\bar{I}_k}) g_{+a}(x) \times e^{\frac{i}{h}(S_{\Omega_k,I_k} \circ \pi^{-1}_{\Omega_k,I_k}(x_{I_k},\xi_{\bar{I}_k}) + \langle x_{\bar{I}}, \xi_{\bar{I}}\rangle - \varphi_+(x, \sqrt{\lambda}\omega^+))}\, dx\, đ_h\xi_{\bar{I}_k}. \tag{3.3.11}$$

Here γ_k is the Maslov index of the chart Ω_k, defined as the index of a chain of charts joining Ω_k to Λ_{-b} (see Definitions 3.3.4 i) and D.3), S_{Ω_k,I_k} denotes the local generating function on Ω_k obtained from the generating function $S \in C^\infty(\Lambda)$ (see Definitions 3.3.4 iii) and 2.3.2 ii)), π_{Ω_k,I_k} is the notation of Convention 2.4.4 iv) and $\mathsf{D}_{\Omega_k,I_k} \in C^\infty(\Omega_k)$ is defined via

$$\mathsf{D}_{\Omega_k,I_k} \circ \iota = \big|\det d(\pi_{\Omega_k,I_k} \circ \iota)\big| \tag{3.3.12}$$

Due to the cut-off function $e_k \in C_0^\infty(\Omega_k)$, the $(x_{I_k}, \xi_{\overline{I}_k})$-support of the integrand of G_k lies in $\pi_{\Omega_k, I_k}\Omega_k$ and we can perform a coordinate transformation

$$\iota^{-1} \circ \pi_{\Omega_k, I_k}^{-1} : (x_{I_k}, \xi_{\overline{I}_k}) \mapsto (s, z). \tag{3.3.13}$$

Noting (3.3.12), we obtain

$$G_k(t; \omega_+, h) = (2\pi h)^{-|\overline{I}_k|/2} \int e^{\frac{i}{h}\Phi_k(x_{\overline{I}}, s, z; \omega_+)} \psi_k(s, z, x_{\overline{I}})\, ds\, dz\, dx_{\overline{I}_k} \tag{3.3.14}$$

with

$$\psi_k(s, z, x_{\overline{I}}) := \big(\mathsf{D}^{\frac{1}{2}}_{\Omega_k, I_k}\psi_0\big) \circ \iota(s, z)\tau(s)\zeta_k(z) g_{+a}(\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}}; h, \omega_+), \tag{3.3.15}$$

$$\begin{aligned}\Phi_k(x_{\overline{I}}, s, z; \omega_+) := {}& S_{\Omega_k, I_k} \circ \iota(s, z) - \varphi_+((\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}}), \sqrt{\lambda}\omega_+) \\ &+ \langle x_{\overline{I}}, \boldsymbol{\xi}_{\overline{I}}(s, z; \lambda)\rangle.\end{aligned} \tag{3.3.16}$$

3.3.6. Lemma *For $k = 1, \dots, M$ and some fixed $\delta \in (0, 1/4)$, $i_k \in \mathcal{N}$ as in Definition 3.3.1 and $\Sigma_{i_k}(\delta), \Sigma_{i_k} \subset S^{n-1}$ given in Definition 2.4.2, we define*

$$\sigma_k \in C_0^\infty(\Sigma_{i_k}) \qquad \text{with } 0 \le \sigma_k \le 1 \text{ and } \sigma_k = 1 \text{ on } \Sigma_{i_k}(\delta), \tag{3.3.17}$$

Let $G_k(t; \cdot\,, h) \in C^\infty(S^{n-1} \setminus \{\omega \colon |\omega - \omega_-| \le \varepsilon\})$ be defined by (3.3.14). Then

$$G_k(t; \cdot\,, h) = \sigma_k^2(\,\cdot\,) \cdot G_k(t; \cdot\,, h) + O(h^\infty). \tag{3.3.18}$$

Proof. It is sufficient to show that the phase Φ_k of (3.3.16) has no stationary points on $\operatorname{supp}\psi_k \cap \operatorname{supp}(1 - \sigma_k)$. We will calculate the first derivatives of Φ_k with respect to the variables of integration in (3.3.14). First,

$$\frac{\partial \Phi_k}{\partial x_{\overline{I}_k}} = -\nabla_{x_{\overline{I}_k}}\varphi_+((\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}_k}), \sqrt{\lambda}\omega_+) + \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda) \tag{3.3.19}$$

Note that since S_{Ω_k, I_k} is a local generating function on Ω_k, we have

$$dS_{\Omega_k, I_k} = \xi_I\, dx_I - x_{\overline{I}}\, d\xi_{\overline{I}} \tag{3.3.20}$$

(see (2.3.4)) and hence the chain rule gives

$$\begin{aligned}\frac{\partial \Phi_k}{\partial s} &= \frac{\partial}{\partial s} S_{\Omega_k, I_k} \circ \iota(s, z) - \langle \nabla_{x_{I_k}}\varphi_+((\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}_k}), \sqrt{\lambda}\omega_+), \partial_s \boldsymbol{x}_{I_k}(s, z; \lambda)\rangle + \langle x_{\overline{I}_k}, \partial_s \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda)\rangle \\ &= \langle \boldsymbol{\xi}_{I_k}(s, z; \lambda) - \nabla_{x_{I_k}}\varphi_+((\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}_k}), \sqrt{\lambda}\omega_+), \partial_s \boldsymbol{x}_{I_k}(s, z; \lambda)\rangle \\ &\quad + \langle x_{\overline{I}_k} - \boldsymbol{x}_{\overline{I}_k}(s, z; \lambda), \partial_s \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda)\rangle.\end{aligned} \tag{3.3.21}$$

In the same way, for $m = 1, \dots, n-1$ we obtain

$$\begin{aligned}\frac{\partial \Phi_k}{\partial z_m} &= \langle \boldsymbol{\xi}_{I_k}(s, z; \lambda) - \nabla_{x_{I_k}}\varphi_+((\boldsymbol{x}_{I_k}(s, z; \lambda), x_{\overline{I}_k}), \sqrt{\lambda}\omega_+), \partial_{z_m} \boldsymbol{x}_{I_k}(s, z; \lambda)\rangle \\ &\quad + \langle x_{\overline{I}_k} - \boldsymbol{x}_{\overline{I}_k}(s, z; \lambda), \partial_{z_m} \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda)\rangle.\end{aligned} \tag{3.3.22}$$

Since the map (3.3.13) is a diffeomorphism on $\pi_{\Omega_k, I_k}\Omega_k$, the set of vectors

$$\left\{ \begin{pmatrix} \partial_s \boldsymbol{x}_{I_k}(s, z; \lambda) \\ \partial_s \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda) \end{pmatrix}, \begin{pmatrix} \partial_{z_m} \boldsymbol{x}_{I_k}(s, z; \lambda) \\ \partial_{z_m} \boldsymbol{\xi}_{\overline{I}_k}(s, z; \lambda) \end{pmatrix} \right\}_{1 \le m \le n-1} \tag{3.3.23}$$

is a basis of $T_{\pi_{\Omega_k, I_k} \circ \iota(s,z)}\mathbb{R}^n$. Using this in (3.3.21) and (3.3.22) at any stationary point $(x_{\overline{I}_k}, s, z)$ of $\Phi_k(\,\cdot\,, \cdot\,, \cdot\,; \omega_+)$ we have with (3.3.19)

$$x_{\overline{I}_k} = \boldsymbol{x}_{\overline{I}_k}(s, z; \lambda), \qquad \boldsymbol{\xi}_\infty(s, z; \lambda) = \nabla\varphi_+(\boldsymbol{x}_\infty(s, z; \lambda), \sqrt{\lambda}\omega_+). \tag{3.3.24}$$

Now $(s, z, x_{\overline{I}}) \in \operatorname{supp}\psi_k$ implies $s \in \operatorname{supp}\tau$ and $z \in \operatorname{supp}\zeta_k$. Hence $s > s_+$ and we can apply Lemma 2.2.5 to deduce $\omega_+ = \omega_+(z; \lambda)$. But $z \in \operatorname{supp}\zeta_k$ means $\omega_+(z; \lambda) \in \Sigma_{i_k}(\delta)$ by Definition 3.3.1. Since $\sigma_k = 1$ on $\Sigma_{i_k}(\delta)$, we have shown that the phase Φ_k of (3.3.16) has no stationary points on $\operatorname{supp}\psi_k \cap \operatorname{supp}(1 - \sigma_k)$. ☐

3.3.7. Lemma *For $k = 1, \dots, M$ and $i_k \in \mathcal{N}$ we define*

$$u_k \in C_0^\infty(\mathbb{R}^{n-1}) \qquad \text{with } u_k = 1 \text{ on } \pi^{(i_k)}_{S^+_\lambda(Z_k), J_k}(S^+_\lambda(Z_k)). \tag{3.3.25}$$

Here we have used the notation of Convention 2.4.4 iii) and Definition 3.3.1. Let $\omega_+ = \theta = (\theta_1, \dots, \theta_n)$ and $\theta' = \theta_{\mathcal{N}_{i_k}}$ for short. Then

$$\begin{aligned} &\big(\sigma_k(\,\cdot\,)^2 \cdot G_k(t;\,\cdot\,,h)\big) \circ \chi_{i_k}^{-1}(\theta') \\ &= \sigma_k \circ \chi_{i_k}^{-1}(\theta') \mathcal{F}_h^{-1}\big\{u_k(\theta_{J_k},\,\cdot\,)\mathcal{F}_h[(\sigma_k \cdot G_k(t;\,\cdot\,,h)) \circ \chi_{i_k}^{-1}(\theta_{J_k},\,\cdot\,)]\big\}\big|_{\theta_{\mathcal{N}_{i_k}\setminus J_k}} + O(h^\infty), \end{aligned} \tag{3.3.26}$$

Proof. The first term on the right–hand side of (3.3.26) is explicitly given by

$$\int \psi_k(s, z, x_{\bar I}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}, m_{\mathcal{N}_{i_k}\setminus J_k}; \theta') e^{\frac{i}{h}\Phi_k(s,z,x_{\bar I_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}, m_{\mathcal{N}_{i_k}\setminus J_k};\theta')}\, ds\, dz\, đ_h x_{\bar I_k}\, đ_h \vartheta_{\mathcal{N}_{i_k}\setminus J_k} đ_h m_{\mathcal{N}_{i_k}\setminus J_k}$$

where by (3.3.14)

$$\begin{aligned} &\psi_k(s, z, x_{\bar I}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}, m_{\mathcal{N}_{i_k}\setminus J_k}; \theta') \\ &:= (\sigma_k \circ \chi_{i_k}^{-1})(\theta') \cdot u_k(\theta_{J_k}, m_{\mathcal{N}_{i_k}\setminus J_k}) \cdot (\sigma_k \circ \chi_{i_k}^{-1})(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}) \\ &\quad \times \big(\mathsf{D}^{\frac12}_{\Omega_k, I_k} f_{0b} \circ g_{-t}\big) \circ \iota(s,z) \tau(s) \zeta_k(z) g_{0a}(\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I}; h, \chi_{i_k}^{-1}(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})) \end{aligned} \tag{3.3.27}$$

and

$$\begin{aligned} \Phi_k(s, z, x_{\bar I_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}, m_{\mathcal{N}_{i_k}\setminus J_k}; \theta') &:= S_{\Omega_k, I_k} \circ \iota(s,z) - \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})) \\ &\quad + \langle x_{\bar I_k}, \boldsymbol{\xi}_{\bar I_k}(s,z;\lambda)\rangle + \langle m_{\mathcal{N}_{i_k}\setminus J_k}, \theta_{\mathcal{N}_{i_k}\setminus J_k} - \vartheta_{\mathcal{N}_{i_k}\setminus J_k}\rangle. \end{aligned} \tag{3.3.28}$$

As in the proof of Lemma 3.3.6 we will show that the phase $\Phi_k(\,\cdot\,;\theta') \in C^\infty(\mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}^{|\bar I_k|} \times \mathbb{R}^{2|\mathcal{N}_{i_k}\setminus J_k|}$ has no stationary points on the support of $\psi_k(\,\cdot\,;\theta') \cap \operatorname{supp}(1-u_k)$.

We now calculate the stationary points of $\Phi_k(\,\cdot\,;\theta')$, obtaining

$$0 = \frac{\partial \Phi_k}{\partial m_{\mathcal{N}_{i_k}\setminus J_k}} = \theta_{\mathcal{N}_{i_k}\setminus J_k} - \vartheta_{\mathcal{N}_{i_k}\setminus J_k} \tag{3.3.29}$$

at the stationary point. Furthermore, differentiating and then using (3.3.29), at the stationary point we have

$$0 = \frac{\partial \Phi_k}{\partial x_{\bar I_k}} = -\nabla_{x_{\bar I_k}} \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta')) + \boldsymbol{\xi}_{\bar I_k}(s,z;\lambda) \tag{3.3.30}$$

and

$$0 = \frac{\partial \Phi_k}{\partial \vartheta_{\mathcal{N}_{i_k}\setminus J_k}} = -\frac{\partial \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta'))}{\partial \theta_{\mathcal{N}_{i_k}\setminus J_k}} - m_{\mathcal{N}_{i_k}\setminus J_k}. \tag{3.3.31}$$

Analogously to (3.3.21) we obtain

$$\begin{aligned} 0 = \frac{\partial \Phi_k}{\partial s} &= \frac{\partial}{\partial s} S_{\Omega_k, I_k} \circ \iota(s,z) - \langle \nabla_{x_{I_k}} \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})), \partial_s \boldsymbol{x}_{I_k}(s,z;\lambda)\rangle \\ &\quad + \langle x_{\bar I_k}, \partial_s \boldsymbol{\xi}_{\bar I_k}(s,z;\lambda)\rangle \\ &= \langle \boldsymbol{\xi}_{I_k}(s,z;\lambda) - \nabla_{x_{I_k}} \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta')), \partial_s \boldsymbol{x}_{I_k}(s,z;\lambda)\rangle \\ &\quad + \langle x_{\bar I_k} - \boldsymbol{x}_{\bar I_k}(s,z;\lambda), \partial_s \boldsymbol{\xi}_{\bar I_k}(s,z;\lambda)\rangle. \end{aligned} \tag{3.3.32}$$

where we have inserted $\theta_{\mathcal{N}_{i_k}\setminus J_k} = \vartheta_{\mathcal{N}_{i_k}\setminus J_k}$ after differentiating. In the same way, for $m = 1, \dots, n-1$

$$\begin{aligned} 0 &= \langle \boldsymbol{\xi}_{I_k}(s,z;\lambda) - \nabla_{x_{I_k}} \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda), x_{\bar I_k}), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta')), \partial_{z_m} \boldsymbol{x}_{I_k}(s,z;\lambda)\rangle \\ &\quad + \langle x_{\bar I_k} - \boldsymbol{x}_{\bar I_k}(s,z;\lambda), \partial_{z_m} \boldsymbol{\xi}_{\bar I_k}(s,z;\lambda)\rangle. \end{aligned} \tag{3.3.33}$$

As in (3.3.24), the equations (3.3.30), (3.3.32) and (3.3.33) imply

$$x_{\bar I_k} = \boldsymbol{x}_{\bar I_k}(s,z;\lambda), \qquad \boldsymbol{\xi}(s,z;\lambda) = \nabla \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta')). \tag{3.3.34}$$

Now we have $s > s_+$, hence Lemma 2.2.5 yields

$$\chi_{i_k}(\omega_+(z;\lambda)) = \theta'. \tag{3.3.35}$$

Moreover, (3.3.31) and (2.4.5) imply

$$m_{\mathcal{N}_{i_k}\setminus J_k} = l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda). \tag{3.3.36}$$

It follows from (3.3.35) and (3.3.36) that at the stationary point we have

$$\pi^{(i_k)}_{S^+_\lambda(Z_k),J_k} \circ S^+_\lambda(z) = (\omega_{J_k}(z;\lambda), l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda)) = (\theta_{J_k}, m_{\mathcal{N}_{i_k}\setminus J_k}), \tag{3.3.37}$$

i.e., there can be no stationary point on the support of $1-u_k$. □

By (3.3.26) we have

$$G_k(t;\,\cdot\,,h)\circ\chi^{-1}_{i_k}(\theta') = \int \sigma_k\circ\chi^{-1}_{i_k}(\theta')\mathcal{F}_h^{-1}\{u_k(\theta_{J_k},\,\cdot\,)\mathcal{I}_k(t,s,\theta_{J_k},\,\cdot\,)\}\big|_{\theta_{\mathcal{N}_{i_k}\setminus J_k}}\,ds + O(h^\infty) \tag{3.3.38}$$

where (noting $|\overline{I}|_k = |J_k|$)

$$\begin{aligned}\mathcal{I}_k(t,s,\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) = (2\pi h)^{-(n-1)/2}\int u_k(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k})\cdot(\sigma_k\circ\chi^{-1}_{i_k})(\theta_{J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k})\psi_k(s,z,x_{\overline{I}})\\ \times e^{\frac{i}{h}\Phi_k(s,z,x_{\overline{I}_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k};m_{\mathcal{N}_{i_k}\setminus J_k},\theta_{J_k})}\,dz\,dx_{\overline{I}_k}\,d\vartheta_{\mathcal{N}_{i_k}\setminus J_k}\end{aligned} \tag{3.3.39}$$

with

$$\begin{aligned}\Phi_k(z,x_{\overline{I}_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k};s,\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) := S_{\Omega_k,I_k}\circ\iota(s,z) - \varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda),x_{\overline{I}_k}),\sqrt{\lambda}\chi^{-1}_{i_k}(\theta_{J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k}))\\ + \langle x_{\overline{I}_k},\boldsymbol{\xi}_{\overline{I}_k}(s,z;\lambda)\rangle - \langle m_{\mathcal{N}_{i_k}\setminus J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k}\rangle.\end{aligned} \tag{3.3.40}$$

The following results are preparatory to evaluating $\mathcal{I}_k$ using the method of stationary phase.

3.3.8. Lemma *Let $\Phi_k(z,x_{\overline{I}_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k};s,m_{\mathcal{N}_{i_k}\setminus J_k},\theta_{J_k})$ be given by (3.3.40), $(s,\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k})\in\operatorname{supp}\tau\times\operatorname{supp}u_k$. Then $\Phi_k(\,\cdot\,;s,m_{\mathcal{N}_{i_k}\setminus J_k},\theta_{J_k})$ has a unique critical point on $\operatorname{supp}\psi_k\cap\operatorname{supp}\sigma_k\circ\chi^{-1}_{i_k})(\theta_{J_k},\,\cdot\,)$ given by*

$$z = (S^+_\lambda)^{-1}\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}),\qquad x_{\overline{I}_k} = \boldsymbol{x}_{\overline{I}_k}(s,z;\lambda),\qquad \vartheta_{\mathcal{N}_{i_k}\setminus J_k} = \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda). \tag{3.3.41}$$

For short, we shall write $\Phi_k = \Phi_k(\,\cdot\,;s,m_{\mathcal{N}_{i_k}\setminus J_k},\theta_{J_k})$ and denote the values of Φ_k at such a stationary point by $\Phi_k|_{\text{stat.pt.}}$, also using this subscript to denote derivatives of Φ_k evaluated at the stationary point (3.3.41).

Let F_+ be the global generating function on $\mathcal{L}_+$ of Lemma 2.3.8, $(\Gamma_k,\pi^{(i_k)}_{\Gamma_k,J_k})$ given in Definition 3.3.1 and F_{Γ_k,J_k} the local generating function derived from F_+ as in (2.3.3). Then at the stationary point,

$$\Phi_k|_{\text{stat.pt.}} = F_{\Gamma_k,J_k}\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) \tag{3.3.42}$$

using the notation Convention 2.4.4 iii) for $\pi^{(i_k)}_{\Gamma_k,J_k}$.

Proof. The stationary points of Φ_k are given by

$$0 = \frac{\partial\Phi_k}{\partial x_{\overline{I}_k}} = -\nabla_{x_{\overline{I}_k}}\varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda),x_{\overline{I}_k}),\sqrt{\lambda}\chi^{-1}_{i_k}(\theta_{J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k})) + \boldsymbol{\xi}_{\overline{I}_k}(s,z;\lambda), \tag{3.3.43}$$

$$0 = \frac{\partial\Phi_k}{\partial\vartheta_{\mathcal{N}_{i_k}\setminus J_k}} = -\frac{\partial\varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda),x_{\overline{I}_k}),\sqrt{\lambda}\chi^{-1}_{i_k}(\theta_{J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k}))}{\partial\vartheta_{\mathcal{N}_{i_k}\setminus J_k}} - m_{\mathcal{N}_{i_k}\setminus J_k}, \tag{3.3.44}$$

and for $m=1,\dots,n-1$,

$$\begin{aligned}0 = \frac{\partial\Phi_k}{\partial z_m} = \langle\boldsymbol{\xi}_{I_k}(s,z;\lambda) - \nabla_{x_{I_k}}\varphi_+((\boldsymbol{x}_{I_k}(s,z;\lambda),x_{\overline{I}_k}),\sqrt{\lambda}\chi^{-1}_{i_k}(\theta_{J_k},\vartheta_{\mathcal{N}_{i_k}\setminus J_k})),\partial_{z_m}\boldsymbol{x}_{I_k}(s,z;\lambda)\rangle\\ + \langle x_{\overline{I}_k} - \boldsymbol{x}_{\overline{I}_k}(s,z;\lambda),\partial_{z_m}\boldsymbol{\xi}_{\overline{I}_k}(s,z;\lambda)\rangle.\end{aligned} \tag{3.3.45}$$

Note that for any $s\in\operatorname{supp}\tau$ the map $\iota_s|_{Z_k}\colon Z_k\to\Omega_k\cap\Lambda_s$ is a diffeomorphism, and the restriction of π_{Ω_k,I_k} to $\Omega_k\cap\Lambda_s$ remains a diffeomorphism, too. Hence the set of vectors

$$\left\{\begin{pmatrix}\partial_{z_m}\boldsymbol{x}_{I_k}(s,z;\lambda)\\ \partial_{z_m}\boldsymbol{\xi}_{\overline{I}_k}(s,z;\lambda)\end{pmatrix}\right\}_{1\le m\le n-1} \tag{3.3.46}$$

is a basis of $T_{\pi_{\Omega_k,I_k}\circ\iota(s,z)}\mathbb{R}^{n-1}$. Then by (3.3.45) and (3.3.43) at any stationary point $(z, x_{\bar{I}_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})$ of Φ_k we have

$$x_{\bar{I}_k} = \boldsymbol{x}_{\bar{I}_k}(s,z;\lambda), \qquad \boldsymbol{\xi}_\infty(s,z;\lambda) = \nabla\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\chi_{i_k}^{-1}(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})). \tag{3.3.47}$$

Now $(s,z,x_{\bar{I}}) \in \operatorname{supp}\psi_k$ implies $s \in \operatorname{supp}\tau$ and $z \in \operatorname{supp}\zeta_k$. Hence $s > s_+$ and we can apply Lemma 2.2.5 to deduce

$$\omega_+(z;\lambda) = \chi_{i_k}^{-1}(\theta_{J_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}), \qquad \text{hence} \qquad \vartheta_{\mathcal{N}_{i_k}\setminus J_k} = \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda). \tag{3.3.48}$$

Moreover, (3.3.44), (3.3.47) and (3.3.48) with (2.4.5) imply

$$m_{\mathcal{N}_{i_k}\setminus J_k} = l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda). \tag{3.3.49}$$

Now (3.3.45), (3.3.48) and (3.3.49) together imply (3.3.41).

Thus there exists a single stationary point determined by s and $(\theta_{J_k}, m_{\mathcal{N}_{i_k}\setminus J_k})$. Inserting (3.3.41) into (3.3.40) we obtain

$$\begin{aligned}\Phi_k|_{\text{stat.pt.}} = {}& S_{\Omega_k,I_k}\circ\iota(s,z) - \varphi_+((\boldsymbol{x}_\infty(s,z;\lambda)), \sqrt{\lambda}\omega_+(z;\lambda)) \\ &+ \langle \boldsymbol{x}_{\bar{I}_k}(s,z;\lambda), \boldsymbol{\xi}_{\bar{I}_k}(s,z;\lambda)\rangle - \langle l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda), \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda)\rangle.\end{aligned} \tag{3.3.50}$$

Now using the definition (2.3.3) of the local generating function S_{Ω_k,I_k}, we have

$$\Phi_k|_{\text{stat.pt.}} = S\circ\iota(s,z) - \varphi_+((\boldsymbol{x}_\infty(s,z;\lambda)), \sqrt{\lambda}\omega_+(z;\lambda)) - \langle l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda), \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda)\rangle. \tag{3.3.51}$$

Furthermore, by (2.3.17) and (2.3.19),

$$\Phi_k|_{\text{stat.pt.}} = F\circ\iota(s,z;\lambda) - \langle l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda), \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda)\rangle. \tag{3.3.52}$$

$$= F_+\circ S_\lambda^+(z;\lambda) - \langle l_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda), \omega_{\mathcal{N}_{i_k}\setminus J_k}(z;\lambda)\rangle. \tag{3.3.53}$$

where F_+ is the global generating function on $\mathcal{L}_+$. But again using (2.3.3), we see that

$$\Phi_k|_{\text{stat.pt.}} = F_{\Gamma_k,J_k}\circ S_\lambda^+(z;\lambda), \tag{3.3.54}$$

where F_{Γ_k,J_k} is the local generating function on $(\Gamma_k, \pi^{(i_k)}_{\Gamma_k,J_k})$, defined in Definition 3.3.1. Now at the stationary point, z is given by $z = (S_\lambda^+)^{-1}\circ(\pi^{i_k}_{\Gamma_k,J_k})^{-1}(\theta_{J_k}, m_{\mathcal{N}_{i_k}\setminus J_k})$, so we obtain (3.3.42) from (3.3.54). □

In order to analyse the integrand oif I_k in (3.3.39) at the point of stationary phase, we need some results of Robert and Tamura.

3.3.9. Lemma [24, Eq. (3.6), (3.7)] *We have*

$$g_{-b}(x;h,\omega_-) = ihg_{0b}(x) + g_b(x;h)\cdot h^2, \qquad g_{+a}(x;h,\omega_-) = ihg_{0a}(x) + g_a(x;h)\cdot h^2 \tag{3.3.55}$$

where $g_b, g_a \in C_0^\infty(\mathbb{R}^n_x)$ *with* $\sup_{x\in\mathbb{R}^n}|g_a(x;h)|, \sup_{x\in\mathbb{R}^n}|g_b(x;h)| \le C$ *uniformly for* $h\in[0,1]$. *and*

$$g_{0a}(x) = \chi_{0a}(x)a_{+0}(x,\sqrt{\lambda}\omega_+), \qquad \chi_{0a}(x) := \langle\nabla_x\varphi_+(x;\sqrt{\lambda}\omega_+), \nabla\chi_a(x)\rangle, \tag{3.3.56}$$

$$g_{0b}(x) = \chi_{0b}(x)b_{-0}(x,\sqrt{\lambda}\omega_-), \qquad \chi_{0b}(x) := \langle\nabla_x\varphi_-(x;\sqrt{\lambda}\omega_-), \nabla\chi_b(x)\rangle. \tag{3.3.57}$$

3.3.10. Lemma [24, Lemmas 4.2-4.4] *Let* $z\in Z_\varepsilon$. *Then on the support of* f_{0b} *we have*

$$\mathsf{D}^{\frac{1}{2}}_{\Omega_0,\mathcal{N}}\circ\iota(s,z)\cdot g_{0b}(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_-) = \frac{1}{\sqrt{2}}\lambda^{\frac{1}{4}}\frac{\partial}{\partial s}\chi_b(\boldsymbol{x}_\infty(s,z;\lambda)) \tag{3.3.58}$$

Furthermore, let

$$A(x,\xi) := \left(\frac{\partial^2\varphi_+(x,\xi)}{\partial x\partial\xi}\right). \tag{3.3.59}$$

Then for $z\in Z_\varepsilon$ *and* $s > S_0 + T_1 - 1$,

$$g_{0a}(\boldsymbol{x}_\infty(s,z;\lambda)) = \frac{\partial}{\partial s}\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))\sqrt{\det A(\boldsymbol{x}_\infty(s;z,\lambda), \sqrt{\lambda}\omega_+(z;\lambda))} \tag{3.3.60}$$

Proof. Following Robert and Tamura, we will give the proof of (3.3.58) in order to check that the differing factor of 2 (they define $p(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$) in the hamiltonian system does not influence the result. We set

$$f_{0b} \circ \iota(s,z) = \mathsf{D}^{\frac{1}{2}}_{\Omega_0,\mathcal{N}} \circ \iota(s,z) \cdot b_{-0}(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_-)\chi_{0b}(\boldsymbol{x}_\infty(s,z;\lambda)) \tag{3.3.61}$$

where $\chi_{0b}(x) = \langle \nabla_x\varphi_-(x;\sqrt{\lambda}\omega_-), \nabla\chi_b(x)\rangle$. It follows from (3.3.61), (3.3.57), (3.1.11), Theorem 3.1.4 and Definition 3.1.5 that

$$\operatorname{supp} f_{0b} \cap \mathcal{T}_{Z_\varepsilon} \subset (\pi_x|_{\Lambda_-})^{-1}\Sigma'_{-b}. \tag{3.3.62}$$

Hence by (2.2.18), (1.1.6) and the chain rule we have

$$\begin{aligned}\chi_{0b}(\boldsymbol{x}_\infty(s,z;\lambda)) &= \langle \boldsymbol{\xi}_\infty(s,z;\lambda), \nabla\chi_b(\boldsymbol{x}_\infty(s,z;\lambda))\rangle = \frac{1}{2}\langle \partial_s\boldsymbol{x}_\infty(s,z;\lambda), \nabla\chi_b(\boldsymbol{x}_\infty(s,z;\lambda))\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial s}\chi_b(\boldsymbol{x}_\infty(s,z;\lambda)).\end{aligned} \tag{3.3.63}$$

We now show that

$$b_{-0}(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_-)^{-2} = e^{2\int_{-\infty}^{s}(\Delta_x\varphi_-)(\boldsymbol{x}(\tau;z,\lambda),\sqrt{\lambda}\omega_-)\,d\tau} = \frac{1}{2\sqrt{\lambda}}\mathsf{D}_{\Omega_0,\mathcal{N}} \circ \iota(s,z) \tag{3.3.64}$$

By (3.3.62), we need to show (3.3.64) only for $\boldsymbol{x}_\infty(s,z;\lambda) \in \Sigma_-(R_b,\sigma_3,\sqrt{\lambda}\omega_-)$, where (by Proposition 3.1.2 and Theorem 3.1.4) b_{-0} solves

$$2\langle \nabla_x\varphi_-, \nabla_x b_{-0}\rangle + (\Delta_x\varphi_-)b_{-0} = 0 \qquad\qquad \text{with } b_{-0} \to 1 \text{ as } |x| \to \infty. \tag{3.3.65}$$

Noting that $|\nabla_x\varphi_\pm(x,\xi) - \xi| \le c\cdot|x|^{-1}$ for some $c > 0$ and sufficiently large $|x|$, we apply the method of characteristics to obtain a representation of b_{-0}. For $t < 0$ we have the characteristic curve $r_-(t;x,\xi)$

$$\frac{dr_-}{dt} = 2\nabla_x\varphi_-(r_-,\xi), \qquad r_-(0;x,\xi) = x$$

and $F_-(t;x,\xi)$ given by

$$\frac{\partial F_-(t;x,\xi)}{\partial t} = -\Delta\varphi_-(r_-(t;x,\xi),\xi) \qquad \text{hence} \qquad F_-(t;x,\xi) = -\int_{-\infty}^{t}\Delta\varphi_-(r_-(\tau;x,\xi))\,d\tau. \tag{3.3.66}$$

Then (3.3.65) becomes

$$\frac{\partial b_{-0}(r_-(t;x,\xi),\xi)}{\partial t} = b_{-0}\frac{\partial F_-}{\partial t} \qquad \text{implying} \qquad b_{-0}(r_-(t;x,\xi),\xi) = e^{F_-(t;x,\xi)}. \tag{3.3.67}$$

Thus $b_{-0}(x,\sqrt{\lambda}\omega_-) = b_{-0}(r_-(0;x,\xi)\xi) = e^{F_-(0;x,\xi)}$. Inserting $\xi = \sqrt{\lambda}\omega_-$ and $x = \boldsymbol{x}_\infty(s,z;\lambda)$ and taking the inverse, we obtain the first equality in (3.3.64).

Now let $y \in \Sigma'_{-b}$. Then by Definition 3.1.5, $y = \boldsymbol{x}_\infty(s,z;\lambda)$ for some $s < s_-$, $z \in \mathbb{H}$. Using the notation of Definition 1.1.2,

$$\frac{\partial}{\partial t}\boldsymbol{x}(t;y,\nabla_x\varphi_-) = 2\boldsymbol{\xi}(t;y,\nabla_x\varphi_-) = 2\nabla_x\varphi_-(\boldsymbol{x}(t;y,\nabla_x\varphi_-),\sqrt{\lambda}\omega_-), \tag{3.3.68}$$

where for short we have abbreviated $\nabla_x\varphi_-(\boldsymbol{x}(t;y,\nabla_x\varphi_-),\sqrt{\lambda}\omega_-)$ by $\nabla_x\varphi_-$. Then, by Liouville's Theorem, we have

$$\left|\det\frac{\partial\boldsymbol{x}(t;y,\nabla_x\varphi_-)}{\partial y}\right| = e^{2\int_0^t(\Delta_x\varphi_-)(\boldsymbol{x}(\tau;y,\nabla_x\varphi_-),\sqrt{\lambda}\omega_-)\,d\tau}$$

Now $\boldsymbol{x}(t;y,\nabla_x\varphi_-) = \boldsymbol{x}_\infty(t+s,z;\lambda)$, so the chain rule yields

$$\left|\det\frac{\partial\boldsymbol{x}(t;y,\nabla_x\varphi_-)}{\partial y}\right| = \left|\det\frac{\partial\boldsymbol{x}_\infty(t+s;z,\lambda)}{\partial(s,z)}\right| \cdot \left|\det\frac{\partial\boldsymbol{x}_\infty(s;z,\lambda)}{\partial(s,z)}\right|^{-1}$$

and thus

$$\left|\det\frac{\partial\boldsymbol{x}_\infty(s;z,\lambda)}{\partial(s,z)}\right| = \left|\det\frac{\partial\boldsymbol{x}_\infty(t+s;z,\lambda)}{\partial(s,z)}\right| e^{2\int_{t+s}^{s}(\Delta_x\varphi_-)(\boldsymbol{x}_\infty(\tau;z,\lambda),\sqrt{\lambda}\omega_-)\,d\tau}. \tag{3.3.69}$$

Now by (1.2.11),

$$\lim_{t\to-\infty} \det\left(\frac{\partial \boldsymbol{x}_\infty(t+s;z,\lambda)}{\partial(s,z)}\right) = 2\sqrt{\lambda} \tag{3.3.70}$$

Taking the limit $t \to -\infty$ in (3.3.69) we thus obtain the second half of (3.3.64). □

3.3.11. Lemma *Let ψ_k be given by (3.3.15). Let $A(x,\xi)$ be given by (3.3.59). Then for $(s, m_{\mathcal{N}_{i_k}\setminus J_k}, \theta_{J_k}) \in \operatorname{supp}\sigma_k \times \operatorname{supp} u_k$ and $(z, x_{\overline{I}_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})$ determined by (3.3.41) we have*

$$\begin{aligned}\psi_k(s,z,x_{\overline{I}}) = \frac{1}{\sqrt{2}}(ih)^2\lambda^{\frac14}\partial_s\chi_b(\boldsymbol{x}_\infty(s-t,z;\lambda))\partial_s\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))\zeta_k(z)\mathsf{D}^{\frac12}_{\Omega_k,I_k}\circ\iota(s,z)\\ \times\sqrt{\det A(\boldsymbol{x}_\infty(s;z,\lambda),\sqrt{\lambda}\omega_+(z;\lambda))} + h^2R_2(s,t,z,x_{\overline{I}};h),\end{aligned} \tag{3.3.71}$$

where $\|R_2(\,\cdot\,;h)\|_\infty < c\cdot h$ for some $c>0$ and $0<h<1$.

Proof. Since $z \in Z_\varepsilon$ at the stationary point (3.3.41), the result for the main term follows immediately from Lemma 3.3.10. Note that $\tau(s)$ does not appear in (3.3.71) since by (3.1.19), (3.1.20) and Definition 3.3.3 we have

$$\tau = 1 \qquad \text{on} \qquad \operatorname{supp}\tau\cap\{x\colon R_a \le |\boldsymbol{x}_\infty(\,\cdot\,,z;\lambda)| \le R_a+1\},\ z\in\operatorname{supp}\zeta_k. \tag{3.3.72}$$

and by (3.1.5) we have $\operatorname{supp}\nabla\chi_a \subset \{x\colon R_a \le |x| \le R_a+1\}$. The estimate of the error R_2 follows from the estimates in Corollary 3.2.17 and Lemma 3.3.9. □

For $(\Gamma_k, \pi^{(i_k)}_{\Gamma_k,J_k})$ defined in Definition 3.3.1 and using the notation of Convention 2.4.4 iii) for $\pi^{(i_k)}_{\Gamma_k,J_k}$ we define

$$\mathsf{E}_{\Gamma_k,J_k}\circ S^+_\lambda = \left|\det d(\pi^{(i_k)}_{\Gamma_k,J_k}\circ S^+_\lambda)\right|. \tag{3.3.73}$$

Denoting by $g^{(i_k)}$ the matrix representation of the metric tensor of S^{n-1} in the coordinates of the chart $(\Sigma_{i_k}, \chi_{i_k})$, we define

$$1\mathsf{g}_{\Sigma_{i_k}}\circ\omega_+(\,\cdot\,;\lambda) := |\det(g^{(i_k)}\circ\omega_+(\,\cdot\,;\lambda))|. \tag{3.3.74}$$

3.3.12. Lemma *Let $\Phi_k(z, x_{\overline{I}_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k}; s, m_{\mathcal{N}_{i_k}\setminus J_k}, \theta_{J_k})$ be defined by (3.3.40). Let $A(x,\xi)$ be given by (3.3.59), $\mathsf{D}_{\Omega_k,I_k}$ by (3.3.12), $\mathsf{E}_{\Gamma_k,J_k}$ by (3.3.73) and $\mathsf{g}_{\Sigma_{i_k}}$ by (3.3.74). Then for $(s, m_{\mathcal{N}_{i_k}\setminus J_k}, \theta_{J_k}) \in \operatorname{supp}\sigma_k\times\operatorname{supp}u_k$ we have*

$$\begin{aligned}&|\det\operatorname{Hess}\Phi_k(\,\cdot\,,\,\cdot\,,\,\cdot\,;s,m_{\mathcal{N}_{i_k}\setminus J_k},\theta_{J_k})|_{\text{stat.pt.}}\\ &= \frac12\lambda^{\frac{n-2}{2}}\mathsf{D}_{\Omega_k,I_k}\circ\iota(s,z)\cdot\mathsf{E}_{\Gamma_k,J_k}\circ S^+_\lambda(z)\cdot(\mathsf{g}_{\Sigma_{i_k}}\circ S^+_\lambda(z;\lambda))^{\frac12}\cdot|\det A(\boldsymbol{x}_\infty(s,z,\lambda),\sqrt{\lambda}\omega_+(z;\lambda)))|\end{aligned} \tag{3.3.75}$$

where $(z, x_{\overline{I}_k}, \vartheta_{\mathcal{N}_{i_k}\setminus J_k})$ are determined by (3.3.41).

The proof of Lemma 3.3.12 relies primarily on involved but elementary manipulations of block matrices. It can be found in Appendix C. We can now use the results of Lemmas 3.3.8, 3.3.11 and 3.3.12 to evaluate the integrals $\mathcal{I}_k$ of (3.3.39) using the method of stationary phase for parameter–dependent oscillatory integrals.

3.3.13. Lemma [13] *Let $\phi(\,\cdot\,,\,\cdot\,)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R})$, assume $d\phi(\,\cdot\,,0)|_0 = 0$ and $|\det\operatorname{Hess}\phi(\,\cdot\,,0)|_0 \neq 0$. Denote by $x(y)$ the solution of the equation $d\phi(\,\cdot\,,y)|_x = 0$ with $x(0)=0$ given by the implicit function theorem. Then for $u\in C_0^\infty(K)$ for some $K\ni(0,0)$*

$$\left|\int u(x,y)e^{\frac{i}{h}\phi(x,y)}\,dx - \frac{(2\pi/h)^{\frac n2}}{|\det\operatorname{Hess}\phi(\,\cdot\,,y)|_{x(y)}}e^{i\frac{\pi}{2}\operatorname{sgn}\operatorname{Hess}\phi(\,\cdot\,,0)|_0}e^{\frac{i}{h}\phi(x(y),y)}u(x(y),y)\right| \le c\cdot h^{\frac n2+1} \tag{3.3.76}$$

for some $c>0$.

We obtain

3.3.14. Lemma *There exists a constant $C > 0$ such that*

$$\Big|\mathcal{I}_k(t,s,\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) - (ih)^2(2\pi h)^{\frac{n-1}{2}}\lambda^{\frac{3-n}{4}}e^{i\frac{\pi}{2}\beta_k}\partial_s\chi_b(\boldsymbol{x}_\infty(s-t,z;\lambda))\partial_s\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))$$
$$\times\,\zeta_k(z)(\mathsf{g}_{\Sigma_{i_k}}^{-\frac14}\mathsf{E}_{\Gamma_k,J_k}^{-\frac12})\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k})e^{\frac{i}{h}F_{\Gamma_k,J_k}\circ\pi^{-1}_{\Gamma_k,J_k}(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k})}\Big| \le C\cdot h^{\frac{n-1}{2}+3} \tag{3.3.77}$$

with $z = (S^+_\lambda)^{-1}\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\theta_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k})$ *and* $\beta_k := \operatorname{sgn}\operatorname{Hess}\Phi_k|_{\text{stat.pt.}}$.

Note that by (3.3.7) and (3.3.38) we have

$$G_0(\omega_-,\omega_+;\lambda,h) = \frac{1}{ih}\sum_{k=1}^{M}e^{i\frac{\pi}{2}\gamma_k}\int_{T_1}^{T_0}\int_{S_1+T_1}^{S_0+T_0}\sigma_k\circ\chi_{i_k}^{-1}(\omega_{\mathcal{N}_{i_k}})$$
$$\times\,\mathcal{F}_h^{-1}\{u_k(\omega_{J_k},\,\cdot\,)\mathcal{I}_k(t,s,\omega_{J_k},\,\cdot\,)\}\big|_{\omega_{\mathcal{N}_{i_k}\setminus J_k}}\,dt\,ds + O(h^{\frac{n-1}{2}+2}) \tag{3.3.78}$$

where γ_k was introduced in (3.3.11) and we have written $\omega_+ = (\omega_1,\dots,\omega_n)$, $\omega_I = (\omega_i)_{i\in I}$ for $I\in\mathcal{N}$ and $\mathcal{N}_{i_k} = \mathcal{N}\setminus\{i_k\}$. Inserting (3.3.77) into (3.3.78) we obtain

$$G_0(\omega_-,\omega_+;\lambda,h)$$
$$= ih(2\pi h)^{\frac{n-1}{2}}\lambda^{\frac{3-n}{4}}\sum_{k=1}^{M}e^{i\frac{\pi}{2}(\gamma_k+\beta_k)}\sigma_k\circ\chi_{i_k}^{-1}(\omega_{\mathcal{N}_{i_k}})\mathcal{F}_h^{-1}\{(\mathsf{g}_{\Sigma_{i_k}}^{-\frac14}\mathsf{E}_{\Gamma_k,J_k}^{-\frac12})\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\omega_{J_k},\,\cdot\,)$$
$$\times\,\zeta_k\circ(S^+_\lambda)^{-1}\circ(\pi^{(i_k)}_{\Gamma_k,J_k})^{-1}(\omega_{J_k},\,\cdot\,)\mathcal{G}(\omega_{J_k},\,\cdot\,)e^{\frac{i}{h}F_{\Gamma_k,J_k}\circ\pi^{-1}_{\Gamma_k,J_k}(\omega_{J_k},\,\cdot\,)}\}\big|_{\omega_{\mathcal{N}_{i_k}\setminus J_k}} + O(h^{\frac{n-1}{2}+2}) \tag{3.3.79}$$

with

$$\mathcal{G}(\omega_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) = \int_{T_1}^{T_0}\int_{S_1+T_1}^{S_0+T_0}\partial_s\chi_b(\boldsymbol{x}_\infty(s-t,z;\lambda))\partial_s\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))\,ds\,dt. \tag{3.3.80}$$

We have omitted the function u_k since $u_k\circ\pi^{(i_k)}_{\Gamma_k,J_k}\circ S^+_\lambda = 1$ in $\operatorname{supp}\zeta_k$. Now since $\partial_s\chi_b(\boldsymbol{x}_\infty(s-t,z;\lambda)) = -\partial_t\chi_b(\boldsymbol{x}_\infty(s-t,z;\lambda))$, we have

$$\begin{aligned}\mathcal{G}(\omega_{J_k},m_{\mathcal{N}_{i_k}\setminus J_k}) &= -\int_{S_1+T_1}^{S_0+T_0}(\chi_b(\boldsymbol{x}_\infty(s-T_1,z;\lambda)) - \chi_b(\boldsymbol{x}_\infty(s-T_0,z;\lambda)))\partial_s\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))\,ds\\ &= \int_{S_1+T_1}^{S_0+T_0}(0-1)\partial_s\chi_a(\boldsymbol{x}_\infty(s,z;\lambda))\,ds\\ &= 1.\end{aligned} \tag{3.3.81}$$

Here we have used (3.1.5) and Definitions 3.1.5, 3.1.6. With (D.105) we hence obtain from (3.3.79) and (3.3.81) that

$$G_0(\omega_-,\omega_+;\lambda,h) = ih(2\pi h)^{\frac{n-1}{2}}\lambda^{\frac{3-n}{4}}\sum_{k=1}^{M}e^{i\frac{\pi}{2}(\gamma_k+\beta_k)}K_{\Gamma_k,J_k}[\zeta_k\circ(S^+_\lambda)^{-1}](\omega_+) + O(h^{\frac{n+1}{2}+1}) \tag{3.3.82}$$

where K_{Γ_k,J_k} denotes a local Maslov operator on $(\Gamma_k,\pi^{(i_k)}_{\Gamma_k,J_k})$ (see Definition 3.3.1) defined using the global generating function F_+ of (2.3.19), the global coordinate map $S^+_\Lambda\colon\mathbb{R}^{n-1}\to\mathcal{L}_+$ and the cut-off functions $\zeta_k\circ(S^+_\lambda)^{-1}\in C_0^\infty(\Gamma_k)$ and $\sigma_k\in C_0^\infty(\Sigma_{i_k})$.

Inserting (3.3.82) into (3.1.8)

$$f(\omega_-,\omega_+;\lambda,h) = \sum_{k=1}^{M}e^{-(n-1+\gamma_k+\beta_k)i\frac{\pi}{2}}K_{\Gamma_k,J_k}[\zeta_k\circ(S^+_\lambda)^{-1}](\omega_+) + O(h). \tag{3.3.83}$$

Recall that γ_k, introduced into the formula in (3.3.11), is the Maslov index of the chain of charts joining Ω_k to Λ_{-b} and β_k is the sign of the Hessian in (3.3.77).

3.3.15. Remark The formula (3.3.83) is the pinnacle of our direct calculations. Unfortunately, the Hessian of Lemma 3.3.12 is too complicated for the author to evaluate its sign explicitly. We will therefore pass to some structural arguments top complete the proof of Theorem 1.

The scattering amplitude f is naturally independent of the concrete choice of the covering $\{Z_k\}_{k\geq 1}$ of Z_ε in Definition 3.3.1 and the subordinate partition of unity $\{e_k\}$ in Definition 3.3.3. Both can be slightly modified (within the confines of Proposition 2.4.6) without changing f. It follows that

$$e^{-(n-1+\gamma_k+\beta_k)i\frac{\pi}{2}} K_{\Gamma_k,J_k} = e^{-(n-1+\gamma_{k'}+\beta_{k'})i\frac{\pi}{2}} K_{\Gamma_{k'},J_{k'}} + O(h) \qquad \text{on } \Gamma_k \cap \Gamma_{k'},\ k,k' \geq 1, \tag{3.3.84}$$

where for $j = k, k'$, γ_j is the Maslov index of $\iota((S_0 + T_1 -, S_1 + T_0 + 1) \times Z_k$ and β_j is the sign of the corresponding Hessian Φ_j.

Now fix any open set $\Gamma_0 \subset \mathcal{L}_+$ such that $\operatorname{rank} d\pi_\omega|_p = n-1$ for all $p \in \Gamma_0$ and for any lagrangian chart $(\Gamma_j, \pi^{(i)}_{\Gamma_j,J_j})$ on $\mathcal{L}_+$ denote by δ_j the Maslov index of a chain of charts joining Γ_0 and Γ_j. Then for any two sets Γ_k and $\Gamma_{k'}$ we have

$$e^{-\delta_k i\frac{\pi}{2}} K_{\Gamma_k,J_k} = e^{-\delta_{k'} i\frac{\pi}{2}} K_{\Gamma_{k'},J_{k'}} + O(h) \qquad \text{on } \Gamma_k \cap \Gamma_{k'},\ k,k' \geq 1, \tag{3.3.85}$$

by general Maslov theory [18, 20, 27]. Comparing (3.3.84) and (3.3.85) we obtain

$$n - 1 + \gamma_k + \beta_k = \delta_k + c_0 \qquad \text{for any choice of } \Gamma_0 \text{ (which induces } \delta_k). \tag{3.3.86}$$

We define a global Maslov operator on $\mathcal{L}_+$ in the following way (see Definition D.1):

3.3.16. DEFINITION *We use Notation 2.4.4 and define a global Maslov operator $K_{\mathcal{L}_+}$ on $\mathcal{L}_+$ using the following objects.*

i) *The atlas $\{(\Sigma_k, \chi_k)\}$ on S^{n-1} of Definition 2.4.2,*
ii) *A lagrangian atlas $(\Gamma_k, \pi^{(i_k)}_{\Gamma_k,J_k})_{k\geq 0}$ on $\mathcal{L}_+$ such that*
   – *for $k = 1, \ldots, M$ the charts coincide with those of Definition 3.3.1 ii),*
   – *for each k there exists some i_k such that each $\pi_\omega \Gamma_k \subset \Sigma_{i_k}(\delta)$, where δ was fixed in Lemma 3.3.6,*
   – *and Γ_0 is well-projected onto S^{n-1}.*
iii) *The global coordinate map $S^+_\lambda \colon \mathbb{R}^{n-1} \to \mathcal{L}_+$ of Theorem 2.1.4,*
iv) *The global generating function F_+ of (2.3.19),*
v) *A partition of unity $\{\zeta_k \circ (S^+_\lambda)^{-1}\}$, $\zeta_k \in C_0^\infty(\mathbb{R}^{n-1})$ subordinate to the covering $\{\Gamma_k\}$, where ζ_k coincides with the functions defined in Definition 3.3.3 for $k = 1, \ldots, M$.*
vi) *The functions $\sigma_k \in C_0^\infty(\Sigma_k)$ (note that $\sigma_{i_k} = 1$ on $\pi_\omega \Omega_k$).*

It follows from (3.3.83) and (3.3.86) that for the global Maslov operator of Definition 3.3.16

$$f(\omega_-, \omega_+; \lambda, h) = e^{ic_0\frac{\pi}{2}} \cdot K_{\mathcal{L}_+}[1](\omega_+) + O(h), \tag{3.3.87}$$

where c_0 is fixed by the choice of Γ_0 in Definition 3.3.16.

3.3.17. REMARK Clearly, the constant c_0 in (3.3.87) is not very explicit. It arises because any Maslov operator on $\mathcal{L}_+$ is determined only up to a factor $e^{i\frac{\pi}{2}\mathbb{Z}_4}$ by the choice of Γ_0. Through an analysis of the case where $(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega_+$ is well-projected, we can make this constant a little more explicit. It is impractical to use our construction in this case, as in Definition 3.3.1 lagrangian coordinates $\{\omega_1, \ldots, \omega_n\} \setminus \{\omega_{i_k}\}$ on $\Gamma_k = S^+_\lambda(Z_k)$ induce lagrangian coordinates $\{x_{i_k}, \xi_1, \ldots \xi_n\} \setminus \{\xi_{i_k}\}$ on Ω_k. Alternatively, however, it is more practical in this case to take $\{x_1, \ldots, x_n\}$ as lagrangian coordinates on Ω_k, which is also possible. This is the strategy employed in [24] and instead of repeating their construction, we will compare (3.3.87) to [24, Theorem 1] to investigate c_0.

3.3.18. LEMMA *Let $2 \leq n \leq 6$ and assume that $\mathcal{L}_+$, $\dim \mathcal{L}_+ = n-1$, is in general position. Denote by $\pi_\omega \colon T^*S^{n-1} \to S^{n-1}$ the canonical projection onto the base. Then for any $\omega \in S^{n-1}$ with $\omega \neq \omega_-$ there are only finitely many $z^{(k)} \in \mathbb{H}$ such that $(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega = S^+_\lambda(z^{(k)})$.*

PROOF. It follows from the estimate (1.2.16b) that for any $\varepsilon > 0$ the set

$$\overline{Z_\varepsilon} = \{z \in \mathbb{R}^{n-1} \colon |\omega_+(z;\lambda) - \omega_-| \geq \varepsilon\} \tag{3.3.88}$$

is compact. Choosing ε sufficiently small, we have $(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega \subset S^+_\lambda(\overline{Z_\varepsilon})$, which is compact by continuity of S^+_λ. By the General Position Hypothesis, $\mathcal{L}_+$ has only stable singularities and $\dim \mathcal{S}(\Lambda) \leq n-1$. Hence $(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega$ contains only isolated points. Since $S^+_\lambda(\overline{Z_\varepsilon})$ is compact, there can be only finitely many of such points. $\square$

We first show that the non–singular points on $\mathcal{L}_+$ coincide with regular scattering directions in the sense of [24]. We will assume that the manifold $\mathcal{L}_+$ is in general position, which allows us to use Lemma 3.3.18.

3.3.19. LEMMA *The scattering direction $\omega_+ \in S^{n-1}$, $\omega_+ \neq \omega_-$ is regular in the sense of [24] if and only if all of the (finite number of) points $(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega_+$ are well-projected onto S^{n-1}.*

PROOF. Assume that ω_+ is regular. Then by the implicit function theorem there are finitely many points $z^{(1)}, \dots, z^{(l)}$ such that $\omega_+(z^{(k)}; \lambda) = \omega_+$. Furthermore, for each $z^{(1)}$ we have

$$\widehat{\sigma}(z^{(k)}; \lambda) := \Big|\det\Big(\omega_+, \frac{\partial \omega_+}{\partial z_1}, \dots, \frac{\partial \omega_+}{\partial z_1}\Big)\Big|_{z^{(k)}} \neq 0 \tag{3.3.89}$$

Let (Σ, χ) be some chart on S^{n-1} such that $\omega_+ \in \Sigma$, $\chi\colon \Sigma \ni \omega \mapsto \theta_{\mathcal{N}'} \in \mathbb{R}^{n-1}$. Denoting by g the metric tensor on the sphere we have

$$\widehat{\sigma}(z^{(k)}; \lambda) = (\mathsf{g}_\Sigma \circ \omega_+(z^{(k)}; \lambda))^{\frac{1}{2}} \cdot \mathsf{E}_{\Gamma_k, \mathcal{N}'} \circ S^+_\lambda(z^{(k)}) \tag{3.3.90}$$

by (3.3.73), (3.3.74), (C.96). It follows that $\widehat{\sigma}(z^{(k)}; \lambda) \neq 0$ (i.e., ω_+ is regular) if and only if $\mathsf{E}_{\Gamma_k, \mathcal{N}'} \circ S^+_\lambda(z^{(k)}) \neq 0$ (i.e., $S^+_\lambda(z^{(k)})$ lies in a chart on $\mathcal{L}_+$ that is well-projected onto S^{n-1}).

Conversely, by Lemma 3.3.18 there are only finitely many points $z^{(1)}, \dots, z^{(l)}$ such that $S^+_\lambda(\pi_\omega|_{\mathcal{L}_+})^{-1}\omega_+$, hence $\omega_+(z^{(k)}; \lambda) = \omega_+$ only for $k = 1, \dots, l$. Furthermore, $\widehat{\sigma}(z^{(k)}; \lambda) \neq 0$ by the previous argument, so ω_+ is regular □

Assume that $\omega_+ \neq \omega_-$ is regular and lies in some chart (Σ, χ), where $\chi\colon \omega \mapsto \omega_{\mathcal{N}'}$ for some $n-1$-tuple $\mathcal{N}' \subset \mathcal{N}$. Assume that $\omega_+(z; \lambda) = \omega_+$ precisely for $z = z^{(0)}, \dots, z^{(l)}$ with some $l \in \mathbb{N}$. Let $z^{(k)} \in \Gamma_k \subset \mathcal{L}_+$, $k = 0, \dots, l$, where by Lemma 3.3.19 we can assume that each Γ_k is a lagrangian chart well-projected onto S^{n-1}. We construct a Maslov operator $K_{\mathcal{L}_+}$ on $\mathcal{L}_+$ using the charts Γ_k, $k = 0, \dots, l$ with lagrangian coordinates $\omega_{\mathcal{N}'}$ together with arbitrary additional charts, but otherwise as in Defintion 3.3.16 i), iii)-vi). We can construct $K_{\mathcal{L}_+}$ in such a way that by (3.3.87) we have

$$f(\omega_-, \omega_+; \lambda, h) = e^{-i\frac{\pi}{2}(\gamma_k + c_0)} \sum_{k=0}^{l} K_{\Gamma_k, \mathcal{N}'}[1] + O(h), \tag{3.3.91}$$

where γ_j is the Maslov index joining Γ_j to Γ_0. Since each Γ_k is well-projected onto S^{n-1}, we have with (D.105) for $\theta = \chi(\omega_+)$,

$$f(\omega_-, \omega_+; \lambda, h) = \sum_{k=1}^{l} e^{-i\frac{\pi}{2}(\gamma_k + c_0)} e^{\frac{i}{h} F_{\Gamma_k, \mathcal{N}'} \circ \pi_{\Gamma_k, \mathcal{N}'}(\theta)} (\mathsf{g}_\Sigma \circ \chi^{-1}(\theta); \lambda))^{-\frac{1}{4}} \cdot (\mathsf{E}_{\Gamma_k, \mathcal{N}'} \circ \pi_{\Gamma_k, \mathcal{N}'}(\theta))^{-\frac{1}{2}} + O(h),$$

Using (2.3.18) and (3.3.90), we obtain

$$f(\omega_-, \omega_+; \lambda, h) = \sum_{k=1}^{l} \widehat{\sigma}(z^{(k)}; \lambda)^{-\frac{1}{2}} e^{-i\frac{\pi}{2}(\gamma_k + c_0)} e^{\frac{i}{h}(2\int_{-\infty}^{\infty}(|\boldsymbol{\xi}_\infty(\tau, z^{(k)}; \lambda)|^2 - \lambda)\, d\tau - \langle r_+(z^{(k)}; \lambda), \sqrt{\lambda}\omega_+(z^{(k)}, \lambda)\rangle)} + O(h)$$

We can compare this to Robert and Tamura's formula [24, (0.8), (0.9)], and obtain

$$\mu_j = \gamma_j + c_0, \tag{3.3.92}$$

where μ_j is the Maslov index of the trajectory $\mathcal{T}_z$. We have not introduced the concept of the Maslov index of a trajectory and refer the reader to [18, §7]. Since $\gamma_0 = 0$, it follows that c_0 is equal to μ_0, the Maslov index of any trajectory $\mathcal{T}_z$ with $S^+_\lambda(z) \in \Gamma_0$. This completes the proof of Theorem 1.

3.4. Perspectives and remarks

We will make some remarks on Theorem 1 and further studies on this subject. A formula identical to (15) for the case where V is compactly supported can be obtained from Protas' article through a detailed analysis of the first term in [21, Theorem 2, 2)], see [12]. It is not surprising that the formula carries over to the case where $\mathcal{L}_+ \subset T^*S^{n-1}$ is induced by a short-range potential. The effect that caustics have on the asymptotics of the scattering amplitude is discussed above in Section 3.5.

It would be very interesting in practice to explore the relationship between caustics in Λ and caustics in $\mathcal{L}_+$ more deeply. The following questions arise naturally:

i) For some sufficiently large $R > 0$, denote $U_R := \{z \in \mathbb{H} : |z| > R\}$. Assuming that $\mathcal{L}_+$ is in general position, is then $\mathcal{L}_+ \cap S^+_\lambda(U_R)$ free of caustics?
ii) Can the uniformity with respect to T, T_0, T_1 in Propositions 2.4.6 and 2.4.7 be improved? In particular, does there exist a uniform time $T > 0$ independent of $z \in \mathbb{H}$ such that the rank of $d\pi_x|_p$ does not change for $p \in \mathcal{T}^+_{z,T}$?
iii) Under which conditions do caustics occur in Λ and $\mathcal{L}_+$ if $V \neq 0$? (Alexandrova [2, page 1510] claims that caustics always occur if $V \in C_0^\infty(\mathbb{R}^n_x)$, $V \neq 0$, but she is apparently not referring to stable singularities that remain when $\mathcal{L}_+$ is shifted into general position.)

Michel [19] imposes the weakened Energy Hypothesis

Assumption ($\mathbb{H}$) For all $z \in \mathbb{H}$, $\lim_{s\to\infty} |\boldsymbol{x}_\infty(s; z, \lambda)| = \infty$.

This assumption is sufficient to imply the assertions in Definition 1.2.3, hence all of the results of Chapters 1 and 2 remain valid, as they depend only on the estimates of the classical trajectories in Propositions 1.2.7 and 1.2.10. Under additional assumptions on the "analyticity at infinity" of the potential V and the behaviour of the resolvent $R(\zeta, P)$ Michel proceeds to obtain the results of Robert and Tamura [24] for regular scattering directions. In fact, is clear from the Proof of [19, Theorem 1.3] that Michel's results can be extended to caustics in the same way as Robert and Tamura's, yielding Theorem 1 under these weakened conditions.

3.5. On caustics at infinity

Using Theorem 1 the leading term of the asymptotics of the scattering amplitude can be calculated even in the presence of "caustics at infinity" (i.e., singularities of the projection $\pi_\omega : \mathcal{L}_+ \to S^{n-1}$). These occur whenever the map $z \mapsto \omega_\infty(z; \lambda)$ is not an immersion, i.e., the "Regular Condition" of [24] is not fulfilled. In general, the scattering amplitude at a caustic angle will diverge as $h \to 0$, and (15) allows us to determine the rate of divergence, which essentially depends on the geometry of $\mathcal{L}_+$.

For a lagrangian manifold in general position (see the General Position Hypothesis on page 83), the possible types of caustics have been classified according to their local generating functions in low dimensions ($n = 1, \dots, 5$ for a manifold $\Lambda \in T^*M$, $\dim M = n$), while in higher dimensions ($n \geq 6$) this type of classification is essentially impossible [6]. Details on this problem can be found in [4, 6, 7] with a summary in [8, Appendix 12]. The oscillating integrals occurring in the Maslov operator $K_{\mathcal{L}_+}$ then have a certain asymptotic blow-up as $h \to 0$, depending on the type of singularity. The exponent in h of this blow-up was calculated for the classified types of singularities by Arnol'd [5]. It is hence possible to give the leading term of the asymptotic behaviour as $h \to 0$ of the scattering amplitude for scattering in $\mathbb{R}^n$ if $n = 2, \dots, 6$ (note that $\mathcal{L}_+ \in T^*S^{n-1}$). This classification relies essentially on the General Position Hypothesis (see Appendix D).

Arnol'd's classification of caustics (lagrangian singularities) in some open subset Ω of a lagrangian manifold $\Lambda \subset T^*M$, $\dim M = n$, is according to the form of the local generating function $S_{\Omega,I}$, cf. Definition 2.3.2. It is assumed that Λ is in general position and that the singularity occurs at $(x,\xi)(p) = (0,0)$. For $n = 1$ and $n = 2$ the following stable singularities exist:

$n \geq 1$	A_1	$S_{\Omega,\mathcal{N}\setminus\{1\}} \circ \pi^{-1}_{\Omega,\mathcal{N}\setminus\{1\}}(x_{\mathcal{N}\setminus\{1\}}, \xi_1) = \xi_1^2$	(a non-singular region)
$n \geq 1$	A_2	$S_{\Omega,\mathcal{N}\setminus\{1\}} \circ \pi^{-1}_{\Omega,\mathcal{N}\setminus\{1\}}(x_{\mathcal{N}\setminus\{1\}}, \xi_1) = \pm\xi_1^3$	(a fold)
$n \geq 2$	A_3	$S_{\Omega,\mathcal{N}\setminus\{1\}} \circ \pi^{-1}_{\Omega,\mathcal{N}\setminus\{1\}}(x_{\mathcal{N}\setminus\{1\}}, \xi_1) = \pm\xi_1^4 + x_2\xi_1^2$	(a tuck with a cusp)

While these singularities may occur in any dimension higher than 1 or 2, respectively, more complicated singularities appear only for $n \geq 3$. We will now discuss the physical applications of the classification in more detail. In the case of a non-singular region (A_1) we have

$$x_1 = -\frac{\partial S_{\Omega,\mathcal{N}\setminus\{1\}}}{\partial \xi_1} = -2\xi_1 \qquad \text{on } \Omega \text{ in the case } (A_1), \tag{3.5.1}$$

so it is clear that the set Ω is well-projected onto M. At singularities of type A_2 and A_3 we have $\operatorname{rank} d\pi|_\Lambda = n - 1$, so it is possible to choose lagrangian coordinates of the form (x_I, ξ_j), $I = \mathcal{N} \setminus \{j\}$,

$j \in \mathcal{N}$, on Ω. (For simplicity, $j = 1$ in the table above.) Then there exists a function $\eta_j(x_I, \xi_j)$ such that

$$\Psi\colon \pi_{\Omega,I}\Omega \to \mathbb{R}^n, \qquad (x_I, \xi_j) \mapsto (x_I, \eta_j) \tag{3.5.2}$$

is an embedding and $\widetilde{S}_{\Omega,I}(x_I, \eta_j) := S_{\Omega,I} \circ \Psi^{-1}(x_I, \eta_j) = S_{\Omega,I}(x_I, \xi_j(x_I, \eta_j))$ is of the form A_2 or A_3 above, with $\mathcal{N} \setminus \{1\}$ replaced by I.

3.5.1. Scattering in $\mathbb{R}^2$ and $\mathbb{R}^3$. By Theorem 1 the leading term in the asymptotics of the scattering amplitude is determined through the geometry of $\mathcal{L}_+ \subset T^*S^{n-1}$. We will see that, more precisely, the leading term is given through the asymptotics of integrals of the type

$$I(x, h) = (2\pi h)^{-\frac{1}{2}} \int e^{\frac{i}{h} S_{\Omega,\mathcal{N}\setminus\{1\}} \circ \pi^{-1}_{\Omega,\mathcal{N}\setminus\{1\}}(x_{\mathcal{N}\setminus\{1\}}, \xi_1)} u_0(x_{\mathcal{N}\setminus\{1\}}, \xi_1)\, d\xi_1, \qquad u_0 \in C_0^\infty(\mathbb{R}). \tag{3.5.3}$$

The asymptotics of such integrals for stable singularities have been studied by Arnol'd in [5] and more extensively by Guillemin and Sternberg [11, §9]. For $n = 2, 3$ we will make this more explicit.

Assume that $\mathcal{L}_+ \subset T^*S^{n-1}$, $2 \le n \le 3$ and apply Definition 1. Fix some $\omega_+ \in S^{n-1}$ and assume that $(\pi_\omega|_{sL_+})^{-1}(\omega_+) = S^+_\lambda(z_0)$ for some $z_0 = z_0(\omega_+) \in \mathbb{H}$. Let $S^+_\lambda(z_0) \in \Gamma_k$ for some $k \in \mathbb{N}$ and assume that $(x_{\mathcal{N}\setminus\{i_k, j_k\}}, \xi_{j_k})$ are canonical coordinates on Γ_k, where i_k was defined in Definition 1. Then by Theorem 1,

$$f(\omega_-, \omega_+; \lambda, h) = e^{i(\mu_0 + \gamma_k)\frac{\pi}{2}} \cdot K_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}}[e_k(\,\cdot\,)](\omega_+) + O(h). \tag{3.5.4}$$

Now using (D.105) and the charts of Definition 2.4.2 we have for $\omega \in \pi_\omega \Gamma_k$

$$\begin{aligned} K_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}}[e_k(\,\cdot\,)] \circ \chi^{-1}_{i_k}(\omega_{\mathcal{N}\setminus\{i_k\}}) &= \int e^{\frac{i}{h} F_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} \circ (\pi^{(i_k)}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}})^{-1}(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, l_{j_k})} \\ &\times (\mathsf{g}^{\frac{1}{4}}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} \mathsf{D}^{-\frac{1}{2}}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} e_k) \circ (\pi^{(i_k)}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}})^{-1}(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, l_{j_k})\, dl_{j_k}. \end{aligned} \tag{3.5.5}$$

By the above considerations, we can find an embedding Ψ_k as described above so that

$$\Phi_k(\omega_{\mathcal{N}\setminus\{i_k, 1\}}, \eta_{j_k}) := F_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} \circ (\pi^{(i_k)}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}})^{-1} \circ \Psi_k^{-1}(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, \eta_{j_k}) \tag{3.5.6}$$

has the form A_1, A_2 or A_3. Performing a transformation of variables $l_{j_k} \mapsto \eta_{j_k}$ in the integral, we obtain

$$f(\omega_-, \,\cdot\,; \lambda, h) \circ \chi^{-1}_{i_k}(\omega_{\mathcal{N}\setminus\{i_k\}}) = e^{i(\mu_0 + \gamma_k)\frac{\pi}{2}} \cdot \int A(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, \eta_{j_k}) e^{\Phi_k(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, \eta_{j_k})}\, d\eta_{j_k} + O(h) \tag{3.5.7}$$

where the amplitude is

$$A(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, \eta_{j_k}) = (\mathsf{g}^{\frac{1}{4}}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} \mathsf{D}^{-\frac{1}{2}}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}} e_k) \circ (\pi^{(i_k)}_{\Gamma_k, \mathcal{N}\setminus\{i_k, j_k\}})^{-1} \circ \Psi_k^{-1}(\omega_{\mathcal{N}\setminus\{i_k, j_k\}}, \eta_{j_k}) \tag{3.5.8}$$

$$\times \frac{\partial l_{j_k}(\omega_{\mathcal{N}\setminus\{i_k, 1\}}, \eta_{j_k})}{\partial \eta_{j_k}} \tag{3.5.9}$$

and the phase function Φ_k is given by (3.5.6) and has one of the forms A_1, A_2 or A_3. The asymptotics of such integrals can be analysed as in [11, §9], allowing the determination of the principal term of the asymptotics of $f(\omega_-, \omega_+; \lambda, h)$ once $\mathcal{L}_+$ is known.

Summarising, for scattering in $\mathbb{R}^2$ we have

$$f(\omega_-, \omega_+; \lambda, h) \sim \begin{cases} \text{const} & \text{if } \omega_+ \text{ is regular,} \\ h^{-\frac{1}{6}} & \text{if } \omega_+ \text{ is a fold point,} \end{cases} \qquad h \to 0, \tag{3.5.10}$$

and in $\mathbb{R}^3$ we have

$$f(\omega_-, \omega_+; \lambda, h) \sim \begin{cases} \text{const} & \text{if } \omega_+ \text{ is regular,} \\ h^{-\frac{1}{6}} & \text{if } \omega_+ \text{ is a fold point,} \\ h^{-\frac{1}{4}} & \text{if } \omega_+ \text{ is a cusp point,} \end{cases} \qquad h \to 0. \tag{3.5.11}$$

No other cases can generically occur in these dimensions, and the factor in front of h^γ ($\gamma = 0, -\frac{1}{6}$ or $-\frac{1}{4}$) can be calculated explicitly when $\mathcal{L}_+$ is known.

3.5.2. An Example. For scattering in two-dimensional configuration space, caustics at infinity occur in $\mathcal{L}_+ \subset T^*S^1$. Hence only singularities of type A_1 and A_2 can occur. In order to further illustrate the way in which singularities enter into the asymptotics of the scattering amplitude, we will construct an explicit situation involving a fold (A_2) singularity, which gives rise to a scattering amplitude that behaves as an Airy function.

3.5.1. Definition *On S^1 we define a chart (Σ_1^+, χ_1) by*

$$\Sigma_1^+ := \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x| = 1,\ x_1 > 0\}, \qquad \chi_1(x_1, x_2) = \arctan(x_1/x_2). \tag{3.5.12}$$

We denote the induced chart on the cotangent bundle by $(T^*\Sigma_1^+, \widetilde{\chi}_1)$. We define open sets

$$Z \subset \mathbb{R}, \qquad \Gamma := S_\lambda^+(Z) \subset \mathcal{L}_+ \cap T^*\Sigma_1^+, \qquad \Sigma := \pi_\omega \Gamma \subset S^{n-1} \tag{3.5.13}$$

where $\mathcal{L}_+$ is the lagrangian manifold with coordinate map S_λ^+ of Theorem 2.1.4 and $\pi_\omega \colon T^*S^1 \to S^1$ denotes the canonical projection onto the base. We make the following assumptions:

i) $(\pi_\omega|_{\mathcal{L}_+})^{-1}\Sigma = \Gamma$,
ii) *For $z \in Z$,*

$$\theta_2(z) = \frac{\pi}{4} - z^2, \qquad l_2(z) = z, \qquad (\theta_2(z), l_2(z)) := \widetilde{\chi}_1 \circ S_\lambda^+(z). \tag{3.5.14}$$

iii) *We define $Z' = (\pi/4 - \varepsilon, \pi/4 + \varepsilon) \subset Z$ for some $\varepsilon > 0$, and set $\Gamma' := S_\lambda^+(Z') \subset \Gamma$, $\Sigma' := \pi_\omega \Gamma' \subset \Sigma$.*
iv) *On Γ we have the lagrangian chart $\pi_{\Gamma,\emptyset} \colon (\omega_2, l_2) \mapsto l_2$ (see Definition 2.3.2). Again for simplicity we assume that the local generating function $F_{\Gamma,\emptyset}$ is given by*

$$F_{\Gamma,\emptyset} \circ \pi_{\Gamma,\emptyset}^{-1}(l_2) = \frac{1}{3} l_2^3 - \frac{\pi}{4} l_2 \tag{3.5.15}$$

(in general, it may differ by an additive constant).

By Remark D.6, these choices fix $K_{\Gamma'}$ up to a factor $e^{\frac{i}{h}c_1 + i\frac{\pi}{2}c_2}$ and additive functions of order $O(h)$. In the situation of Definition 3.5.1, the fold-point is at

$$p_0 = (\omega_0, L_0) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, 0\right) \in \mathcal{L}_+ \cap T^*\Sigma_1^+, \tag{3.5.16}$$

We denote the Airy function by

$$\mathrm{Ai}[x] := \frac{1}{\pi}\int_0^\infty \cos(t^3/3 + xt)\, dt. \tag{3.5.17}$$

3.5.2. Proposition *In the situation of Definition 3.5.1, for some $\gamma \in \mathbb{Z}$,*

$$f(\omega_-, \omega_+; \lambda, h) = e^{i\frac{\pi}{2}\gamma}\sqrt{2\pi}\, h^{-\frac{1}{6}}\, \mathrm{Ai}[h^{\frac{1}{3}}(\theta - \pi/4)] + O(h^{\frac{1}{6}}), \qquad \omega_+ \in \Sigma',\ \theta = \chi_1(\omega_+). \tag{3.5.18}$$

Proof. By Theorem 1, the scattering amplitude is given by

$$f(\omega_-, \omega_+; \lambda, h) = e^{i\mu_0\frac{\pi}{2}} \cdot K_{\mathcal{L}_+}[1](\omega_+) + O(h) \tag{3.5.19}$$

for some $\mu_0 \in \{0, 1, 2, 3\}$. Since $(\pi_\omega|_{\mathcal{L}_+})^{-1}\Sigma = \Gamma$ we can construct $K_{\mathcal{L}_+}$ in such a way that only the local Maslov operator on Γ contributes to $K_{\mathcal{L}_+}[1](\omega_+)$ and we can take the Maslov index of Γ to equal zero. Then we have

$$f(\omega_-, \omega_+; \lambda, h) = e^{i\mu_0\frac{\pi}{2}} \cdot K_{\Gamma,\emptyset}[e(\,\cdot\,)](\omega_+) + O(h) \tag{3.5.20}$$

where $e(\,\cdot\,) \in C_0^\infty(\Gamma)$ and by (D.105),

$$K_{\Gamma,\emptyset}[e](\chi_1^{-1}(\theta)) = g(\chi_1^{-1}(\theta))\mathcal{F}_h^{-1}\left[e^{\frac{i}{h}F_{\Gamma,\emptyset}\circ\pi_{\Gamma,\emptyset}^{-1}(\,\cdot\,)}(e \cdot \mathsf{g}_{\Gamma;\emptyset}^{\frac{1}{4}}\mathsf{D}_{\Gamma,\emptyset}^{-\frac{1}{2}}) \circ \pi_{\Gamma,\emptyset}^{-1}(\,\cdot\,)\right]\Big|_\theta. \tag{3.5.21}$$

Here $g \in C_0^\infty(\Sigma_1^+)$, $g = 1$ on Σ, $\mathsf{g}_{\Gamma;\emptyset} \in C^\infty(\Gamma)$ is the determinant of the matrix representation of the metric tensor of the sphere, evaluated at $\pi_\omega p$ for $p \in \Gamma$, and

$$\mathsf{D}_{\Gamma,\emptyset} \circ S_\lambda^+(z) = \det\left(\frac{\partial l_2}{\partial z}\right) = 1, \qquad z \in Z. \tag{3.5.22}$$

Furthermore, we can choose $e \in C^\infty(\Gamma)$ such that $e = 1$ on Γ' and

$$\mathsf{g}_{\Gamma;\emptyset} \circ S^+_\lambda(z) = \left| \left\langle \frac{\partial \chi_1^{-1}(\theta)}{\partial \theta}, \frac{\partial \chi_1^{-1}(\theta)}{\partial \theta} \right\rangle \right|_{\theta=\theta(z)} = |\cos^2\theta(z) + \sin^2\theta(z)| = 1. \tag{3.5.23}$$

Then it follows that for $\theta \in \chi_1(\Sigma')$,

$$\begin{aligned} K_{\Gamma,\emptyset}[e(\,\cdot\,)] \circ \chi_1^{-1}(\theta) &= \int_{Z'} e^{\frac{i}{h}(\langle l,\theta\rangle + F_{\Gamma,\emptyset}\circ\pi^{-1}_{\Gamma,\emptyset}(l)}\, đ_h l + O(h^\infty) \\ &= (2\pi h)^{-1/2} \int_{-\varepsilon}^{\varepsilon} e^{\frac{i}{h}(\frac{1}{3}l^3 + l(\theta - \frac{\pi}{4})}\, dl + O(h^\infty) \end{aligned} \tag{3.5.24}$$

where we have inserted (3.5.15). Now

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} e^{\frac{i}{h}(\frac{1}{3}l^3 + l(\theta - \frac{\pi}{4})}\, dl &= 2\int_0^{\varepsilon} \cos\left(\frac{1}{3h}l^3 + \frac{l}{h}\left(\theta - \frac{\pi}{4}\right)\right) dl \\ &= 2h^{\frac{1}{3}} \int_0^{h^{-\frac{1}{3}}\varepsilon} \cos\left(\frac{1}{3}l^3 + lh^{\frac{1}{3}}\left(\theta - \frac{\pi}{4}\right)\right) dl \end{aligned} \tag{3.5.25}$$

where we have substituted in the integral. It is not hard to see that

$$\left| \int_{h^{-\frac{1}{3}}\varepsilon}^{\infty} \cos\left(\frac{1}{3}l^3 + lh^{\frac{1}{3}}\left(\theta - \frac{\pi}{4}\right)\right) dl \right| \le c \cdot h^{\frac{1}{3}} \tag{3.5.26}$$

for some constant $c > 0$. it follows that

$$K_{\Gamma,\emptyset}[e(\,\cdot\,)] \circ \chi_1^{-1}(\theta) = \sqrt{2\pi}h^{-\frac{1}{6}}\,\mathrm{Ai}[h^{\frac{1}{3}}(\theta - \pi/4)] + O(h^{\frac{1}{6}}). \tag{3.5.27}$$

The (3.5.27) together with (3.5.20) implies (3.5.18). □

APPENDIX A

Some multi–index calculus

For easy reference, this section summarises some well–known results that are referred to in the main sections.

We will use the standard notation for multi–indices $\alpha = (\alpha_i)_{i=1}^n \in \mathbb{N}^m$, in particular, we define

$$\left.\begin{aligned}
&\alpha! := \alpha_1! \cdots \alpha_m! \qquad |\alpha| := \alpha_1 + \cdots + \alpha_m \quad \alpha + \beta := (\alpha_i + \beta_i)_{i=1}^m, \qquad \alpha, \beta \in \mathbb{N}^m,\\
&\partial_t^\alpha := \partial_{t_1}^{\alpha_1} \dots \partial_{t_n}^{\alpha_m}, \qquad \partial_{t_i}^{\alpha_i} := \frac{\partial^{\alpha_i}}{\partial t_i^{\alpha_i}}, \qquad t = (t_1, \dots, t_m) \in \mathbb{R}^m,\\
&\partial_t^\alpha x := (\partial_t^\alpha x_1, \dots, \partial_t^\alpha x_n) \qquad x = (x_1, \dots, x_n) \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)
\end{aligned}\right\} \tag{A.28}$$

We state the analogue of the generalised product rule in multi–index notation. We define the product symbol "$\odot$" by

$$x \odot y := \begin{cases} x \cdot y & \text{if } x \in \mathbb{R} \text{ and } y \in \mathbb{R}^n \text{ or vice–versa,} \\ \langle x, y \rangle & \text{if } x, y \in \mathbb{R}^n. \end{cases} \tag{A.29}$$

For functions the corresponding product is assumed to be defined point-wisely.

A.1. Lemma (Leibnitz Rule) *Let $u \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, $v \in C^\infty(\mathbb{R}^m, \mathbb{R}^l)$ be smooth functions. Then for any multi–index $\alpha \in \mathbb{N}^m$,*

$$\partial_t^\alpha [u(t) \odot v(t)] = \sum_{\substack{(\beta,\gamma) \in \mathbb{N}^m \times \mathbb{N}^m \\ \beta + \gamma = \alpha}} \frac{\alpha!}{\beta! \gamma!} \partial_t^\beta u(t) \odot \partial_t^\gamma v(t). \tag{A.30}$$

A.2. Corollary *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^{n-1}$ be an open set and $\phi \in C^\infty(\Omega, \mathbb{R}^n)$, $\psi \in C^\infty(\Omega, \mathbb{R}^m)$ be smooth functions. Assume that there exist $k, l \in \mathbb{Z}$ such that for any $\alpha \in \mathbb{N}$ and multi–index $\beta \in \mathbb{N}^{n-1}$ there exist constants $C_{\phi;\Omega,\alpha,\beta}, C_{\psi;\Omega,\alpha,\beta} > 0$ such that*

$$|\partial_z^\beta \partial_s^\alpha \phi(s,z)| \le C_{\phi;\Omega,\alpha,\beta} \cdot s^{k-\alpha}, \qquad \text{and} \qquad |\partial_z^\beta \partial_s^\alpha \psi(s,z)| \le C_{\psi;\Omega,\alpha,\beta} \cdot s^{l-\alpha} \tag{A.31}$$

for $(s,z) \in \Omega$. Then there exists a constant $C_{\phi\odot\psi;\Omega,\alpha,\beta} > 0$ so that

$$|\partial_z^\beta \partial_s^\alpha (\phi(s,z) \odot \psi(s,z))| \le C_{\phi\odot\psi;\Omega,\alpha,\beta} \cdot s^{k+l-\alpha} \qquad \text{for } (s,z) \in \Omega.$$

The analogue of the Leibnitz rule for repeated application of the chain rule is Fáa di Bruno's formula, which we will cite for the special case of the composition of functions $\mathbb{R}^m \to \mathbb{R}^n$, $R^n \to \mathbb{R}$:

A.3. Lemma (Fáa di Bruno's Formula) *Let $\Omega \subset \mathbb{R}^n$, $x \in C^\infty(\mathbb{R}^m, \Omega$ and $V \in C^\infty(\Omega)$. Then for $\beta \in \mathbb{N}^m \setminus \{0\}$,*

$$\partial_t^\beta (V \circ x)(t) = \sum_{j=1}^{|\beta|} \sum_{\substack{\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_j) \\ \alpha_i \in \mathbb{N}^m \setminus \{0\}, \\ i=1,\dots,j \\ \alpha_1 + \cdots + \alpha_j = \beta}} c_{\beta,\boldsymbol{\alpha}} \prod_{k=1}^{j} \langle \partial_t^{\alpha_k} x, \nabla_y \rangle V(y)|_{y=x(t)}, \tag{A.32}$$

where the coefficients $c_{\beta,\boldsymbol{\alpha}}$ are given by

$$c_{\beta,\boldsymbol{\alpha}} = \frac{\beta!}{\alpha_1! \cdots \alpha_j!} \frac{1}{m_1! \cdots m_j!}$$

and $m_l \in \mathbb{N}$ denotes the number of multi–indices in $(\alpha_1, \dots, \alpha_j)$ that are equal to some $\alpha_l \in \{\alpha_1, \dots, \alpha_j\}$.

An immediate consequence is the following result, A.4. Lemma *Let $V \in C^\infty(\mathbb{R}^n)$ be a smooth function with decay at infinity such that for any multi–index $\alpha \in \mathbb{N}^n$ there exist constants C_α so that*

$$|\partial_x^\alpha V(x)| \le C_\alpha \langle x\rangle^{-\varrho-|\alpha|}, \qquad \text{for some } \varrho > 0. \tag{A.33}$$

Let $\Omega \subset \mathbb{R}\times\mathbb{R}^{n-1}$ be an open set and $\psi \in C^\infty(\Omega,\mathbb{R}^n)$ a smooth function. Assume that there exist weight functions $\rho_1,\rho_2 \in C^\infty(\Omega,\mathbb{R}_+)$ such that for some $k\in\mathbb{N}$ and some $\beta\in\mathbb{N}^{n-1}$ there exists a constant $c_{k,\beta}$ such that the estimate

$$|\partial_s^j\partial_z^\delta\psi(s,z)| \le c_{k,\beta}\rho_1(s,z)^{-j}\rho_2(s,z)^{-|\beta|}\langle\psi(s,z)\rangle \quad \text{for all } (j,\delta)\in\mathbb{N}\times\mathbb{N}^{n-1},\ j\le k,\ |\delta|\le|\beta| \tag{A.34}$$

holds. Then there exists a constant depending on α and $k+|\beta|$ such that for $(s,z)\in\Omega$,

$$\left|\partial_s^k\partial_z^\beta(\partial_x^\alpha V)(\psi(s,z))\right| \le C(\alpha,k+|\beta|)\cdot\rho_1(s,z)^{-k}\rho_2(s,z)^{-|\beta|}\langle\psi(s,z)\rangle^{-\varrho-|\alpha|}. \tag{A.35}$$

Proof. For $k+|\beta|=0$, the estimate (A.35) is simply a reformulation of (A.33). For $k+|\beta|\neq 0$, we apply the chain rule (A.32),

$$\begin{aligned}\left|\partial_s^k\partial_z^\beta(\partial_x^\alpha V)(\psi(s,z))\right| &= \Bigg|\sum_{j=1}^{k+|\beta|}\sum_{\substack{l=(l_1,\dots,l_j),\\ \boldsymbol{\delta}=(\delta_1,\dots,\delta_j),\\ l_i\in\mathbb{N}\setminus\{0\},\ \delta_i\in\mathbb{N}^n\setminus\{0\},\\ \sum l_i=k,\ \sum\delta_i=\beta}} c_{k+|\beta|,(l,\boldsymbol{\delta})}\Big(\prod_{i=1}^j\langle\partial_s^{l_i}\partial_z^{\delta_i}\psi(s,z),\nabla_y\rangle\Big)(\partial_x^\alpha V)(y)|_{y=\psi(s,z)}\Bigg| \\ &\le \sum_{j=1}^{k+|\beta|}\sum_{l,\boldsymbol{\delta}} c_{k+|\beta|,(l,\boldsymbol{\delta})}\max_{|\gamma|=j}|\partial_y^\gamma(\partial_x^\alpha V)(y)|_{y=\psi(s,z)}|\prod_{i=1}^j|\partial_s^{l_i}\partial_z^{\delta_i}\psi(s,z))|. \end{aligned}\tag{A.36}$$

Now (A.33) implies

$$\max_{|\alpha|\le m}|(\nabla\partial_x^\alpha V)(x)| \le C_{m+1}\langle x\rangle^{-\varrho-1-m}, \qquad C_m := \sqrt{n}\max_{|\alpha|\le m} C_\alpha. \tag{A.37}$$

Then applying (A.37) and inserting the estimates (A.34) into (A.36), we obtain

$$\begin{aligned}&\left|\partial_s^k\partial_z^\beta(\nabla V)(\psi(s,z))\right| \qquad (A.38)\\ &\le \sum_{j=1}^{k+|\beta|}\sum_{l,\boldsymbol{\delta}} c_{k+|\beta|,(l,\boldsymbol{\delta})}C_{j+|\alpha|}\langle\psi(s,z)\rangle^{-\varrho-j-|\alpha|}\prod_{i=1}^j c_{j,\beta}\rho_1(s,z)^{-l_i}\rho_2(s,z)^{-|\delta_i|}\langle\psi(s,z)\rangle \\ &\le \sum_{j=1}^{k+|\beta|}\sum_{l,\boldsymbol{\delta}} c_{k+|\beta|,(l,\boldsymbol{\delta})}C_{j+|\alpha|}\langle\psi(s,z)\rangle^{-\varrho-|\alpha|}c_{j,\beta}^j\rho_1(s,z)^{-\sum l_i}\rho_2(s,z)^{-\sum|\delta_i|} \\ &= \langle\psi(s,z)\rangle^{-\varrho-1}\rho_1(s,z)^{-k}\rho_2(s,z)^{-|\beta|}\sum_{j=1}^{k+|\beta|}C_{j+|\alpha|}c_{j,\beta}^j\sum_{l,\boldsymbol{\delta}}c_{k+|\beta|,(l,\boldsymbol{\delta})} \qquad (A.39)\end{aligned}$$

With

$$C(\alpha,k+|\beta|) := \sum_{j=1}^{k+|\beta|}C_{j+|\alpha|}c_{j,\beta}^j\sum_{\substack{l=(l_1,\dots,l_j),\\ \boldsymbol{\delta}=(\delta_1,\dots,\delta_j),\\ l_i\in\mathbb{N}\setminus\{0\},\ \delta_i\in\mathbb{N}^n\setminus\{0\},\\ \sum l_i=k,\ \sum\delta_i=\beta}} c_{k+|\beta|,(l,\boldsymbol{\delta})} \tag{A.40}$$

(A.39) becomes the assertion (A.35). □

We will need some basic estimate on the integral of $\langle\cdot\rangle^{-\alpha}$ for $\alpha\in\mathbb{N}$.

A.5. Lemma *For $\alpha > 1$ we have*

$$\int_{-\infty}^0\langle\tau\rangle^{-\alpha}\,d\tau \le \frac{\alpha}{\alpha-1} \qquad \text{and} \qquad \int_{-\infty}^t\langle\tau\rangle^{-\alpha}\,d\tau \le \frac{\sqrt{2}}{\alpha-1}\langle t\rangle^{1-\alpha} \qquad \text{for } t\le -1. \tag{A.41}$$

Furthermore, if $\alpha > 2$ we have

$$\int_{-\infty}^0\int_{-\infty}^t\langle\tau\rangle^{-\alpha}\,d\tau dt \le \frac{1}{2}\frac{\alpha}{\alpha-2} \tag{A.42}$$

and

$$\int_{-\infty}^{s}\int_{-\infty}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt \le \begin{cases}\frac{2}{(\alpha-2)(\alpha-1)}\langle s\rangle^{2-\alpha} & \text{if } s\le -1,\\ c_\alpha(1+s) & \text{if } s>0.\end{cases} \tag{A.43}$$

with some constant $c_\alpha > 0$.

Proof. Using the following basic estimates for $\alpha > 1$,

$$\int_{-\infty}^{-1}\langle\tau\rangle^{-\alpha}\,d\tau = \int_{1}^{\infty}\langle\tau\rangle^{-\alpha}\,d\tau \le \int_{1}^{\infty}\tau^{-\alpha}\,d\tau = \frac{1}{\alpha-1} \tag{A.44}$$

and

$$\int_{-1}^{t}\langle\tau\rangle^{-\alpha}\,d\tau \le \int_{-1}^{t}d\tau = t+1 \qquad \text{for } t\le 0, \tag{A.45}$$

we obtain the first assertion in (A.41). Let $|\tau|\ge 1$. Then

$$\frac{d}{d\tau}\langle\tau\rangle^{1-\alpha} = |1-\alpha|\langle\tau\rangle^{-\alpha}\frac{\tau}{\sqrt{1+\tau^2}} = -(\alpha-1)\langle s\rangle^{-\alpha}\frac{\tau}{\sqrt{1+\tau^2}} \ge \frac{\alpha-1}{\sqrt{2}}\langle\tau\rangle^{-\alpha} \qquad \text{if } \tau\le -1. \tag{A.46}$$

Now the second assertion in (A.41) follows from

$$\langle t\rangle^{1-\alpha} = \int_{0}^{t}\frac{d}{d\tau}\langle\tau\rangle^{1-\alpha}\,d\tau \ge \frac{\alpha-1}{\sqrt{2}}\int_{0}^{\tau}\langle\tau\rangle^{-\alpha}\,d\tau \qquad \text{for } t\le -1. \tag{A.47}$$

For $-1\le s\le 0$ we have

$$\begin{aligned}\int_{-\infty}^{s}\int_{-\infty}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt &= \int_{-\infty}^{-1}\int_{-\infty}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt + \int_{-1}^{s}\int_{-\infty}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt\\ &= \int_{1}^{\infty}\int_{-\infty}^{-t}\langle\tau\rangle^{-\alpha}\,d\tau dt + \int_{-1}^{s}\int_{-\infty}^{-1}\langle\tau\rangle^{-\alpha}\,d\tau dt + \int_{-1}^{s}\int_{-1}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt\\ &\le \int_{1}^{\infty}\int_{t}^{\infty}\langle\tau\rangle^{-\alpha}\,d\tau dt + \frac{1}{\alpha-1}\int_{-1}^{s}dt + \int_{-1}^{s}(t+1)dt\\ &\le \int_{1}^{\infty}\int_{t}^{\infty}\tau^{-\alpha}\,d\tau dt + \frac{1}{\alpha-1}(s+1) + \frac{1}{2}(s+1)^2\\ &= \frac{1}{\alpha-1}\int_{1}^{\infty}t^{1-\alpha}\,dt + \frac{1}{\alpha-1}(s+1) + \frac{1}{2}(s+1)^2\\ &= \frac{1}{\alpha-1}\frac{1}{\alpha-2} + \frac{1}{\alpha-1}(s+1) + \frac{1}{2}(s+1)^2\end{aligned} \tag{A.48}$$

For $s=0$ we have

$$\int_{-\infty}^{0}\int_{-\infty}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt \le \frac{1}{\alpha-1}\frac{1}{\alpha-2} + \frac{1}{\alpha-1} + \frac{1}{2} = \frac{1}{2} + \frac{1}{\alpha-2} \le \frac{1}{2}\frac{\alpha}{\alpha-2} \tag{A.49}$$

Then (A.42) follows from (A.42) by setting $s=0$. Now for $s\ge 1$ we have

$$\begin{aligned}\int_{0}^{s}\int_{0}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt &= \int_{0}^{1}\int_{0}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt + \int_{1}^{s}\int_{0}^{1}\langle\tau\rangle^{-\alpha}\,d\tau dt + \int_{1}^{s}\int_{1}^{t}\langle\tau\rangle^{-\alpha}\,d\tau dt\\ &\le \int_{0}^{1}\int_{0}^{t}d\tau dt + \int_{1}^{s}dt + \int_{1}^{s}\int_{1}^{t}\tau^{-\alpha}\,d\tau dt\\ &= \frac{1}{2} + s - 1 + \frac{1}{\alpha-1}\int_{1}^{s}\left(1-t^{1-\alpha}\right)dt\\ &= s - \frac{1}{2} + \frac{1}{\alpha-1}(s-1) + \frac{1}{\alpha-1}\frac{1}{\alpha-2}(s^{2-\alpha}-1)\end{aligned}$$

Since $\alpha > 2$ and $s \geq 1$, we have $(s^{2-\alpha} - 1) < 1$, so

$$\begin{aligned}\int_0^s \int_0^t \langle \tau \rangle^{-\alpha} \, d\tau dt &\leq s - \frac{1}{2} + \frac{1}{\alpha - 1}\left(s - 1 + \frac{1}{\alpha - 2}\right) \\ &= \frac{\alpha}{\alpha - 1}s - \frac{1}{2} + \frac{1}{\alpha - 2} - \frac{1}{\alpha - 1}\end{aligned} \tag{A.50}$$

Note that by (A.41) we have

$$\int_0^s \int_{-\infty}^0 \langle \tau \rangle^{-\alpha} \, d\tau dt \leq \frac{\alpha}{\alpha - 1}s \tag{A.51}$$

Now from (A.49), (A.50) and (A.51) we obtain for $s > 1$

$$\begin{aligned}\int_{-\infty}^s \int_{-\infty}^t \langle \tau \rangle^{-\alpha} \, d\tau dt &= \int_{-\infty}^0 \int_{-\infty}^t \langle \tau \rangle^{-\alpha} \, d\tau dt + \int_0^s \int_{-\infty}^0 \langle \tau \rangle^{-\alpha} \, d\tau dt + \int_0^s \int_0^t \langle \tau \rangle^{-\alpha} \, d\tau dt \\ &\leq \frac{2\alpha}{\alpha - 1}s + \frac{2}{\alpha - 2} + \frac{1}{\alpha - 1}\end{aligned} \tag{A.52}$$

This shows (A.43).

□

An immediate application of Lemma A.5 is the following result, which we will need in Section 1.2.

A.6. Lemma *Let $\omega_- \in \mathbb{R}^n$ and $\check{z} \perp \omega_-$, as in Convention 1.2.1. Then for any $\gamma > 0$ and $\alpha > 2$, there exists a constant $C_{\alpha,\gamma} > 0$ so that for all $\check{z}$ and all $s \leq 0$, the estimate*

$$\langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle^{\gamma} \int_{-\infty}^s \int_{-\infty}^t \langle 2\sqrt{\lambda}\omega_- \tau + \check{z} \rangle^{-\alpha-\gamma} \, d\tau \, dt \leq C_{\alpha,\gamma} \langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle^{2-\alpha}. \tag{A.53}$$

holds.

Proof. The orthogonality of ω_- and $\check{z}$ gives

$$\langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle = \langle z \rangle \langle 2\sqrt{\lambda}\langle z \rangle^{-1} s \rangle. \tag{A.54}$$

so

$$\begin{aligned}&\langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle^{\gamma} \int_{-\infty}^s \int_{-\infty}^t \langle 2\sqrt{\lambda}\omega_- \tau + \check{z} \rangle^{-\alpha-\gamma} \, d\tau \, dt \\ &\qquad = \langle 2\sqrt{\lambda}\langle z \rangle^{-1} s \rangle^{\gamma} \langle z \rangle^{-\alpha} \int_{-\infty}^s \int_{-\infty}^t \langle 2\sqrt{\lambda}\langle z \rangle^{-1} \tau \rangle^{-\alpha-\gamma} \, d\tau \, dt.\end{aligned} \tag{A.55}$$

We substitute twice in the integral, $\tau' = 2\sqrt{\lambda}\langle z \rangle^{-1}\tau$ and $t' = 2\sqrt{\lambda}\langle z \rangle^{-1} t$, to obtain

$$\begin{aligned}&\langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle^{\gamma} \int_{-\infty}^s \int_{-\infty}^t \langle 2\sqrt{\lambda}\omega_- \tau + \check{z} \rangle^{-\alpha-\gamma} \, d\tau \, dt \\ &\qquad = \langle 2\sqrt{\lambda}\langle z \rangle^{-1} s \rangle^{\gamma} \langle z \rangle^{-\alpha} (2\sqrt{\lambda}\langle z \rangle^{-1})^{-2} \int_{-\infty}^{2\sqrt{\lambda}\langle z \rangle^{-1} s} \int_{-\infty}^t \langle \tau \rangle^{-\alpha-\gamma} \, d\tau \, dt.\end{aligned} \tag{A.56}$$

We consider two cases and show that in each case, there exists a constant allowing the right–hand side of (A.56) to be estimated by $\langle 2\sqrt{\lambda}\omega_- s + \check{z} \rangle^{2-\alpha}$.

i) $(2\sqrt{\lambda}\langle z \rangle^{-1} s \leq -1)$ We can apply (A.43) to the right–hand side of (A.56). Then

$$\int_{-\infty}^{2\sqrt{\lambda}\langle z \rangle^{-1} s} \int_{-\infty}^t \langle \tau \rangle^{-\alpha-\gamma} \, d\tau \, dt \leq \frac{2}{(\alpha - 2)(\alpha - 1)} \langle \sqrt{\lambda}\langle z \rangle^{-1} s \rangle^{2-\alpha} \tag{A.57}$$

and by (A.56) with (A.54),

$$\begin{aligned}
&\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{\gamma} \int_{-\infty}^{s} \int_{-\infty}^{t} \langle 2\sqrt{\lambda}\omega_- \tau + \check{z}\rangle^{-\alpha-\gamma}\, d\tau\, dt \\
&\leq \frac{2}{(\alpha-2)(\alpha-1)} \langle 2\sqrt{\lambda}\langle z\rangle^{-1} s\rangle^{\gamma} \langle z\rangle^{-\alpha} (2\sqrt{\lambda}\langle z\rangle^{-1})^{-2} \langle \sqrt{\lambda}\langle z\rangle^{-1} s\rangle^{2-\alpha-\gamma} \\
&= \frac{}{2\lambda(\alpha-2)(\alpha-1)} \langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{2-\alpha}. \qquad \text{(A.58)}
\end{aligned}$$

This shows (A.53) in the first case.

ii) $(-1 \leq 2\sqrt{\lambda}\langle z\rangle^{-1} s \leq 0)$ By (A.42), noting $s \leq 0$,

$$\int_{-\infty}^{2\sqrt{\lambda}\langle z\rangle^{-1} s} \int_{-\infty}^{t} \langle \tau\rangle^{-\alpha-\gamma}\, d\tau\, dt \leq \int_{-\infty}^{0} \int_{-\infty}^{t} \langle \tau\rangle^{-\alpha-\gamma}\, d\tau\, dt \leq \frac{\alpha}{2(\alpha-2)}. \qquad \text{(A.59)}$$

Furthermore, $\langle 2\sqrt{\lambda}\langle z\rangle^{-1} s\rangle^{\gamma} \leq 2^{\gamma}$, so with (A.59) we have from (A.56),

$$\langle 2\sqrt{\lambda}\omega_- s + \check{z}\rangle^{\gamma} \int_{-\infty}^{s} \int_{-\infty}^{t} \langle 2\sqrt{\lambda}\omega_- \tau + \check{z}\rangle^{-\alpha-\gamma}\, d\tau\, dt \qquad \text{(A.60)}$$

$$= \frac{2^{\gamma}\alpha}{8\lambda(\alpha-2)} \langle z\rangle^{2-\alpha}. \qquad \text{(A.61)}$$

This shows (A.53) in the second case.

□

APPENDIX B

Some properties of smooth maps

We will repeat here some well–known results on the properties of smooth maps which will be used in the main text. First, we repeat from [22, V.3]

B.1. Definition *Let $\langle M, \varrho\rangle$ be a metric space. A map $T\colon M \to M$ for which there is a $K < 1$ such that $\varrho(Tx, Ty) \le K\varrho(x, y)$ is called a strict contraction.*

Contraction Mapping Principle *A strict contraction on a complete metric space has a unique fixed point.*

B.2. Lemma *Let B be a Banach space, $M \subset B$ a closed convex subset and $T\colon M \to M$ a C^1 mapping. Then T is a strict contraction and thus has a unique fixed point in M if $\|DT\| < 1$.*

Proof. We need to show that $\|Tx - Ty\| \le K\|x - y\|$ for $x, y \in M$ and some $K < 1$. Since M is convex, the set $\{z \in B\colon z = \lambda x + (1-\lambda)y,\ \lambda \in [0,1]\} \subset M$. Then by the fundamental theorem of calculus (cf., e.g., cite[(1.8)]Taylor),

$$Tx - Ty = \int_0^1 DT(\lambda x + (1-\lambda)y)(x - y)\, d\lambda, \tag{B.62}$$

so

$$\|Tx - Ty\| \le \|DT\| \cdot \|x - y\|. \tag{B.63}$$

and the result follows. □

We will frequently need criteria for smooth maps to be embeddings. In general, we quote from [].

B.3. Lemma *Let M, N be smooth manifolds and $T\colon M \to N$ an injective immersion. Then T is an embedding if*

i) *T^{-1} is continuous, i.e., T maps any open set onto an open set,*
ii) *or T is proper, i.e., the pre–image of any compact set is compact.*

B.4. Lemma *Let $\Omega \subset \mathbb{R}^n$ be an open convex set and $f, g\colon \Omega \to \mathbb{R}^n$ smooth functions, $\lambda_{\min}(x) > 0$ the smallest eigenvalue of $Df|_x$.*

i) *If $Df|_x$ is a positive self-adjoint matrix for all $x \in \Omega$ and $\lambda_{\inf} := \inf_{x\in\Omega} \lambda_{\min}(x)$ is positive, then f is an embedding.*
ii) *If Df is as in i) then there exists some $\varepsilon > 0$ so that if $Dg = Df + R$ with $\|R\| \le \varepsilon$, then g is an embedding.*

Proof.
i) By definition, f is an immersion. Furthermore, it is well–known (e.g., [25, Proposition 3.2]) that f is injective if Df is positive and Ω is convex. It remains to show that the inverse function f^{-1} is continuous. Since Df is self–adjoint, the largest eigenvalue of $Df^{-1}|_x$ is bounded by $1/\lambda_{\inf}$ and the continuity follows immediately by applying (B.63) to $T = f^{-1}$.
ii) The condition on R guarantees that Dg and Dg^{-1} are both positive with smallest eigenvalues bounded away from 0.

□

APPENDIX C

Proof of Lemma 3.3.12

The proof of Lemma 3.3.12 was inspired by Protas [21], who obtained an analogous result for the case of a compactly supported potential $V \in C_0^\infty(\mathbb{R}^n)$. We will deal with rather cumbersome block matrices, and in order to maintain legibility, we first introduce some notation.

C.1. Convention *Let $I, J \subset \mathcal{N} = \{1, \ldots, n\}$, ordered such that $I = (i_k)_{k=1}^{|I|}$ and $J = (j_k)_{k=1}^{|J|}$ with $i_k < i_{k+1}$, $j_k < j_{k+1}$.*

i) *We will understand $A = (a_{ij'})_{i \in I, j' \in J}$ to refer to an $(|I| \times |J|)$-matrix given by $A = (a_{i_k j_{l'}})_{1 \le k \le |I|, 1 \le l' \le |J|}$. A reference to the ith row or (j')th column of $(a_{ij'})$ is understood to be to the kth row or the (l') column, respectively, of $(a_{i_k j_{l'}})$.*
ii) *Whenever A is expressed as $(f(i,j'))_{i \in I, j' \in J}$, the unprimed variable is understood to refer to the rows of A, while the primed variable numbers the columns.*
iii) *Within block matrices, we will often write simply $f(i,j')$ to express $(f(i,j'))_{i \in I, j' \in J}$ in a space-conserving fashion. The range ($i \in I$ and $j' \in J$) of the variables will be given in the text.*

We shall denote the values of functions at a stationary point by $|_{stat.\ pt.}$.

Proof of Lemma 3.3.12. In order to lighten the notation, we will omit the subscript k from the formulae, writing, e.g., $J \subset \mathcal{N}_i$ instead of $J_k \subset \mathcal{N}_{i_k}$. In the notation of Convention C.1, the Hessian of Φ is given by the block matrix

$$\operatorname{Hess} \Phi = \begin{pmatrix} \frac{\partial^2 \Phi}{\partial x_{\bar\imath} \partial x_{\bar\imath'}} & \frac{\partial^2 \Phi}{\partial x_{\bar\imath} \partial \vartheta_{\bar\jmath'}} & \frac{\partial^2 \Phi}{\partial x_{\bar\imath} \partial z_{m'}} \\ \frac{\partial^2 \Phi}{\partial \vartheta_{\bar\jmath} \partial x_{\bar\imath'}} & \frac{\partial^2 \Phi}{\partial \vartheta_{\bar\jmath} \partial \vartheta_{\bar\jmath'}} & \frac{\partial^2 \Phi}{\partial \vartheta_{\bar\jmath} \partial z_{m'}} \\ \frac{\partial^2 \Phi}{\partial z_m \partial x_{\bar\imath'}} & \frac{\partial^2 \Phi}{\partial z_m \partial \vartheta_{\bar\jmath'}} & \frac{\partial^2 \Phi}{\partial z_m \partial z_{m'}} \end{pmatrix} \tag{C.64}$$

for $\bar\jmath, \bar\jmath' \in \overline{J}$, $\bar\imath, \bar\imath' \in \overline{I}$, $m, m' \in \mathcal{N}'$, writing $\mathcal{N}' := \{1, \ldots, n-1\}$. Note that by (2.2.19), for $m = 1, \ldots, n-1$ and $j \in \mathcal{N}$,

$$\frac{\partial \boldsymbol{\xi}_j}{\partial z_m} = \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_k \partial x_j} \frac{\partial \boldsymbol{x}_k}{\partial z_m} + \sqrt{\lambda} \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial \xi_k \partial x_j} \frac{\partial \omega_k}{\partial z_m}, \qquad \frac{\partial \boldsymbol{\xi}_j}{\partial s} = \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_k \partial x_j} \frac{\partial \boldsymbol{x}_k}{\partial s}. \tag{C.65}$$

Calculating the derivatives directly from (3.3.40) and using (C.65) we have for $m' \in \mathcal{N}'$,

$$\begin{aligned} \frac{\partial \Phi}{\partial z_{m'}} &= \langle x_{\overline{I}} - \boldsymbol{x}_{\overline{I}}(s,z;\lambda), \partial_{z_{m'}} \nabla_{x_{\overline{I}}} \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z;\lambda))(s,z;\lambda)\rangle \\ &\quad + \big\langle \nabla_{x_I} \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z;\lambda)) - \nabla_{x_I} \varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i \setminus J})), \\ &\quad \partial_{z_{m'}} \boldsymbol{x}_I(s,z;\lambda)\big\rangle. \end{aligned} \tag{C.66}$$

At the stationary point (3.3.41) we have

$$\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i \setminus J})) = \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z;\lambda)) \tag{C.67}$$

so for $m, m' \in \mathcal{N}_i$,

$$\begin{aligned}\frac{\partial^2\Phi}{\partial z_m \partial z_{m'}}\Big|_{\text{stat. pt.}} &= \langle \partial_{z_m}(x_{\overline{I}} - \boldsymbol{x}_{\overline{I}}(s,z;\lambda)), \partial_{z_{m'}} \nabla_{x_{\overline{I}}}\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z;\lambda))(s,z;\lambda)\rangle \\ &\quad + \langle \partial_{z_m}(\nabla_{x_I}\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\omega_+(z;\lambda)) \\ &\quad - \nabla_{x_I}\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))), \partial_{z_{m'}}\boldsymbol{x}_I(s,z;\lambda)\rangle \\ &= \sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ i\in I}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial\omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_i} - \sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ \overline{\imath}\in\overline{I}}} \frac{\partial \boldsymbol{x}_{\overline{\imath}}}{\partial z_m}\frac{\partial\omega_k}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}}} - \sum_{\overline{\imath},\overline{\imath}'\in\overline{I}} \frac{\partial \boldsymbol{x}_{\overline{\imath}}}{\partial z_m}\frac{\partial \boldsymbol{x}_{\overline{\imath}'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}}\end{aligned} \tag{C.68}$$

Furthermore,

$$\frac{\partial\Phi}{\partial\vartheta_{\mathcal{N}_i\setminus J}} = -\frac{\partial\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))}{\partial\vartheta_{\mathcal{N}_i\setminus J}} - m_{\mathcal{N}_i\setminus J},$$

so for $\overline{\jmath}, \overline{\jmath}' \in \mathcal{N}_i \setminus J$, $m' \in \mathcal{N}_i$, $\overline{\imath}' \in \overline{I}$ we obtain

$$\frac{\partial^2\Phi}{\partial\vartheta_{\overline{\jmath}}\partial\vartheta_{\overline{\jmath}'}} = -\frac{\partial^2\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))}{\partial\vartheta_{\overline{\jmath}}\partial\vartheta_{\overline{\jmath}'}}, \tag{C.69}$$

$$\frac{\partial^2\Phi}{\partial\vartheta_{\overline{\jmath}}\partial z_{m'}} = -\sum_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))}{\partial\vartheta_{\overline{\jmath}}\partial x_i} \tag{C.70}$$

$$\frac{\partial^2\Phi}{\partial\vartheta_{\overline{\jmath}}\partial x_{\overline{\imath}'}}, = -\frac{\partial^2\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))}{\partial\vartheta_{\overline{\jmath}}\partial x_{\overline{\imath}'}}. \tag{C.71}$$

We have

$$\frac{\partial\Phi}{\partial x_{\overline{I}}} = \boldsymbol{\xi}_{\overline{I}}(s,z;\lambda) - \nabla_{x_{\overline{I}}}\varphi_+((\boldsymbol{x}_I(s,z;\lambda), x_{\overline{I}}), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J})),$$

so for $\overline{\imath}, \overline{\imath}' \in \overline{I}$, $m \in \mathcal{N}_i$,

$$\frac{\partial^2\Phi}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}} = -\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}}, \tag{C.72}$$

$$\begin{aligned}\frac{\partial^2\Phi}{\partial z_m\partial x_{\overline{\imath}'}} &= \frac{\partial\boldsymbol{\xi}_{\overline{\imath}'}}{\partial z_m} - \sum_{\substack{\overline{\imath}\in\overline{I}\\ i\in I}}\frac{\partial\boldsymbol{x}_i}{\partial z_m}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_i} \\ &= \sum_{k\in\mathcal{N}}\frac{\partial\boldsymbol{x}_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial x_k\partial x_{\overline{\imath}'}} + \sqrt{\lambda}\sum_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}'}} - \sum_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_m}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\overline{\imath}'}} \\ &= \sum_{\overline{\imath}''\in\overline{I}}\frac{\partial\boldsymbol{x}_{\overline{\imath}''}}{\partial z_m}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}''}\partial x_{\overline{\imath}'}} + \sqrt{\lambda}\sum_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}'}},\end{aligned} \tag{C.73}$$

where we have applied (C.65). Inserting (C.68)-(C.73) into (C.64), we obtain

$$\operatorname{Hess}\Phi|_{\text{stat. pt.}} = \left(\begin{array}{c|c|c} -\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial\vartheta_{\overline{\jmath}'}\partial x_{\overline{\imath}}} & \sum_{\overline{\imath}''\in\overline{I}}\frac{\partial\boldsymbol{x}_{\overline{\imath}''}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}''}\partial x_{\overline{\imath}}} + \sqrt{\lambda}\sum_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}}} \\ \hline -\frac{\partial^2\varphi_+}{\partial\vartheta_{\overline{\jmath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial\vartheta_{\overline{\jmath}}\partial\vartheta_{\overline{\jmath}'}} & -\sum_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\vartheta_{\overline{\jmath}}\partial x_i} \\ \hline \begin{array}{c}\sum_{\overline{\imath}''\in\overline{I}}\frac{\partial\boldsymbol{x}_{\overline{\imath}''}}{\partial z_m}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}''}\partial x_{\overline{\imath}'}} \\ +\sqrt{\lambda}\sum_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}'}}\end{array} & -\sum_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\vartheta_{\overline{\jmath}'}\partial x_i} & \begin{array}{c}\sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ i\in I}}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial\omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_i} \\ -\sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ \overline{\imath}\in\overline{I}}}\frac{\partial\boldsymbol{x}_{\overline{\imath}}}{\partial z_m}\frac{\partial\omega_k}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\overline{\imath}}} \\ -\sum_{\overline{\imath},\overline{\imath}'\in\overline{I}}\frac{\partial\boldsymbol{x}_{\overline{\imath}}}{\partial z_m}\frac{\partial\boldsymbol{x}_{\overline{\imath}'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}}\end{array}\end{array}\right).$$

For derivatives of φ_+ with respect to x- and ξ-variables we have omitted the arguments which are taken at the stationary point (3.3.41), i.e.,

$$\frac{\partial^2 \varphi_+}{\partial x_i \partial \xi_k} = \frac{\partial\text{‘}2\varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J})}{\partial x_i \partial \xi_k}. \tag{C.74}$$

On the other hand, derivatives of φ_+ with respect to ϑ have not yet been evaluated with the chain rule and we have written

$$\frac{\partial \varphi_+}{\partial \vartheta_{\overline{\jmath}}} = \frac{\partial \varphi_+(\boldsymbol{x}_\infty(s,z;\lambda), \sqrt{\lambda}\chi_i^{-1}(\theta_J, \vartheta_{\mathcal{N}_i\setminus J}))}{\partial \vartheta_{\overline{\jmath}}}. \tag{C.75}$$

for short. We will now perform elementary row and column manipulations, denoting similar matrices by "$\sim$". For all $\overline{\imath}' \in \overline{I}$ and all $m' \in \mathcal{N}_i$, we multiply the $(\overline{\imath}')$th column in the first block column by $\frac{\partial \boldsymbol{x}_{\overline{\imath}'}}{\partial z_{m'}}$ and add it to the (m')th column of the third block column. The resulting matrix reads

$$\operatorname{Hess}\Phi|_{\text{stat. pt.}} \sim \left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}'}\partial x_{\overline{\imath}}} & \sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \omega_k}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}}} \\
\hline
-\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}}\partial \vartheta_{\overline{\jmath}'}} & -\sum\limits_{k\in\mathcal{N}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}} \partial x_k} \\
\hline
\begin{array}{c} \sum\limits_{\overline{\imath}''\in\overline{I}} \frac{\partial \boldsymbol{x}_{\overline{\imath}''}}{\partial z_m} \frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}''}\partial x_{\overline{\imath}'}} \\ +\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}'}} \end{array} & -\sum\limits_{i\in I} \frac{\partial \boldsymbol{x}_i}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}'}\partial x_i} & \begin{array}{c} \sqrt{\lambda}\sum\limits_{k,k'\in\mathcal{N}} \frac{\partial \boldsymbol{x}_{k'}}{\partial z_{m'}} \frac{\partial \omega_k}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{k'}} \\ -\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ \overline{\imath}\in\overline{I}}} \frac{\partial \boldsymbol{x}_{\overline{\imath}}}{\partial z_m}\frac{\partial \omega_k}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}}} \end{array}
\end{array}\right)$$

Next, for all $\overline{\imath} \in \overline{I}$ and all $m \in \mathcal{N}_i$ we multiply the $(\overline{\imath})$th row in the first block row by $\frac{\partial \boldsymbol{x}_{\overline{\imath}}}{\partial z_m}$ and add it to the mth row of the third block row, obtaining

$$\operatorname{Hess}\Phi|_{\text{stat. pt.}} \sim \left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}'}\partial x_{\overline{\imath}}} & \sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \omega_k}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}}} \\
\hline
-\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}}\partial \vartheta_{\overline{\jmath}'}} & -\sum\limits_{k\in\mathcal{N}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}} \partial x_k} \\
\hline
\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \omega_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}'}} & -\sum\limits_{k\in \mathcal{N}} \frac{\partial \boldsymbol{x}_k}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \vartheta_{\overline{\jmath}'}\partial x_k} & \sqrt{\lambda}\sum\limits_{k,k'\in\mathcal{N}} \frac{\partial \boldsymbol{x}_{k'}}{\partial z_{m'}} \frac{\partial \omega_k}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{k'}}
\end{array}\right) \tag{C.76}$$

At the stationary point $\vartheta_{\overline{J}} = \theta_{\overline{J}}$, and we will from now on write θ_k instead of ϑ_k setting additionally $\theta' = \theta_{\mathcal{N}_i}$. Noting that

$$\frac{\partial \omega_j}{\partial z_{m'}} = \sum_{k\in\mathcal{N}_i} \frac{\partial \chi_j^{-1}(\theta')}{\partial \theta_k} \frac{\partial \theta_k}{\partial z_{m'}} \tag{C.77}$$

we evaulate derivatives of the type (C.75) to

$$\frac{\partial^2\varphi_+}{\partial \theta_{\overline{\jmath}} \partial x_{k'}} = \sqrt{\lambda}\sum_{k\in\mathcal{N}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{\overline{\jmath}}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{k'}}, \qquad \frac{\partial^2\varphi_+}{\partial \theta_{\overline{\jmath}} \partial \theta_{\overline{\jmath}'}} = \sqrt{\lambda}\sum_{k\in\mathcal{N}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{\overline{\jmath}}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial \theta_{\overline{\jmath}'}}. \tag{C.78}$$

Hence the matrix in (C.76) is similar to

$$\left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\overline{\imath}}\partial x_{\overline{\imath}'}} & -\frac{\partial^2\varphi_+}{\partial \theta_{\overline{\jmath}'}\partial x_{\overline{\imath}}} & \sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \omega_k}{\partial z_{m'}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}}} \\
\hline
-\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{\overline{\jmath}}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}'}}, & -\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{\overline{\jmath}}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial \theta_{\overline{\jmath}'}} & -\sqrt{\lambda}\sum\limits_{k,k'\in\mathcal{N}} \frac{\partial \boldsymbol{x}_{k'}}{\partial z_{m'}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{\overline{\jmath}}} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{k'}} \\
\hline
\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ k''\in\mathcal{N}_i}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{k''}} \frac{\partial \theta_{k''}}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{\overline{\imath}'}} & -\sum\limits_{k\in\mathcal{N}} \frac{\partial \boldsymbol{x}_k}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \theta_{\overline{\jmath}'}\partial x_k} & \sqrt{\lambda}\sum\limits_{\substack{k,k'\in\mathcal{N}\\ k''\in\mathcal{N}_i}} \frac{\partial \boldsymbol{x}_{k'}}{\partial z_{m'}} \frac{\partial \chi_k^{-1}(\theta')}{\partial \theta_{k''}} \frac{\partial \theta_{k''}}{\partial z_m} \frac{\partial^2\varphi_+}{\partial \xi_k \partial x_{k'}}
\end{array}\right)$$

(We will occasionally omit "$\operatorname{Hess}\Phi|_{\text{stat. pt.}} \sim$" in order to save space)

For all $\bar\imath' \in \bar I$ and all $m' \in \mathcal{N}'$ we now subtract the $(\bar\imath')$th column of the leftermost block column, multiplied with $\frac{\partial \boldsymbol{x}_{\bar\imath'}}{\partial z_{m'}}$, from the (m')th column of the rightermost block column, obtaining

$$\left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & -\frac{\partial^2\varphi_+}{\partial \theta_{\bar\jmath'}\partial x_{\bar\imath}} & \sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath}} + \sum\limits_{\bar\imath'\in\bar I}\frac{\partial \boldsymbol{x}_{\bar\imath'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} \\ \hline
-\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath'}}, & -\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial \theta_{\bar\jmath'}} & -\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{j'}} \\ \hline
\sqrt{\lambda}\sum\limits_{\substack{k'\in\mathcal{N}\\ k\in\mathcal{N}_i}}\frac{\partial\chi_{k'}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{k'}\partial x_{\bar\imath'}} & -\sum\limits_{k\in\mathcal{N}}\frac{\partial \boldsymbol{x}_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\theta_{\bar\jmath'}\partial x_k} & \sqrt{\lambda}\sum\limits_{\substack{k'\in\mathcal{N}\\ k\in\mathcal{N}_i\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_{k'}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{k'}\partial x_{j'}}
\end{array}\right)$$

For all $\bar\jmath \in \bar J$ and $m \in \mathcal{N}'$ we further add the $(\bar\jmath)$th row of the middle block row, multiplied by $\frac{\partial\theta_{\bar\jmath}}{\partial z_m}$ to the mth row of the lowermost block row, obtaining

$$\left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & -\frac{\partial^2\varphi_+}{\partial \theta_{\bar\jmath'}\partial x_{\bar\imath}} & \sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\omega_k}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath}} + \sum\limits_{\bar\imath'\in\bar I}\frac{\partial \boldsymbol{x}_{\bar\imath'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} \\ \hline
-\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{\bar\imath'}}, & -\sqrt{\lambda}\sum\limits_{j\in\mathcal{N}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial \theta_{\bar\jmath'}} & -\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{j'}} \\ \hline
\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{\bar\imath'}} & \begin{array}{c} -\sum\limits_{j\in\mathcal{N}}\frac{\partial \boldsymbol{x}_j}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\theta_{\bar\jmath'}\partial x_j} \\ -\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ \bar\jmath\in\bar J}}\frac{\partial\theta_{\bar\jmath}}{\partial z_m}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial\theta_{\bar\jmath'}} \end{array} & \sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{j'}}
\end{array}\right)$$

From (2.4.5) we obtain

$$\frac{\partial l_{\bar\jmath'}^{(i)}(z)}{\partial z_m} = \frac{\partial^2\varphi_+}{\partial z_m\partial\theta_{\bar\jmath'}} = \sum_{j\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_j\partial\theta_{\bar\jmath'}}\frac{\partial \boldsymbol{x}_j}{\partial z_m} + \sum_{\substack{j\in\mathcal{N}\\ k\in\mathcal{N}_i}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_j\partial\theta_{\bar\jmath'}}, \tag{C.79}$$

allowing us to rewrite the matrix as

$$\left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & -\frac{\partial^2\varphi_+}{\partial \theta_{\bar\jmath'}\partial x_{\bar\imath}} & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial \boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}} \\ \hline
-\sqrt{\lambda}\sum\limits_{j\in\mathcal{N}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{\bar\imath'}}, & -\sqrt{\lambda}\sum\limits_{j\in\mathcal{N}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial \theta_{\bar\jmath'}} & -\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{j'}} \\ \hline
\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{\bar\imath'}} & -\frac{\partial l_{\bar\jmath'}}{\partial z_m} + \sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ \bar\jmath\in J}}\frac{\partial\theta_{\bar\jmath}}{\partial z_m}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial\theta_{\bar\jmath'}} & \sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{j'}}
\end{array}\right)$$

We can now add the entire leftermost block column to the middle block column, obtaining

$$\operatorname{Hess}\Phi|_{\text{stat. pt.}} \sim \left(\begin{array}{c|c|c}
-\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & (*) & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial \boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}} \\ \hline
-\sqrt{\lambda}\sum\limits_{j\in\mathcal{N}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{\bar\imath'}}, & (*) & -\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{j'}} \\ \hline
\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{\bar\imath'}} & -\frac{\partial l_{\bar\jmath'}}{\partial z_m} & \sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ k\in\bar I\\ j'\in I}}\frac{\partial \boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_{j}^{-1}(\theta')}{\partial\theta_k}\frac{\partial\theta_k}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_{j}\partial x_{j'}}
\end{array}\right) \tag{C.80}$$

where "$(*)$" represents a block matrix of suitable size whose precise expression will be irrelevant to us. Note that $\bar I = J$ by Definition 3.3.1, so

$$\sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ \bar\imath\in\bar I}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_{\bar\imath}}\frac{\partial\theta_{\bar\imath}}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath'}} = \begin{pmatrix}\frac{\partial\theta_{j'}}{\partial z_m} & \frac{\partial l_{\bar\jmath'}}{\partial z_m}\end{pmatrix}\begin{pmatrix}\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_j}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath'}} \\ 0\end{pmatrix} \tag{C.81}$$

and

$$\sqrt{\lambda}\sum_{\substack{k\in\mathcal{N}\\ \bar\imath\in\bar I\\ i'\in I}}\frac{\partial \boldsymbol{x}_{i'}}{\partial z_{m'}}\frac{\partial \chi_k^{-1}(\theta')}{\partial\theta_{\bar\imath}}\frac{\partial\theta_{\bar\imath}}{\partial z_m}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{i'}} = \begin{pmatrix}\frac{\partial\theta_{j'}}{\partial z_m} & \frac{\partial l_{\bar\jmath'}}{\partial z_m}\end{pmatrix}\begin{pmatrix}\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ i'\in I}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_j}\frac{\partial\boldsymbol{x}_{i'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{i'}}\\ 0\end{pmatrix} \tag{C.82}$$

for $\bar\imath', j, j' \in J$, $\bar\jmath' \in \bar J$, $m, m' \in \mathcal{N}_i$. It follows from (C.80), (C.81) and (C.82) that

$$|\det \operatorname{Hess}\Phi|_{\text{stat. pt.}} = \left|\det\begin{pmatrix}\frac{\partial\theta_{j'}}{\partial z_m} & \frac{\partial l_{\bar\jmath'}}{\partial z_m}\end{pmatrix}\right| \times\left|\det\left(\begin{array}{c|c|c} -\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & (*) & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}}\\ \hline -\sqrt{\lambda}\sum\limits_{j\in\mathcal{N}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{\bar\imath'}}, & (*) & -\sqrt{\lambda}\sum\limits_{\substack{j\in\mathcal{N}\\ j'\in I}}\frac{\partial\boldsymbol{x}_{j'}}{\partial z_{m'}}\frac{\partial\chi_j^{-1}(\theta')}{\partial\theta_{\bar\jmath}}\frac{\partial^2\varphi_+}{\partial\xi_j\partial x_{j'}}\\ \hline -\sqrt{\lambda}\sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_j}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath'}} & 0 & -\sqrt{\lambda}\sum\limits_{\substack{k\in\mathcal{N}\\ i'\in I}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_j}\frac{\partial\boldsymbol{x}_{i'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{i'}}\\ \hline 0 & \mathbb{1}_{|\bar J|} & 0\end{array}\right)\right| \tag{C.83}$$

Here $\mathbb{1}_{|\bar J|}$ denotes the $|\bar J|\times|\bar J|$ unit matrix. We can expand the block matrix along the lowest block row, and also consolidate the middle two block rows into a single one, obtaining

$$|\det \operatorname{Hess}\Phi|_{\text{stat. pt.}} = \lambda^{\frac{n-1}{2}}\mathsf{E}_{\Gamma_k,J_k}\circ S_\lambda^+(z)\cdot|\det M| \tag{C.84}$$

where

$$\mathsf{E}_{\Gamma_k,J_k}\circ S_\lambda^+ = \left|\det d(\pi^{(i_k)}_{\Gamma_k,J_k}\circ S_\lambda^+)\right| = \left|\det\begin{pmatrix}\frac{\partial\theta_J}{\partial z}\\ \frac{\partial l_{\bar J}}{\partial z}\end{pmatrix}\right| \tag{C.85}$$

and M is the $(n-1+|\bar I|)\times(n-1+|\bar I|)$ matrix

$$M = \left(\begin{array}{c|c} -\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}}\\ \hline \sum\limits_{k\in\mathcal{N}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_l}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{\bar\imath'}} & \sum\limits_{\substack{k\in\mathcal{N}\\ i'\in I}}\frac{\partial\chi_k^{-1}(\theta')}{\partial\theta_l}\frac{\partial\boldsymbol{x}_{i'}}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial\xi_k\partial x_{i'}}\end{array}\right) \tag{C.86}$$

with $l \in \mathcal{N}_i$. We now define the $(n+|\bar I|)\times(n+|\bar I|)$ matrices

$$M_1 := \left(\begin{array}{c|c|c} -\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & \mathbb{1}_{|\bar I|} & 0\\ \hline \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath'}\partial\xi_k}\frac{\partial\omega_k}{\partial\theta_m} & 0 & \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{i'}\partial\xi_k}\frac{\partial\omega_k}{\partial\theta_m}\\ \hline \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath'}\partial\xi_k}\omega_k & 0 & \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{i'}\partial\xi_k}\omega_k\end{array}\right), \tag{C.87}$$

$$M_2 := \left(\begin{array}{c|c|c} \mathbb{1}_{|\bar I|} & 0 & 0\\ \hline \mathbb{0} & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}} & \sum\limits_{\bar\imath'\in\bar I}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath'}\partial x_{\bar\imath}}\frac{\partial\boldsymbol{x}_{\bar\imath'}}{\partial s}\\ \hline 0 & \frac{\partial\boldsymbol{x}_i}{\partial z_{m'}} & \frac{\partial\boldsymbol{x}_i}{\partial s}\end{array}\right) \tag{C.88}$$

whose product is

$$M_1M_2 = \left(\begin{array}{c|c|c} -\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath'}} & \frac{\partial\boldsymbol{\xi}_{\bar\imath}}{\partial z_{m'}} - \sum\limits_{i\in I}\frac{\partial^2\varphi_+}{\partial x_i\partial x_{\bar\imath}}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}} & \sum\limits_{\bar\imath\in\bar I}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath}\partial x_{\bar\imath}}\frac{\partial\boldsymbol{x}_{\bar\imath}}{\partial s}\\ \hline \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{i'}\partial\xi_k}\frac{\partial\omega_k}{\partial\theta_m} & \sum\limits_{\substack{k\in\mathcal{N}\\ i\in I}}\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}}\frac{\partial^2\varphi_+}{\partial x_i\partial\xi_k}\frac{\partial\omega_k}{\partial\theta_m} & \sum\limits_{\substack{k\in\mathcal{N}\\ i\in I}}\frac{\partial^2\varphi_+}{\partial x_i\partial\xi_k}\frac{\partial\omega_k}{\partial\theta_m}\frac{\partial\boldsymbol{x}_i}{\partial s}\\ \hline \sum\limits_{k\in\mathcal{N}}\frac{\partial^2\varphi_+}{\partial x_{\bar\imath'}\partial\xi_k}\omega_k & \sum\limits_{\substack{k\in\mathcal{N}\\ i\in I}}\frac{\partial^2\varphi_+}{\partial x_i\partial\xi_k}\omega_k\frac{\partial\boldsymbol{x}_i}{\partial z_{m'}} & \sum\limits_{\substack{k\in\mathcal{N}\\ i\in I}}\frac{\partial^2\varphi_+}{\partial x_i\partial\xi_k}\omega_k\frac{\partial\boldsymbol{x}_i}{\partial s}\end{array}\right). \tag{C.89}$$

For all $\bar{\imath}' \in \bar{I}$, we add the $(\bar{\imath}')$th column of the leftermost block column, multplied by $\frac{\partial \boldsymbol{x}_{\bar{\imath}}}{\partial s}$, to the rightermost (block) column. We obtain

$$M_1 M_2 \sim \left(\begin{array}{c|c|c} -\frac{\partial^2 \varphi_+}{\partial x_{\bar{\imath}} \partial x_{\bar{\imath}'}} & \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial z_{m'}} - \sum_{i \in I} \frac{\partial^2 \varphi_+}{\partial x_i \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & 0 \\ \hline \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{i'} \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} & \sum_{\substack{k \in \mathcal{N} \\ i \in I}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} \frac{\partial^2 \varphi_+}{\partial x_i \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} & \sum_{k,k' \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{k'} \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} \frac{\partial \boldsymbol{x}_{k'}}{\partial s} \\ \hline \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{\bar{\imath}'} \partial \xi_k} \omega_k & \sum_{\substack{k \in \mathcal{N} \\ i \in I}} \frac{\partial^2 \varphi_+}{\partial x_i \partial \xi_k} \omega_k \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \sum_{k,k' \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{k'} \partial \xi_k} \omega_k \frac{\partial \boldsymbol{x}_{k'}}{\partial s} \end{array}\right). \tag{C.90}$$

Now differentiating (2.2.20), we see that

$$\sum_{k' \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{k'} \partial \xi_k} \frac{\partial \boldsymbol{x}_{k'}}{\partial s} = 2\sqrt{\lambda}\omega_k. \tag{C.91}$$

Since $\langle \omega_+, \omega_+ \rangle = 1$ and $\langle \omega_+, \frac{\partial \omega_+}{\partial \theta_m} \rangle = 0$, inserting (C.91) into (C.90) yields

$$M_1 M_2 \sim \left(\begin{array}{c|c|c} -\frac{\partial^2 \varphi_+}{\partial x_{\bar{\imath}} \partial x_{\bar{\imath}'}} & \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial z_{m'}} - \sum_{i \in I} \frac{\partial^2 \varphi_+}{\partial x_i \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & 0 \\ \hline \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{i'} \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} & \sum_{\substack{k \in \mathcal{N} \\ i \in I}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} \frac{\partial^2 \varphi_+}{\partial x_i \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} & 0 \\ \hline \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{\bar{\imath}'} \partial \xi_k} \omega_k & \sum_{\substack{k \in \mathcal{N} \\ i \in I}} \frac{\partial^2 \varphi_+}{\partial x_i \partial \xi_k} \omega_k \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & 2\sqrt{\lambda} \end{array}\right). \tag{C.92}$$

By expanding the determinant we obtain

$$|\det(M_1 M_2)| = 2\sqrt{\lambda}|\det M|, \tag{C.93}$$

hence (C.84) becomes

$$|\det \operatorname{Hess} \Phi|_{\text{stat. pt.}} = \frac{1}{2}\lambda^{\frac{n-2}{2}} \mathsf{E}_{\Gamma_k, J_k} \circ S_\lambda^+(z) \cdot |\det M_1| \cdot |\det M_2| \tag{C.94}$$

Now by (C.87),

$$|\det M_1| = \left| \det \begin{pmatrix} \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{l'} \partial \xi_k} \frac{\partial \omega_k}{\partial \theta_m} \\ \sum_{k \in \mathcal{N}} \frac{\partial^2 \varphi_+}{\partial x_{l'} \partial \xi_k} \omega_k \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{\partial \omega_{l'}}{\partial \theta_m} \\ \omega_{l'} \end{pmatrix} \right| \cdot |\det A(\boldsymbol{x}_\infty(s,z,\lambda), \sqrt{\lambda}\omega_+(z;\lambda)))| \tag{C.95}$$

for $l' \in \mathcal{N}$, $m \in \mathcal{N}'$ and A given by (3.3.59). Now

$$\begin{pmatrix} \frac{\partial \omega_{l'}}{\partial \theta_m} \\ \omega_{l'} \end{pmatrix}^T \begin{pmatrix} \frac{\partial \omega_{l'}}{\partial \theta_m} \\ \omega_{l'} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{\mu\nu} \end{pmatrix}, \qquad g_{\mu\nu} = \left\langle \frac{\partial \omega_+}{\partial \theta_\mu}, \frac{\partial \omega_+}{\partial \theta_\nu} \right\rangle \tag{C.96}$$

so $(g_{\mu\nu})$ is just the metric tensor on the sphere, expressed in the coordinates of $\mathbb{R}^{n-1}$. It follows that

$$|\det M_1| = (\mathsf{g}_{\Sigma_{i_k}} \circ \omega_+(z;\lambda))^{\frac{1}{2}} \cdot |\det A(\boldsymbol{x}_\infty(s,z,\lambda), \sqrt{\lambda}\omega_+(z;\lambda)))| \tag{C.97}$$

Furthermore, expanding the determinant and applying (C.65),

$$\begin{aligned} |\det M_2| &= \left| \det \left(\begin{array}{c|c} \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial z_{m'}} - \sum_{i \in I} \frac{\partial^2 \varphi_+}{\partial x_i \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \sum_{\bar{\imath}' \in \bar{I}} \frac{\partial^2 \varphi_+}{\partial x_{\bar{\imath}'} \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_{\bar{\imath}'}}{\partial s} \\ \hline \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \frac{\partial \boldsymbol{x}_i}{\partial s} \end{array}\right) \right| \\ &= \left| \det \left(\begin{array}{c|c} \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial z_{m'}} - \sum_{i \in I} \frac{\partial^2 \varphi_+}{\partial x_i \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial s} - \sum_{i \in I} \frac{\partial^2 \varphi_+}{\partial x_i \partial x_{\bar{\imath}}} \frac{\partial \boldsymbol{x}_i}{\partial s} \\ \hline \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \frac{\partial \boldsymbol{x}_i}{\partial s} \end{array}\right) \right| \\ &= \left| \det \left(\begin{array}{c|c} \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial z_{m'}} & \frac{\partial \boldsymbol{\xi}_{\bar{\imath}}}{\partial s} \\ \hline \frac{\partial \boldsymbol{x}_i}{\partial z_{m'}} & \frac{\partial \boldsymbol{x}_i}{\partial s} \end{array}\right) \right| \\ &= \mathsf{D}_{\Omega_k, I_k} \circ \iota(s,z) \end{aligned} \tag{C.98}$$

Combining (C.94), (C.97) and (C.98) then yields (3.3.75). □

APPENDIX D

The canonical Maslov operator

The "canonical" Maslov operator is mathematically a conversion of a function defined on a lagrangian manifold Λ in some cotangent bundle T^*M to a function on M. Physically, it converts a classical object (the lagrangian manifold Λ of integral curves of a hamiltonian vector field $X_p = dp$, $p(x,\xi)$ being a function of configuration space) to a quantum-mechnical object (a function $\psi \in L^2(M)$ solving $P\psi = \lambda\psi$, $P\psi = \int e^{-ix\xi} p(x,\xi)\hat{\psi}(\xi)\, đ\xi$).

The Maslov operator is "canonical" in the sense that, writing a function $\varphi \in C^\infty(\Lambda)$ locally as a function of lagrangian coordinates $(x_I, \xi_{\overline{I}})$ (cf. Definition 2.3.2), a Fourier transform of the ξ_{olI}-variables is performed to obtain a function of the x variables. It is otherwise evident that this procedure is based on coordinate transformations and is therefore dependent on the choice of chart, local coordinates and other data *not* canonically given.

In order to define the Maslov operator, we need to introduce some notation and definitions. A large amount of literature exists on the theory of the Maslov operator (cf., e.g., [18, esp. §8.4], [27], [20]), but we will restrict ourselves here to the essentials necessary for the definition on Λ and $\mathcal{L}_+$.

D.1. Definition *We shall denote by M an n–dimensional riemannian manifold with riemannian metric tensor g. We use the notation of of Definition 2.3.2. We will refer to the following objects as Maslov data:*

- i) *An atlas $\{(\Sigma_k, \chi_k)\}$ on M consisting of open sets $\Sigma_k \subset M$ and maps $\chi_k\colon \Sigma_k \to \mathbb{R}^n$. If $M = \mathbb{R}^n$ we shall always use global (x,ξ)–coordinates on $T^*\mathbb{R}^n$.*
- ii) *A lagrangian atlas $\{(\Omega_m, \pi_{\Omega_m, I_m})\}_{m\geq 0}$, $I_m \subset \mathcal{N} := \{1,\dots,n\}$, on Λ chosen such that for each m there exists an $k_m \in \mathbb{N}$ such that $\Omega_m \subsetneqq T^*\Sigma_{k_m}$. We additionally require that $I_0 = \mathcal{N}$. The General Position Hypothesis ensures that we can always choose an Ω_0 in this way.*
- iii) *A global coordinate map $\iota\colon \mathbb{R}^n \to \Lambda$.*
- iv) *A global generating function S.*
- v) *A partition of unity subordinate to the covering $\{\Omega_m\}$ (i.e., $e_m \in C_0^\infty(\Omega_m)$, $\sum e_m = 1$).*
- vi) *A set of functions $g_m \in C_0^\infty(M)$ such that $g_m = 1$ on $\pi(\Omega_m)$ and $g|_{M\setminus\pi(\Omega_m)}$ has compact support.*

We will assume that all lagrangian manifolds under consideration are in "general position" in the sense of Arnol'd [3, Theorem 2.1] if they are of sufficiently small dimension.

General Position Hypothesis We assume that all lagrangian manifolds under consideration (in particular Λ and $\mathcal{L}_+$ of (1.3.3) and (2.1.17), respectively) are in general position if $n \leq 6$.

D.2. Remark For lagrangian manifolds of dimension less than six, the General Position Hypothesis implies that all singularities are stable, i.e., the rank of $d\pi$ does not change under a sufficiently small perturbation of the manifold in the class of lagrangian manifolds. Denote by $\mathcal{S}(\Lambda) = \{p \in \Lambda\colon \operatorname{rank} d\pi|_p < n\}$ the set of singular points of the n–dimensional lagrangian manifold Λ. Then one of the consequences of the General Position Hypothesis is

$$\dim \mathcal{S}(\Lambda) \leq n - 1. \tag{D.99}$$

The statement (D.99) ensures the existence of some lagrangian atlas $\{(\Omega_m, \pi_{\Omega_m, I_m})\}_{m\geq 0}$ on Λ such that $I_0 = \mathcal{N}$.

Note that for $V \equiv 0$, $\mathcal{L}_+ = T^*_{\omega_-}S^{n-1}$ and is hence *not* in general position.

Following Vainberg[27], we define the Maslov index of a chain of charts.

D.3. Definition & Lemma *Let $\{\Omega_m, \pi_{\Omega_m, I_m}\}$, $I_m \subset \mathcal{N}$, be a lagrangian atlas on Λ.*

i) *We define the Maslov index of a pair of charts by*

$$\gamma(\Omega_m,\Omega_k) = \operatorname{inerdex}\left(\frac{\partial x_{\overline{I}_m}\circ\pi^{-1}_{\Omega_m,I_m}(x_{I_m},\xi_{\overline{I}_m})}{\partial\xi_{\overline{I}_m}}\right)\Bigg|_{(x_{I_m},\xi_{\overline{I}_m})=\pi_{\Omega_m,I_m}p} - \operatorname{inerdex}\left(\frac{\partial x_{\overline{I}_k}\circ\pi^{-1}_{\Omega_k,I_k}(x_{I_k},\xi_{\overline{I}_k})}{\partial\xi_{\overline{I}_k}}\right)\Bigg|_{(x_{I_m},\xi_{\overline{I}_m})=\pi_{\Omega_k,I_k}p} \tag{D.100}$$

for any $p\in\Omega_m\cap\Omega_k$ with $p\notin\Sigma(\Lambda)$. Here inerdex A *(the "inertial index") denotes the number of negative eigenvalues of a matrix A. If $\Omega_m\cap\Omega_k=\emptyset$, we set $\gamma(\Omega_m,\Omega_k)=0$. The index γ is independent of the choice of p.*

ii) *We call $\{\Omega_{m_j}\}_{j=1}^{l}$, $l\in\mathbb{N}$, a chain of charts joining Ω_0 and Ω_k if $\Omega_{m_1}=\Omega_0$, $\Omega_{m_p}=\Omega_k$ and $\Omega_{m_j}\cap\Omega_{m_{j+1}}\neq\emptyset$.*

iii) *We define the index of a chain of charts joining Ω_0 and Ω_k by*

$$\gamma(\Omega_{m_1},\dots,\Omega_{m_l}) = \sum_{j=1}^{l-1}\gamma(\Omega_{m_j},\Omega_{m_{j+1}}) \tag{D.101}$$

If $\Omega_0\cap\mathcal{S}(\Lambda)=\emptyset$, the indices of two chains of charts both joining Ω_0 and Ω_k are identical, so we can define γ_k as the index of any such chain.

D.4. Definition *We make the following definitions:*

i) *The $1/h$–Fourier transform of functions $\varphi\in L^2(\mathbb{R}^n)$ is given by*

$$(\mathcal{F}_h\varphi)(\xi) := (2\pi h)^{-n/2}\int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle x,\xi\rangle}\varphi(x)\,dx, \quad (\mathcal{F}_h^{-1}\varphi)(x) = (2\pi h)^{-n/2}\int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle}\varphi(\xi)\,d\xi. \tag{D.102}$$

For short, we will write $đx_i := (2\pi h)^{-1/2}dx_i$ (where $x=(x_i)$).

ii) $\mathsf{g}_\Omega\circ\iota(y) := \det(g\circ\pi|_\Lambda\circ\iota(y))$

iii) *We define $\mathsf{D}_{\Omega,I}\in C^\infty(\Omega)$ via*

$$\mathsf{D}_{\Omega,I}\circ\iota(y) := \left|\det d(\pi_{\Omega,I}\circ\iota)|_y\right| \qquad y\in\iota^{-1}(\Omega). \tag{D.103}$$

We can now introduce the Maslov operator on Λ:

D.5. Definition *We refer to the notation and definitions of this section as well as Definition 2.3.2. We assume that we have chosen Maslov data as in Definition D.1. Then*

$$K_\Lambda\colon C^\infty(\Lambda)\to C^\infty(M), \qquad K_\Lambda[\varphi] := \sum_m e^{i\frac{\pi}{2}\gamma_m}K_{\Omega_m,I_m}[e_m\varphi], \tag{D.104}$$

with

$$(K_{\Omega,I}[\varphi])\circ\chi^{-1}(x) := g(x)\mathcal{F}_h^{-1}\left[e^{\frac{i}{h}S_{\Omega,I}\circ\pi^{-1}_{\Omega,I}(x_I,\,\cdot\,)}(\mathsf{g}^{\frac14}_{\Omega;I}\mathsf{D}^{-\frac12}_{\Omega,I}\cdot\varphi)\circ\pi^{-1}_{\Omega,I}(x_I,\,\cdot\,)\right]\Big|_{x_{\overline{I}}}. \tag{D.105}$$

defines a Maslov operator on Λ. We shall call $K_{\Omega,I}$ as a local Maslov operator, and will sometimes refer to K_Λ as a global Maslov operator in contradistinction to $K_{\Omega,I}$.

D.6. Remark In order to define a Maslov operator K_Λ on a lagrangian manifold Λ, a wide variety of objects were defined or chosen. Without giving any proofs, we will summarise how these objects and choices influence K_Λ. We consider the riemannian manifold (M,g) and $\Lambda\subset T^*M$ to be fixed.

i) The global coordinate map $\iota\colon\mathbb{R}^n\to\Lambda$ effectively defines a measure on Λ, which fundamentally influences $\mathsf{D}_{\Omega,I}$ and $\mathsf{g}_{\Omega;I}$. Replacing ι by some other global coordinate map would yield an entirely different Maslov operator.

ii) The global generating function S is defined up to an additive constant $c\in\mathbb{R}$, which influences K_Λ by a factor of $e^{\frac{i}{h}c}$.

iii) The atlas $\{(\Sigma_k,\chi_k)\}$ on M and the lagrangian atlas $\{(\Omega_m,\pi_{\Omega_m,I_m})\}_{m\geq 0}$ on Λ, the partition of unity $\{e_m\}$ and the functions $\{g_m\}$ all influence K_λ by terms of order $O(h)$, i.e., if K_λ and K'_λ denote two Maslov operators constructed using different charts and cut–off functions,

$$\|K_\lambda[\varphi]\|_2 = \|K'_\lambda[\varphi]\|_2 + O(h) \qquad \text{for } \varphi\in L^2(M),\text{ as } h\to 0. \tag{D.106}$$

iv) The choice of Ω_0 in the definition of γ_m (the index of the chain of charts joining Ω_m to Ω_0) influences K_Λ by a multiplicative factor of $e^{i\frac{\pi}{2}\mu}$, $\mu \in \{0, 1, 2, 3\}$.

Hence the term "canonical operator" often seen in the literature is actually a misnomer. However, for many applications this arbitrariness is not important, as often any given Maslov operator on Λ can be used to construct approximate solutions to a given problem (e.g., Lemma 3.2.9 and Theorem 3.2.10).

D.7. Remark It is not actually necessary to have a global generating function S and a global coordinate map ι on Λ; instead one can patch together local generating functions and local measures in a suitable way, cf. [20]. However, on $\Lambda \subset T^*\mathbb{R}^n$ and $\mathcal{L}_+ \subset T^*S^{n-1}$ we have established these convenient data and thus make use of them.

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List of Symbols

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