Institut für Mathematik
Arbeitsgruppe "Partielle Differentialgleichungen und Komplexe Analysis"

Operators on Singular Manifolds

Dissertation

zur Erlangung des akademischen Grades
"doctor rerum naturalium"
(Dr. rer. nat.)
in der Wissenschaftsdisziplin "Mathematische Analysis"

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
der Universität Potsdam

von
Xiaojing Lyu

Potsdam, September 2016

Operators on Singular Manifolds

Xiaojing Lyu

February 19, 2017

Abstract

We study the interplay between analysis on manifolds with singularities and complex analysis and develop new structures of operators based on the Mellin transform and tools for iterating the calculus for higher singularities. We refer to the idea of interpreting boundary value problems (BVPs) in terms of pseudo-differential operators with a principal symbolic hierarchy, taking into account that BVPs are a source of cone and edge operator algebras. The respective cone and edge pseudo-differential algebras in turn are the starting point of higher corner theories. In addition there are deep relationships between corner operators and complex analysis. This will be illustrated by the Mellin symbolic calculus.

Contents

Summary

The paper establishes new relations in the analysis of pseudo-differential operators which are based on the Mellin transform with operator-valued symbols. Those may depend here on parameters. This corresponds to the form of differential operators on so-called stratified spaces, here of order of singularity ≤ 2, which corresponds to conical or edge singularities. In stretched coordinates we have a typical degenerate behavior, where in simple cases the q-dimensional covariable η is multiplied by the distance variable $r \geq 0$ to the edge.
The results are developed in 5 Chapters.
The paper also contains necessary tools, cf. Chapter 1 and Chapter 5. Otherwise the manuscript of the dissection is based on 4 published articles [7], [36], [37], [38], partly with coauthors, which are written in connection with diverse projects. Because of the main objectives it turned out that investigations on functional -analytic structures are of degree of generality. In Chapter 2 is studied in detail the connection between asymptotics of distribution close to the singularity and Paley-Wiener effects leads to the continuity of Mellin-operators with meromorphic symbols in such spaces, cf. Theorem 2.4. These investigations are motivated by similar relations for double asymptotics for singularities of second order with 2 distance variables $r, t \geq 0$, which are also studied in Chapter 2. Chapter 3 contains a new characterization of Kegel spaces and subspaces with asymptotics. In this connection we employ order reducing operators and exit-properties are becoming transparent through the involved degenerate Mellin symbols. This becomes particularly clear which is demonstrated in detail in the example of degenerate differential operators, cf. Theorem 3.2, which explicitly shows the connection between exit-behavior and parameter-dependent degenerate differential operators. In Theorem 3.16 it is shown by an example that the behavior of operator-valued edge symbols is characterized by their subordinate symbols. In a characterization of Kegel -spaces on singular cones by order-reducing parameter-dependent edge operators we need such properties in general form. Because of Theorem 3.25 it suffices to reduce considerations to smoothness 0. Another result refers to the order filtration of the edge algebra where Chapter 4 is contains for the first time equality of different characterizations of smoothing edge-operator, cf. Theorem 4.11.

Introduction

This exposition is devoted to new elements of the analysis of (pseudo-) differential operators on manifolds with singularities, more precisely, on stratified spaces in certain categories $\mathfrak{M}_k, k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, where $k = 0$ corresponds to smoothness, $k = 1$ to conical singularities or edge, etc. In particular, a manifold M with smooth boundary belong to $\mathfrak{M}_1$, where the edge is just the boundary, and M is close to ∂M modeled on $\overline{\mathbb{R}}_+ \times G$ for an open set $G \subseteq \partial M$. In this sense some parts of the analysis on a manifold with edge are inspired by the calculus of boundary value problems (BVPs), see the work of Vishik and Eskin [74], [75], [12], and Boutet de Monvel [3]. It is typical in these theories that operators A have a principal symbolic hierarchy, here consisting of an interior symbol $\sigma_\psi(A)(x, \xi)$ living on $\mathrm{int} M$ and a boundary symbol $\sigma_\partial(A)(y, \eta)$ living on the boundary ∂M. The boundary symbol takes values in operators on $\mathbb{R}_+$ and as such it can be interpreted as an operator on the infinite cone $\overline{\mathbb{R}}_+$. It is very fruitful to see this situation as a special case of a “non-trivial” cone

$$X^\triangle = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

for a, say, based on compact element X of $\mathfrak{M}_0$. However, when we assume X to be a manifold with smooth boundary, then $X^\triangle$ is already an element in $\mathfrak{M}_2$, and the respective analysis on such a space just corresponds to BVPs, on a manifold, where the boundary has conical singularities. It is not possible here to give a complete report on corresponding works. Let us mention, for instance, BVPs for differential operators and asymptotics of solutions, cf. Kondratyev [32], calculus of pseudo-differential BVPs with the transmission property on the smooth part of the boundary, cf. Schrohe and Schulze [51], [50]. The latter theory became possible through the progress of pseudo-differential operators on closed manifolds with conical singularities, see Rempel and Schulze [44], [45], and Schulze [58], [57], [61]. Another step towards generalizing of BVPs was the edge calculus of Schulze [53] which consists of a pseudo-differential algebra on a manifold M with edge Y, locally near Y modeled on $X^\triangle$ for a coordinate neighborhood G on Y and a cone $X^\triangle$ for $X \in \mathfrak{M}_0$. This contains also the special case of BVPs without the transmission property at the boundary, cf. Rempel and Schulze, [42], [43], Schulze [54], [55], [11]. Since then the development created BVPs on manifolds with edge which corresponds to the case that X itself is a manifold with boundary, see the monographs of Kapanadze and Schulze [28], Harutyunyan and Schulze [21] and the cited literature there.
Numerous applications also require the analysis of operators on spaces $M \in \mathfrak{M}_k$ for higher $k \in \mathbb{N}$. Here it seems indispensable to create iterative concepts to establishing the respective operator algebras by an inductive process, starting from $M \in \mathfrak{M}_k$ for a given k and to reach the calculus for $N \in \mathfrak{M}_{k+1}$. Let us mention here difference works, devoted to the iterative approach, namely, [56], [63]. In this process it turned out that it is necessary to deepen and to simply the background on operators on manifolds with conical singularities and edges, also from the point of view of operators on manifolds with conical exit to infinity and their relationship with the edge symbolic calculus.
Therefore, a considerable part of this exposition is devoted to a new look at operators on a manifolds with edge.

Chapter 1 contains the necessary material on pseudo-differential operator techniques, in particular, with operator-valued symbols and based on the Fourier and Mellin transform, see **Sections 1.1,1.2,1.5**, formulate weighted cone edge Sobolev spaces, based on Hilbert spaces with group action, see **Sections 1.4,1.6**, and we outline the function theoretic background of the operator-valued Mellin symbolic calculus, see **Sections 1.3,1.7**.

Chapter 2 is devoted to weighted spaces with edge asymptotic in several independent axial variables $r, t \in \mathbb{R}_+$. We characterize the nature of discrete asymptotics, and we establish other elements of the analysis in these spaces which are necessary in the corner pseudo-differential calculus.
This chapter is aimed at studying iterated edge asymptotics on a corner space $\mathbb{R}^p \times B^\triangle$ where B is a manifold with (first order) edge Y and

$$B^\triangle := (\overline{\mathbb{R}}_+ \times B)/(\{0\} \times B)$$

the associated cone with link B. Also the cone $B^\triangle$ is a space with second order singularities, a corner in this case, and iterated discrete asymptotics has been introduced in [56] in connection with elliptic regularity of solution to corner-degenerate elliptic equations. The case of a wedge $\mathbb{R}^p \times B^\Delta$ for $p \neq 0$ is more complicated. A basic tool in pseudo-differential theories for such singularities are Mellin operators with symbols that are meromorphic in the complex w-plane where $w \in \mathbb{C}$ is the Mellin covariable associated with the corner axis variable $t \in \mathbb{R}_+$. The link B in turn has locally near Y also a wedge structure, namely, $\mathbb{R}^q \times X^\Delta$ for a smooth (in our case compact) manifold X. The axial variable for the cone X^Δ will be $r \in \mathbb{R}_+$. We focus here on some essential functional analytic aspects on the interplay between asymptotic effects for $r \to 0$ and $t \to 0$, especially, on the continuity of Mellin pseudo-differential operators in spaces with iterated asymptotics. In order to keep the approach transparent we mainly consider here weighted spaces of smoothness and weight 0. Non-zero smoothness and arbitrary weights are connected with the smoothness/weight case 0 via some reductions of orders. We employed methods of [48].

In **Section 2.1** we briefly consider the asymptotic terms of edge asymptotics belonging to smooth solutions to elliptic edge-degenerate equations. The exponents in the distance variable r to the edge are assumed here to be independent of the edge-variable y, and we formulate everything in terms of constant (in y) discrete asymptotics. In a simple model situation $\mathbb{R}_+$ we show on how spaces with asymptotics are continuously mapped to other such spaces under applying a Mellin pseudo-differential operator with meromorphic symbol, cf. the second assertion of Theorem 2.4. We then illustrate the asymptotic part of the cone calculus, namely, smoothing Mellin plus Green operators, which play a role in analogous form in higher singular pseudo-differential theories.

In **Section 2.2** we develop some material on edge asymptotics for first order singularities as a preparation of similar phenomena that are then studied for second order edges, locally formulated on a wedge $\mathbb{R}^q \times X^\Delta$. In **Section 2.3** we introduce iterated discrete edge asymptotics as an extension of iterated corner asymptotics from [56]. We characterise the shape of singular functions of such asymptotics, cf. formula (2.70), and then we show the continuity of Mellin operators in spaces with iterated asymptotics, cf. the second assertion of Theorem 2.23. In **Section 2.4** we illustrate the relationship between singular functions of second order edge asymptotics and potentials in the second order edge calculus.

Note that the pseudo-differential analysis on spaces with higher order singularities is motivated by new applications in physics, cf. Harutyunyan, Flad, Schneider and Schulze [13] or [16], [14]. Other applications concern models in elasticity and crack theorem [28] or mixed, transmission and crack problems, [21]. The underlying mathematical theory is still challenging and topic of research, with numerous open questions. The edge pseudo-differential calculus goes back to the paper [53] of Schulze. Since then the details are elaborated in a number of systematic monographs, e.g., in [55], [11]. An update concise version is also given by Flad, Harutyunyan, and Schulze in [15].

Chapter 3 is devoted to a new description of Kegel Spaces, published in [37].

A manifold M with smooth edge Y is locally near Y modeled on $X^\Delta \times \Omega$ for a cone X^Δ where X is a smooth manifold and $\Omega \subseteq \mathbb{R}^q$ an open set corresponding to a chart on Y. Compared with pseudo-differential algebras, based on other quantizations of edge-degenerate symbols, we extend the approach with Mellin representations on the r half-axis up to $r = \infty$, the conical exit of

$$X^\wedge = \mathbb{R}_+ \times X \ni (r, x)$$

at infinity. The alternative description of the edge calculus is useful for pseudo-differential structures on manifolds with higher singularities.

The pseudo-differential calculus on a manifold with edge, first established in [53], and later on refined and extended in other papers and monographs, see [54], [10], is motivated by the task to organize a pseudo-differential algbra that contains edge-degenerate differential operators together with the parametrices of elliptic elements. A manifold M with edge Y is a (paracompact) topological space containing Y as a smooth manifold of dimension $q > 0$ such that $M \setminus Y$ itself is smooth and M is locally near Y modeled on a locally trivial X^{Δ}-bundle over Y for a smooth manifold X.
The ellipticity of an operator A in this algebra is determined by a principal symbolic hierarchy $\sigma(A) = (\sigma_0(A), \sigma_1(A))$, where $\sigma_0(A)$ is the homogeneous principal symbol on $s_0(M) := M \setminus Y$ and $\sigma_1(A)$ the operator-valued principal edge symbol, associated with $s_1(M) := Y$ which is responsible for elliptic edge conditions of trace and potential type, cf. [55, Subsection 3.5.2].
Operators of order $\mu \in \mathbb{R}$ in the edge algebra locally near Y in the splitting of variables

$$(r, x, y) \in \mathbb{R}_+ \times X \times \Omega, \quad \Omega \subseteq \mathbb{R}^q, \ q = \dim Y,$$

are (modulo smoothing operators) of the form $A = r^{-\mu}\mathrm{Op}_y(a)$ where

$$\mathrm{Op}_y(a)u(y) := \iint e^{i(y-y')\eta}a(y, \eta)u(y')dy'đ\eta, \ u(y) \in C_0^\infty(X^\wedge),$$

for an operator-valued amplitude function

$$a(y, \eta) := \mathrm{Op}_r(p)(y, \eta),$$
$$p(r, y, \rho, \eta) = \tilde{p}(r, y, r\rho, r\eta), \ \tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})).$$

Here $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ is the space of classical parameter-dependent pseudo-differential operators on X of order μ, with $\lambda \in \mathbb{R}^l$ as parameters. The iterated representation $A = r^{-\mu}\mathrm{Op}_y(\mathrm{Op}_r(p))$ holds without smoothing remainder for an edge-degenerate differential operator

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j(rD_y)^\alpha$$

for coefficients $a_{j\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$, where $\mathrm{Diff}^\nu(X)$ is the space of all differential operators over X of order ν with smooth coefficients (in local coordinates). The typical edge-degenerate behavior of A close to $r = 0$ is the reason of using a description of A via the Mellin transform on $\mathbb{R}_+$ close to zero, while far from $r = 0$, and in particular, for $r \to \infty$, a formulation in terms of the Fourier transform seems more common. This is at least the background for the choice of Mellin-edge quantizations of operator families $\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta})$ in [53], [54], [11]. However, it also makes sense to employ "pure" Mellin representations of edge amplitude functions in the sense of Gil, Schulze, and Seiler [17]. The main issue of the latter paper is to rephrase the operator-valued edge amplitude functions $a(y, \eta)$ from the mixed Fourier/Mellin representation into Mellin form, in order to have an easier proof of the composition behavior of such amplitude functions in the edge symbolic algebra. In the meantime, a proof of the composition result in original Mellin-edge representation has been updated for edge amplitude function in original form, see [67]. Nevertheless, the interplay between Fourier and Mellin representations, also referred to as Mellin quantisation, is rather involved what concerns the behavior of operators for $r \to \infty$, the conical exit of $X^\wedge$ to infinity. Despite of the alternative form of edge amplitude functions of [67] the exit effect for $r \to \infty$ remained a little odd. Therefore, in this article we

investigate edge symbols from the very beginning from the Mellin point of view, up to the conical exit.
In **Section 3.1** we study the exit behavior of edge-degenerate differential operators and consider the relationship between edge-degenerate ellipticity which is a condition close to the singularity, cf. Definition 3.1, and exit ellipticity which refers to $r \to \infty$, cf. (3.12), (3.13), (3.14) below. The proof of Theorem3.2 in elementary terms allows us to understand on how an extra "non-degenerate" parameter ι guarantees conormal ellipticity for any prescribed weight γ, cf. Remark 3.3. Such a parameter is used later on in Section 3.5 in constructing elliptic Mellin edge symbols.
Section 3.2 contains some necessary material on edge amplitude functions, cf. Definition 3.9. Compared with the Mellin edge quantisation (3.35), earlier used in the edge calculus, cf. [55], we focus on the Mellin representation in (3.32) close to the edge, cf. also [17]. In addition we formulate the principal edge symbols $\sigma_1(\cdot)$ that are employed later on for a new description of weighted Kegel spaces at infinity.
The alternative form of edge symbols in Mellin terms makes it desirable to understand the order filtration of the edge calculus in a new way. Although this is not the main topic of this article, in **Section 3.3** we give an idea for zero-order operators on how vanishing of the principal symbols $\sigma_0(\cdot), \sigma_1(\cdot)$ creates symbols of less order of analogous kind as before, but with the original weight data. Here we already see the role of $r \in \mathbb{R}_+$ as a large parameter in Mellin symbols which is multiplied by a fixed $\eta \neq 0$.
In **Section 3.4** we study the relationship between edge-degenerate Mellin and Fourier symbols, and we give a self-contained proof of the fact that our operator functions are symbols with twisted symbolic estimates, between weighted Kegel spaces, cf. Theorem 3.23. The result is known for symbols based on the quantisation (3.35), but the approach in terms of the Mellin transform up to infinity is a new aspect.
In **Section 3.5** we construct a parameter-dependent elliptic holomorphic edge symbol of order $\mu \in \mathbb{R}$, such that the associated Mellin pseudo-differential operator induces an isomorphism between weighted Kegel spaces of arbitrary weight γ, cf. Theorem 3.24, which is attained by taking into account another non-degenerate parameter ϑ sufficiently large that turns the operators parameter-dependent elliptic over a prescribed compact subset of $X^\wedge$. As a consequence we obtain a new intrinsic characterization of $\mathcal{K}^{s,\gamma}(X^\wedge)$ spaces for $r \to \infty$, based on Mellin operators, cf. Theorem 3.26. The remarkable point is that this works in terms of the Mellin transform although the underlying H^s_{cone}-spaces (cf. notation in formula (1.95) below) are based on the Fourier transform.
The analysis of operators on manifolds with conical singularities or edge has attracted many specialists in the past decades, see, in particular, Kondratyev [32], Eskin [12], Egorov and Schulze [11], and many others. The investigations are often motivated by specific applications and the problem of characterizing asymptotics of solutions close to the singularities. Although it is useful to have parametrices of singular elliptic operators within a pseudo-differential algebra, the corresponding algebras for higher singularities (as far as they are elaborated at all) are complicated from the point of view of technicalities. The present article is aimed at contributing new elements to the respective tools. Our new characterization of Kegel spaces over $X^\wedge$ at infinity, cf. the isomorphism (3.86), belong to the ingredients also for the calculus on spaces with higher singularities, to be elaborated in another paper of the authors.

Chapter 4 presents a new characterization of the order filtration of the edge calculus, based on a new representation of edge operators by a specific Mellin quantization, pub-

lished in [38]. By edge algebra we understand a pseudo-differential calculus on a manifold with edge. The operators have a two-component principal symbolic hierarchy which determines operators up to lower order terms. Those belong to a filtration of the edge operator spaces. We give a new characterization of this structure, based on an alternative representation of edge amplitude functions only containing holomorphic edge-degenerate Mellin symbols, besides smoothing Mellin plus Green symbols.
The calculus of operators on a manifold with edge has been introduced in [53] with its principal symbolic hierarchy, including trace and potential operators, that determine ellipticity and parametrices within the calculus. The approach has been inspired by pseudo-differential boundary value problems in the sense of Vishik and Eskin [74],[75], Eskin [12, Subsection 15], Boutet de Monvel [3], and also by Kondratyev [32], Rabinovich[41] on operators on manifolds with conical singularities. The subsequent development has contributed deeper insight and numerous generalisations, see, for instance, Rempel and Schulze [42], the monograph [58], moreover, Dorschfeld [10], Schrohe and Schulze [52], Gil, Schulze and Seiler [18], Schulze [56], Coriasco and Schulze [8], Chang, Habal and Schulze [5], [6], Rungrottheera [47], [48]. The edge calculus is motivated by many interesting applications, see the monographs of Kapanadze and Schulze [28], Harutyunyan and Schulze [21], the article Flad, Harutyunyan [16], or Chang, Qian and B.-W. Schulze [6]. The details remained complicated, and it is therefore desirable to employ alternative representations of the edge symbolic calculus.
The present chapter is aimed at studying edge symbols in terms of holomorphic Mellin symbols, cf. [18], and to derive a filtration of edge operator spaces with respect to orders. Our technique is expected to be helpful also for higher corner pseudo-differential operators where Mellin representations on singular cones up to exits to infinity seem to be most natural. This paper is organized as follows.
In **Section 4.1** we first recall some notation on weighted distribution spaces in terms of the Mellin transform. Concerning basics we refer to Jeanquartier [27] or [54]. Compared with Mellin quantization in [55] the main new aspect is that the present definition of operator-valued edge amplitude functions in Definition 3.9 (iii) refers to holomorphic families (3.33) but not on the Mellin quantization of edge-degenerate operator families (4.5) as is done earlier in studying the edge calculus, cf., for instance, [55]. This has many consequences for managing edge operators, though we do not discuss here all technical changes connected with this modification. The cut-off functions $\omega,\ \omega'$ in this description (3.32) are fixed. We could take into account possible changes, see Remark 4.2, but those only contribute operators far from the edge.
Section 4.2 gives a brief description of spaces of edge operators $L^\mu(M,\boldsymbol{g})$ on a manifold M with edge Y and weight data $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$. The meaning is completely analogous to other expositions on the edge calculus. However, as announced before, we employ the Mellin version of local edge amplitude functions. We formulate the principal symbolic structure $\sigma = (\sigma_0, \sigma_1)$, consisting of the homogeneous principal interior symbol σ_0 which is edge-degenerate and the (twisted homogeneous) edge symbol σ_1. The latter relies on the representation from the article [17]. Concerning the nature of σ_1 on the open stretched model cone $(r,x) \in X^\wedge$ of local wedges we will return to more details in [37]. The remarkable aspect is that the Mellin representation of σ_1 for $r \to \infty$ concerns Fourier based Sobolev spaces, not Mellin ones, which is just the reason for the present new description of the order filtration of the edge calculus.
In **Section 4.3** we establish this filtration, based on the above-mentioned shape of edge amplitude functions. Note that similarly as in boundary value problems which are known

to be a special case of the edge calculus, the aspect of twisted homogeneity make the local symbolic information spread out to the infinite stretched cone $X^\wedge$ assuming for the edge covariable $\eta \neq 0$. Recall that in elliptic operators this effect is responsible for the nature of elliptic edge conditions, either with Shapiro-Lopatinskij ellipticity, or ellipticity with global projection conditions, cf. [55] or [59]. The main results of Section 4.3 are Theorems 4.4 and 4.5 on the filtration of the spaces of edge amplitude functions. This allows us to state the filtration of the spaces of edge operators themselves. In particular, in Theorem 4.10 we see that the space $L^{-\infty}(M, \boldsymbol{g})$ of edge operators of order $-\infty$ admits different equivalent characterizations, namely, (4.46) and (4.47). We will employ some necessary material on the cone algebra, in particular, Kegel spaces with and without asymptotic.
The material in this manuscript motivates future research in elliptic partial differential equations (PDEs) and applications which are connected in models with singular geometry, e.g., conical points and edges. For instance, the Laplacian in polar or cylindrical coordinates gives rise to a representation of derivatives as $r\frac{\partial}{\partial r}$ or $r\frac{\partial}{\partial y_1}, \cdots, r\frac{\partial}{\partial y_n}$, where $r > 0$ is the distance variable to the singularity. The same is true for other differential operators or systems of mechanics and also quantum mechanics. It is in such concrete cases interesting, to compute asymptotics of solutions, obtained in terms of poles and multiplicities of Mellin symbols which are contained in parametrices. To evaluate these data can be very difficult. Another interesting field of future research is to further develop pseudo-differential algebras based on the Mellin transform for singularities of higher order. The present thesis already contains steps for second order singularities. It should be continued and completed.

1 Fourier and Mellin pseudo-differential operators

1.1 Preliminary considerations

An elliptic differential operator A on a smooth Riemannian manifold Ω, say, an open set in $\mathbb{R}^n$, interpreted as a continuous operator $A : C^\infty(\Omega) \to C^\infty(\Omega)$ or $C_0^\infty(\Omega) \to C_0^\infty(\Omega)$, has a pseudo-differential parametrix P such that $PA - 1 =: C_{\mathrm{L}}$ and $AP - 1 =: C_{\mathrm{R}}$ are smoothing operators. Any such operator C is characterized by a kernel $c(x, x') \in C^\infty(\Omega \times \Omega)$ such that $Cu(x) = \int c(x, x')u(x')dx'$, $u \in C_0^\infty(\Omega)$, where dx is the measure associated with the Riemannian metric. In the case $\Omega \subseteq \mathbb{R}^n$ and

$$A = \sum_{|\alpha| \leq \mu} a_\alpha(x) D_x^\alpha \tag{1.1}$$

for smooth coefficients $a_\alpha(x) \in C^\infty(\Omega)$ and order $\mu \in \mathbb{N}$ we have an amptitude function

$$a(x, \xi) = \sum_{|\alpha| \leq \mu} a_\alpha(x) \xi^\alpha, \tag{1.2}$$

$a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n)$, determined by the relation

$$Au(x) = F^{-1} a(x, \xi)(Fu)(\xi), \tag{1.3}$$

$u \in C_0^\infty(\Omega)$, and $x\xi := \sum_{j=1}^{n} x_j \xi_j$ with F being the Fourier transform, and F^{-1} its inverse, cf. Subsection 1.2 below. In particular, we have $D_x^\alpha = F^{-1}\xi^\alpha F$. Incidentally, in order to indicate the correspondence between variables and covariables, we also write $F_{x\to\xi}$ and $F^{-1}_{\xi\to x}$ instead of F and F^{-1}, respectively. Ellipticity of A means that the homogeneous principal part of (1.2)

$$a_{(\mu)}(x,\xi) = \sum_{|\alpha|=\mu} a_\alpha(x)\xi^\alpha \tag{1.4}$$

does not vanish for all $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. Later on, instead of (1.4) we will also write $\sigma_\psi(A)$, called the homogeneous principal symbol of A. Homogeneity in this notation means positive homogeneity, i.e.,

$$\sigma_\psi(A)(x,\delta\xi) = \delta^\mu \sigma_\psi(A)(x,\xi), \tag{1.5}$$

for all $\delta > 0$. The function (1.2) belongs to the space

$$S^\mu(\Omega \times \mathbb{R}^n), \tag{1.6}$$

defined as the set of all $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$ satisfying the system of estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le c\langle\xi\rangle^{\mu-|\beta|} \tag{1.7}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$ for every $K \Subset \Omega$ and all multi-indices $\alpha,\beta \in \mathbb{N}^n$, for constants $c = c(\alpha,\beta,K) > 0$. Here $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$. If we set

$$\begin{aligned} S^{(\nu)}(\Omega \times (\mathbb{R}^n \setminus \{0\})) := \{ & f_{(\nu)}(x,\xi) \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\})) : \\ & f_{(\nu)}(x,\delta\xi) = \delta^\nu f(x,\xi) \quad \text{for all} \quad \delta > 0\}, \end{aligned} \tag{1.8}$$

where $\nu \in \mathbb{R}$, then we have $\chi(\xi)f_{(\nu)}(x,\xi) \in S^\nu(\Omega \times \mathbb{R}^n)$ for any excision function $\chi(\xi)$ (i.e., $\chi \in C^\infty(\mathbb{R}^n)$, and $\chi(\xi) = 0$ for $|\xi| < \varepsilon_0$, $\chi(\xi) = 1$ for $|\xi| > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$). An element $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, $\mu \in \mathbb{R}$, is called a classical symbol of order μ if there are homogeneous components $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$, $j \in \mathbb{N}$, such that

$$r_N(x,\xi) := a(x,\xi) - \sum_{j=0}^N \chi(\xi)a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^n) \tag{1.9}$$

for every $N \in \mathbb{N}$. By

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$$

we denote the space of all classical symbols of order μ.
Now for the differential operator A of order μ we have $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$, and the ellipticity of A entails the existence of a $p(x,\xi) \in S^{-\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$, such that a parametrix P in the above-mentioned sense has the form

$$P = F^{-1}_{\xi\to x} p(x,\xi) F_{x\to\xi},$$

acting as an operator $P : C_0^\infty(\Omega) \to C^\infty(\Omega)$.
For a symbol $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$, $\mu \in \mathbb{R}$, we often write

$$\mathrm{Op}(a)u(x) := \iint e^{i(x-x')\xi} a(x,\xi)u(x')dx'\,đ\xi \tag{1.10}$$

which is the same as the iterated integral $\int e^{ix\xi}a(x,\xi)\big(\int e^{-ix'\xi}u(x')dx'\big)đ\xi$ as soon as $\mu < -n$. However, (1.10) makes sense as an oscillatory integral for arbitrary $\mu \in \mathbb{R}$, and we systematically use this interpretation. More details may be found in any textbook on pseudo-differential operators. Also here we will return to more technique about oscillatory integrals. If necessary, in order to indicate the variable x we also write $\mathrm{Op}_x(a)$ instead of $\mathrm{Op}(a)$. Summing up the discussion so far, the task to express a parametrix P of an elliptic differential operator A of order μ with smooth coefficients gives rise to a pseudo-differential operator $P = \mathrm{Op}(p)$ for a classical symbol $p(x, \xi)$ of order $-\mu$. Since this does not rely on any other specific properties of A it makes sense to talk about spaces of pseudo-differential operators over Ω of any real order μ, and we set

$$L^\mu(\Omega) := \{A = \mathrm{Op}(a) + C : a(x, \xi) \in S^\mu(\Omega \times \mathbb{R}^n),\ C \text{ has a kernel in } C^\infty(\Omega \times \Omega)\}. \quad (1.11)$$

An $A \in L^\mu(\Omega)$ is called classical if $a(x, \xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$. The subspace of classical pseudo-differential operators will be denoted by $L^\mu_{\mathrm{cl}}(\Omega)$.

All this is material is standard. However, the pseudo-differential structure of parametrices of elliptic operators in other contexts is far from being so simple, for instance, boundary value problems or operators on manifolds with singularities. This is just the motivation for our investigations. In this introduction we give an impression on the nature of novelties and surprising effects when we have to quit the common path of pseudo-differential machineries for expressing a parametrix.

Now let A be a differential operator of order μ on a smooth manifold with boundary. Since ellipticity and parametrices are local phenomena (at least for the moment), for convenience we focus in the introduction on the case of a half-space $U \times \overline{\mathbb{R}}_+$ in the variables $x = (y, r)$ for $y \in U$, $r \in \overline{\mathbb{R}}_+$. In this case A induces continuous operators

$$A : C^\infty(U \times \overline{\mathbb{R}}_+) \to C^\infty(U \times \overline{\mathbb{R}}_+),\ C_0^\infty(U \times \overline{\mathbb{R}}_+) \to C_0^\infty(U \times \overline{\mathbb{R}}_+) \quad (1.12)$$

where $C^\infty(U \times \overline{\mathbb{R}}_+) = C^\infty(U \times \mathbb{R})|_{U \times \overline{\mathbb{R}}_+}$, etc. Writing A in the form (1.1), now with coefficients $a_\alpha(x) \in C^\infty(U \times \overline{\mathbb{R}}_+)$, smooth up to the boundary $r = 0$, we have again the homogeneous principal symbol

$$\sigma_\psi(A)(x, \xi) = \sum_{|\alpha| = \mu} a_\alpha(x)\xi^\alpha = \sum_{|\beta| + k = \mu} a_{\beta,k}(y, r)\eta^\beta \tau^k \quad (1.13)$$

of order μ, where we replaced α by (β, k) and ξ by (η, τ), the covariables to (y, r). In addition there is the (twisted) homogeneous boundary symbol

$$\sigma_\partial(A)(y, \eta) := \sum_{|\beta| + k = \mu} a_{\beta,k}(y, 0)\eta^\beta D_r^k. \quad (1.14)$$

By definition (1.14) takes values in differential operators of order μ on the half axis, in this case with constant coefficients, but depending on the parameters (y, η). For the moment we interpret the boundary symbol as a family of continuous operators

$$\sigma_\partial(A)(y, \eta) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+) \quad (1.15)$$

for $\mathcal{S}(\overline{\mathbb{R}}_+) := \mathcal{S}(\mathbb{R})|_{\mathbb{R}_+}$, with $\mathcal{S}(\mathbb{R})$ being the space of Schwartz functions on the real line. Setting

$$\kappa_\delta u(t) := \delta^{1/2} u(\delta t) \quad (1.16)$$

for $\delta > 0$ we obtain a group $\kappa := \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ of isomorphisms $\kappa_\delta : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$. Twisted homogeneity of order μ means

$$\sigma_\partial(A)(y, \delta\eta) = \delta^\mu \kappa_\delta \sigma_\partial(A)(y, \eta)\kappa_\delta^{-1} \tag{1.17}$$

for every $\delta > 0$. We interpret the pair

$$\sigma(A) := (\sigma_\psi(A), \sigma_\partial(A)) \tag{1.18}$$

as the principal symbolic hierarchy of the operator A, associated with the stratification of the underlying manifold with boundary $M := U \times \overline{\mathbb{R}}_+$

$$s(M) := (s_0(M), s_1(M)) \tag{1.19}$$

for $s_0(M) := U \times \mathbb{R}_+$, the open interior, and $s_1(M) := U$, the boundary. The notation comes from the general analysis on singular manifolds M, of singularity order $k \in \mathbb{N}$, characterized by a stratification

$$s(M) := (s_0(M), s_1(M), \ldots, s_k(M)), \tag{1.20}$$

cf. [63]. In this calculus the typical, in general degenerate, operators A have a principal symbolic hierarchy

$$\sigma(A) := (\sigma_0(A), \sigma_1(A), \ldots, \sigma_k(A)), \tag{1.21}$$

with $\sigma_j(A)$ being associated with $s_j(M)$, $j = 0, \ldots, k$. Therefore, later on instead of (1.18) we also write $\sigma(A) := (\sigma_0(A), \sigma_1(A))$.

The ellipticity of A with respect to σ_ψ now means $\sigma_\psi(A)(x, \xi) \neq 0$ for $\xi \neq 0$ up to $t = 0$. In order to formulate an analogue of the above-mentioned parametrix of A, here with a control of smoothing remainders up to the boundary, we have to be aware of different phenomena and difficulties at the same time. Assume that we consider an elliptic operator $\tilde{A}$ on the double $U \times \mathbb{R}$ such that $A = \tilde{A}|_{U\times\mathbb{R}_+}$ and form a parametric $\tilde{P} \in L_{\mathrm{cl}}^{-\mu}(U \times \mathbb{R})$, according to the constructions in the case without boundary. Then, although $P_{\mathrm{int}} := \tilde{P}|_{U\times\mathbb{R}_+}$ is certainly a parametric of A over the open interior $\Omega := U \times \mathbb{R}_+$ we cannot interpret so easily the compositions in $AP - 1$ or $PA - 1$, since we have to explain the action of P on functions occurring in (1.12). Therefore, we need a quantization near the boundary which guarantees a continuous operator

$$C_0^\infty(U \times \overline{\mathbb{R}}_+) \to C^\infty(U \times \overline{\mathbb{R}}_+). \tag{1.22}$$

In the present case, since we are starting with a differential operator $A = \tilde{A}$ the operator $\tilde{P}$ has the transmission property at the boundary $t = 0$, cf. the notation below. Then we set $P := \mathrm{r}^+\tilde{P}\mathrm{e}^+$ where

$$\mathrm{e}^+u = u \ \text{ for } \ t > 0, \quad \mathrm{e}^+u = 0 \ \text{ for } \ t \leq 0, \tag{1.23}$$

and

$$\mathrm{r}^+\tilde{u} = \tilde{u}|_{t>0}$$

for any distribution $\tilde{u}$ on $U \times \mathbb{R}$. Let $L_{\mathrm{tr}}^\mu(U \times \mathbb{R})$ be the space of all $\tilde{A} \in L_{\mathrm{cl}}^\mu(U \times \mathbb{R})$ with the transmission property at $t = 0$, and set

$$L_{\mathrm{tr}}^\mu(U \times \overline{\mathbb{R}}_+) := \{\mathrm{r}^+\tilde{A}\mathrm{e}^+ : \tilde{A} \in L_{\mathrm{tr}}^\mu(U \times \mathbb{R})\}. \tag{1.24}$$

We will assume here $\mu \in \mathbb{Z}$. For general operators $\tilde{A} \in L^{\mu}_{\mathrm{cl}}(U \times \mathbb{R})$ that have not necessarily the transmission property we admit arbitrary orders $\mu \in \mathbb{R}$. This case can be interpreted below as a part of the calculus on a manifold with edge $t = 0$. For an $A = \mathrm{r}^{+}\tilde{A}\mathrm{e}^{+} \in L^{\mu}_{\mathrm{tr}}(U \times \overline{\mathbb{R}}_{+})$ the boundary symbol is defined as

$$\sigma_{\partial}(A)(y,\eta) := \mathrm{r}^{+}\mathrm{op}(\sigma_{\psi}(\tilde{A})|_{t=0})(y,\eta)\mathrm{e}^{+} \tag{1.25}$$

for $\eta \neq 0$. Here, for $b(t,\tau) \in S^{\mu}(\mathbb{R}_{t} \times \mathbb{R}_{\tau})$ we write

$$\mathrm{op}(b)v(t) := \iint e^{i(t-t')\tau} b(t,\tau)v(t')dt'đ\tau. \tag{1.26}$$

In (1.26) we employ the fact that $f_{(\mu)}(y,t,\eta,\tau) \in S^{(\mu)}(U \times \mathbb{R} \times (\mathbb{R}^{n}_{\eta,\tau} \setminus \{0\}))$ entails $f_{(\mu)}(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_{t} \times \mathbb{R}_{\tau})$ for every fixed $y \in U$, $\eta \in \mathbb{R}^{n-1} \setminus \{0\}$. Later on we also write

$$\mathrm{op}^{+}(b)(y,\eta) := \mathrm{r}^{+}\mathrm{op}(b)(y,\eta)\mathrm{e}^{+}. \tag{1.27}$$

Note that pseudo-differential operators (1.25) in truncation quantization belong to history of Mellin operators on a cone which is here the half-axis, cf.[12] for $\mu = 0$ and in $L^{2}(\mathbb{R}_{+})$ and [42] in higher dimensions.

1.2 Fourier and Mellin transform

We first prepare some material on the Fourier and the Mellin transform, including some basic on Sobolev spaces and theorems of Palay-Wiener type. The Fourier transform in $\mathbb{R}^{n}$ is used in the form

$$Fu(\xi) := \int_{\mathbb{R}^{n}} e^{-ix\xi}u(x)dx$$

where $x\xi = \sum_{j=1}^{n} x_{j}\xi_{j}$ with the inverse

$$(F^{-1}g)(x) := \int_{\mathbb{R}^{n}} e^{ix\xi}g(\xi)đ\xi$$

for $đ\xi := (2\pi)^{-n}d\xi$, We also write $\hat{u}(\xi) = Fu(\xi)$. If necessary the involved vanishes are indicated as $F_{x\to\xi}$ and $F^{-1}_{\xi\to x}$ respectively. We systematically use the fact that F induces isomorphisms

$$F : \mathcal{S}(\mathbb{R}^{n}) \to \mathcal{S}(\mathbb{R}^{n}), \tag{1.28}$$

where $\mathcal{S}(\mathbb{R}^{n})$ is the Schwartz space which is Fréchet in the following semi-norm system, i.e.,

$$\mathcal{S}(\mathbb{R}^{n}) := \{u \in C^{\infty}(\mathbb{R}^{n}) : \sup_{x\in\mathbb{R}^{n}}|x^{\alpha}D^{\beta}_{x}u(x)| < \infty \text{ for all } \alpha,\beta \in \mathbb{N}^{n}\}$$

and

$$F : \mathcal{S}'(\mathbb{R}^{n}) \to \mathcal{S}'(\mathbb{R}^{n}), \tag{1.29}$$

with $\mathcal{S}'(\mathbb{R}^{n})$ being the space of tempered distributions in $\mathbb{R}^{n}$. Moreover, (1.1) restricts to an isomorphism

$$F : L^{2}(\mathbb{R}^{n}_{x}, dx) \to L^{2}(\mathbb{R}^{n}_{\xi}, đ\xi)$$

where dx means the Lebesgue measure in $\mathbb{R}^n$ and $đ\xi := (2\pi)^{-n}d\xi$ for the Lebesgue measure $d\xi$ in $\mathbb{R}^n$. More precisely we have

$$\int u(x)\overline{v(x)}dx = \int (Fu)(\xi)\overline{(Fv)(\xi)}đ\xi$$

which is first valid on Schwartz functions u, v and then extends to the respective L^2- spaces.

Definition 1.1. *The Sobolev space $H^s(\mathbb{R}^n)$ of smoothness $s \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm*

$$\|u\|_{H^s(\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |(Fu)(\xi)|^2 đ\xi \Big\}^{1/2} \tag{1.30}$$

for $\langle\xi\rangle := (1 + |\xi|^2)^{1/2}$ and with the scalar product

$$\langle u, v\rangle_{H^s(\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} (Fu)(\xi)\overline{(Fv)(\xi)}đ\xi \Big\}.$$

Alternatively we can define $H^s(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ with respect to the norm (1.30).The space $H^s(\mathbb{R}^n)$ can be equivalently characterized as the set of all $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $(Fu)(\xi)$ is locally integrable and (1.30) is finite. Because of $H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ for every $s \in \mathbb{R}$, we can talk about the support of Sobolev distributions. Let us set

$$\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

and define

$$H_0^s(\overline{\mathbb{R}}^n_+) := \{u \in H^s(\mathbb{R}^n) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}^n_+\}.$$

Moreover, we define

$$H^s(\mathbb{R}^n_+) := \{u|_{\mathbb{R}^n_+} : u \in H^s(\mathbb{R}^n)\}$$

In the following we often write $x' = (x_1, \ldots, x_{n-1})$ and $\xi' = (\xi_1, \ldots, \xi_{n-1})$.

Theorem 1.2.

(i) *Let $f \in H_0^s(\overline{\mathbb{R}}^n_+)$ Then $\hat{f}(\xi', \xi_n + i\sigma) = F(fe^{x_n\sigma})$ is a continues function in $\sigma \le 0$ with values in $FH^s(\mathbb{R}^n)$, for almost all $\xi' \in \mathbb{R}^{n-1}$. The function $f(\xi', \xi_n + i\sigma)$ is holomorphic in $\xi_n + i\sigma$ in the half -plane $\sigma > 0$, and we have the estimate*

$$\int_{-\infty}^{+\infty} |\hat{f}(\xi', \xi_n + i\sigma)|^2 (1 + |\xi'| + |\xi_n| + |\sigma|)^{2s} d\xi' d\xi_n \le C \tag{1.31}$$

for all $\sigma \le 0$ for a constant C which is independent of σ.

(ii) *Conversely let a locally integrable function $\hat{f}(\xi', \xi_n + i\sigma)$ over $\mathbb{R}^n \times (-\infty, 0)$ be given for $\sigma < 0$ satisfying the estimate (1.31) and assume that this function is holomorphic in $\xi_n + i\sigma$ for almost every $\xi' \in \mathbb{R}^{n-1}$. Then there exists an $f \in H_0^s(\overline{\mathbb{R}}^n_+)$ such that $\hat{f}(\xi', \xi_n + i\sigma) = F(fe^{x_n\sigma})$*

For the proof see [12].
Let us now turn to the Mellin transform on $\mathbb{R}_+$ which is an analogue of the Fourier transform. We use the Mellin transform is motivated by degenerate operators and program to control solution of degenerate equation up to $r = 0$.(see the Subsection 2.1)
The definition on functions $u \in C_0^\infty(\mathbb{R}_+)$ is

$$Mu(w) = \int_0^\infty r^{w-1}u(r)dr, \tag{1.32}$$

for the covariable $w \in \mathbb{C}$. Observe that

$$M(r^b u)(w) = (Mu)(w+b) \tag{1.33}$$

for any $b \in \mathbb{C}$ and $M^{-1}wM = -r\frac{\partial}{\partial_r}$. In fact,

$$M(-r\frac{\partial}{\partial_r}u)(w) = \int_0^\infty r^{w-1}(-r\frac{\partial}{\partial_r}u)(r)dr = \int_0^\infty \frac{\partial}{\partial_r}(r^w u(r))dr = w(Mu)(w).$$

For an open set $G \subseteq \mathbb{C}$, let $\mathcal{A}(G)$ be the space of all holomorphic functions in G, considered in the Fréchet topology of uniform convergence on compact subsets. Then M induces a continuous operator

$$M : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C}) \tag{1.34}$$

Setting

$$\Gamma_\beta := \{w \in \mathbb{C} \ : \mathrm{Re}\, w = \beta\}$$

For $u \in L^2(\mathbb{R}_+)$ such that also $r\partial_r u \in L^2(\mathbb{R}_+)$. Here, for the moment, the Mellin transform is regarded as an isomorphism

$$M : L^2(\mathbb{R}_+) \longrightarrow L^2(\Gamma_{1/2}).$$

For any $\gamma \in \mathbb{R}$ we set

$$(M_\gamma u)(w) := Mu(\frac{1}{2} - \gamma + i\rho), \tag{1.35}$$

called the weighted Mellin transform with weight γ. First for $u \in C_0^\infty(\mathbb{R}_+)$ where $Mu(w)$ is an entire function, $M_\gamma u(w) \in \mathcal{S}(\Gamma_{1/2-\gamma})$, and later on extended as an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \longrightarrow L^2(\Gamma_{1/2-\gamma}),$$

with the inverse

$$M_\gamma^{-1}g(r) = \int_{\Gamma_{1/2-\gamma}} r^{-w}g(w)đw, \ đw := (2\pi i)^{-1}dw. \tag{1.36}$$

By $\hat{H}^s(\mathbb{R}_\rho), \ s \in \mathbb{R}$, we denote the image of $H^s(\mathbb{R})$ (the standard Sobolev space on $\mathbb{R}$ of smoothness s) under the one-dimensional Fourier transform. Moreover, define $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ as the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm

$$\Big\{\int_{\Gamma_{\frac{1}{2}-\gamma}} \langle w\rangle^{2s}|Mu(w)|^2 đw\Big\}^{1/2}, \ đw = (2\pi i)^{-1}dw.$$

Note that the weighted Mellin transform M_γ defines an isomorphism

$$M_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}) \tag{1.37}$$

for every $s, \gamma \in \mathbb{R}$, where $\Gamma_{\frac{1}{2}-\gamma}$ on the right hand side is identified with $\mathbb{R}_\rho$ via $w = \frac{1}{2}-\gamma+i\rho \to \rho$. Thus, if $f(w)$ is a symbol on $w \in \Gamma_{\frac{1}{2}-\gamma}$ of order μ, i.e., $f(\frac{1}{2}-\gamma+i\rho) \in S^\mu(\mathbb{R}_\rho)$, the multiplication by f induces a continuous operator

$$f : \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}) \to \hat{H}^{s-\mu}(\Gamma_{\frac{1}{2}-\gamma}).$$

It follows that

$$\mathrm{Op}_M^\gamma(f) := M_\gamma^{-1} f(w) M_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+) \tag{1.38}$$

is a continuous operator for every $s \in \mathbb{R}$ below.
There is a relationship between the Fourier transform on $\mathbb{R}$ and the Mellin transform on $\mathbb{R}_+$. In fact, for $w = i\rho$, $\rho \in \mathbb{R}$, we have in the case $\gamma = 1/2$

$$\begin{aligned} M_{1/2}u(i\rho) &= \int_0^\infty r^{i\rho} u(r)\frac{dr}{r} = \int_0^\infty e^{i\rho \log r} u(r)\frac{dr}{r} \\ &= \int_{-\infty}^{+\infty} e^{-i\rho t} u(e^{-t})dt = F_{t\to\rho}((\chi^{-1})^* u)(\rho) \end{aligned}$$

with $(\chi^{-1})^*$ denoting the pull back of functions under $\chi^{-1} : \mathbb{R} \to \mathbb{R}_+$ for $\chi(r) = -\log r = t$.
If $\Psi_{1/2} : \mathbb{R} \to \Gamma_0$, $\rho \to i\rho$, is the identification of $\mathbb{R}_\rho$ with Γ_0 then $M_{1/2}u(i\rho) = F((\chi)^{-1})^* u(\rho)$. More generally, let

$$(S_\gamma u)(t) := e^{-(\frac{1}{2}-\gamma)t} u(e^{-t}), \tag{1.39}$$

with the inverse

$$(S_\gamma^{-1} v)(r) := r^{\gamma-\frac{1}{2}} v(\log r)$$

then

$$M_\gamma u(\frac{1}{2} - \gamma + i\rho) = \int_0^\infty r^{\frac{1}{2}-\gamma+i\rho} u(r)\frac{dr}{r} = \int_{\mathbb{R}} e^{-i\rho t} e^{-(\frac{1}{2}-\gamma)t} u(e^{-t})dt = (FS_\gamma u)(\rho).$$

Thus $\Psi_\gamma^* M_\gamma u = FS_\gamma$.

Definition 1.3. *The weighted Mellin-Fourier Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ of smoothness $s \in \mathbb{R}$ and weighted $\gamma \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm*

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n} \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \langle w, \xi\rangle^{2s} |(M_{r\to w} F_{x\to\xi} u)(w,\xi)|^2 đw đ\xi \Big\}^{1/2} \tag{1.40}$$

Here $M_{r\to w}$ means the Mellin transform acting on $r \in \mathbb{R}_+$, with the covariable while $M_{w\to r}^{-1}$ means the inverse. We also use notation like $M_{r,\delta\to w}$ and $M_{\gamma,w\to r}^{-1}$, respectively.

Remark 1.4. *The operator*

$$S_{\gamma-n/2} : C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R} \times \mathbb{R}^n)$$

$$S_{\gamma-n/2} : u(r,x) \to e^{-(\frac{n+1}{2}-\gamma)t} u(e^{-t}, x)$$

extends to an isomorphism

$$S_{\gamma-n/2} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n) \to H^s(\mathbb{R} \times \mathbb{R}^n)$$

for all standard Sobolev space given in the Definition 1.1

In fact, the latter observation is a consequence of (1.39). Here the space $H^s(\mathbb{R}\times\mathbb{R}^n)$ is taken as the completion of $C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ with respective to the norm

$$\|u\|_{H^s(\mathbb{R}\times\mathbb{R}^n)} := \Big\{ \int_{\mathbb{R}^n}\int_{\mathbb{R}} \langle\tau,\xi\rangle^{2s} |(F_{(t,x)\to(\tau,\xi)}u(\tau,\xi)|^2 đ\tau đ\xi \Big\}^{1/2}$$

where

$$F_{(t,x)\to(\tau,\xi)}u(\tau,\xi) = \int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-it\tau}e^{-ix\xi}u(t,x)dtdx = F_{t\to\tau}F_{x\to\xi}u(\tau,\xi).$$

Let us formulate an analogue of the Palay-Wiener theorem for the Mellin transform .

To this end for any $a>0$ we set

$$\mathcal{H}^{s,\gamma}_{(0,a)}(\mathbb{R}_+\times\mathbb{R}^n) := \Big\{u\in\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n) : u=0 \text{ for } r>a\Big\},$$

$$\mathcal{H}^{s,\gamma}_{(a,\infty)}(\mathbb{R}_+\times\mathbb{R}^n) := \Big\{u\in\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n) : u=0 \text{ for } 0<r<a\Big\}$$

Theorem 1.5.

(i) *For $u(r,x)\in\mathcal{H}^{s,\gamma}_{(0,a)}(\mathbb{R}_+\times\mathbb{R}^n),\ s,\gamma\in\mathbb{R}, a\in\mathbb{R}_+,$ the function*

$$\tilde{v}(z,\xi) := (M_{r-n/2,r\to z}F_{x\to\xi}u)(\frac{n+1}{2}-\gamma+i\rho,\xi),\ z=\frac{n+1}{2}-\gamma+i\rho, \tag{1.41}$$

extends to a holomorphic function in $\beta+i\rho\in\{z\in\mathbb{C}:\beta>\frac{n+1}{2}-\gamma\}$ with values in $\hat{H}^s(\mathbb{R}^n_\xi) := F_{x\to\xi}H^s(\mathbb{R}^n_x)$ belonging to the space $C(\beta\geq\frac{n+1}{2}-\gamma,\hat{H}^s(\mathbb{R}^n_\xi))$ such that

$$\iint(1+|\beta|+|\rho|+|\xi|)^{2s}|\tilde{v}(\beta+i\rho,\xi)|^2d\rho d\xi\leq Ca^\beta, \tag{1.42}$$

for a $C>0$ independent of $\beta\geq(n+1)/2-\gamma$.

(ii) *Conversely, let $\tilde{v}(\beta+i\rho,\xi)$ be a function that is locally integrable in (ρ,ξ) and holomorphic in $\beta+i\rho$ for $\beta>\frac{n+1}{2}-\gamma$ for almost all $\xi\in\mathbb{R}^n$ and satisfying the estimate (1.42) for a constant $C>0$ independent of β. Then there exists a function $a(r,x)\in\mathcal{H}^{s,\gamma}_{(0,a)}(\mathbb{R}_+\times\mathbb{R}^n)$ such that the relation (1.41) holds.*

For purpose below we explicitly formulate the counterpart of Theorem 1.5.

Theorem 1.6.

(i) *For $u(r,x)\in\mathcal{H}^{s,\gamma}_{(a,\infty)}(\mathbb{R}_+\times\mathbb{R}^n),\ s,\gamma\in\mathbb{R}, a\in\mathbb{R}_+,$ the function*

$$\tilde{v}(z,\xi) := (M_{r-n/2,r\to z}F_{x\to\xi}u)(\frac{n+1}{2}-\gamma+i\rho,\xi),\ z=\frac{n+1}{2}-\gamma+i\rho, \tag{1.43}$$

extends to a holomorphic function in $\beta+i\rho\in\{z\in\mathbb{C}:\beta<\frac{n+1}{2}-\gamma\}$ with values in $\hat{H}^s(\mathbb{R}^n_\xi) := F_{x\to\xi}H^s(\mathbb{R}^n_x)$ belonging to the space $C(\beta\leq\frac{n+1}{2}-\gamma,\hat{H}^s(\mathbb{R}^n_\xi))$ such that

$$\iint(1+|\beta|+|\rho|+|\xi|)^{2s}|\tilde{v}(\beta+i\rho,\xi)|^2d\rho d\xi\leq Ca^\beta, \tag{1.44}$$

for a $C>0$ independent of $\beta\leq(n+1)/2-\gamma$.

(ii) *Conversely, let* $\tilde{v}(\beta+i\rho,\xi)$ *be a function that is locally integrable in* (ρ,ξ) *and holomorphic in* $\beta+i\rho$ *for* $\beta<\frac{n+1}{2}-\gamma$ *for almost all* $\xi\in\mathbb{R}^n$ *and satisfying the estimate* (1.44) *for a constant* $C>0$ *independent of* β. *Then there exists a function* $a(r,x)\in\mathcal{H}^{s,\gamma}_{(a,\infty)}(\mathbb{R}_+\times\mathbb{R}^n)$ *such that the relation* (1.43) *holds.*

Another of the Palay-Wiener theorem refers to L^2-spaces with values in a Hilbert space H.

Theorem 1.7. *Let* M_γ, $\gamma\in\mathbb{R}$ *be the weighted Mellin transform* $M_\gamma: u(r)\to L^2(\Gamma_{\frac{1}{2}-\gamma},H)$ *for* $u(r)\in r^\beta L^2(\mathbb{R}_+,H)$. *Then the following conditions are equivalent.*

(i) $h(w)=M_\gamma u(w)$ *for some* $u(r)\in r^\beta L^2(\mathbb{R}_+,H), \operatorname{supp} u\subseteq[0,a], a>0;$

(ii) h *is holomorphic in* $\operatorname{Re} w>\frac{1}{2}-\gamma$; *moreover,* $h_\delta(\rho):=h(\frac{1}{2}-\gamma+\delta+i\rho)$ *belongs to* $L^2(\mathbb{R}_\rho,H)$ *for every* $\delta>0$ *and there are constants* a, $c>0$ *such that*

$$\|h\|_{L^2(\mathbb{R}_\rho,H)}\le ca^\delta$$

for all $\delta\in\mathbb{R}_+$. *Under the condition* (i), (ii) *we have*

$$\lim_{\delta\to0}h_\delta=h_0$$

in $L^2(\mathbb{R}_\rho,H)$. *Moreover,*

$$\|h_\delta\|_{L^2(\mathbb{R}_\rho,H)}=(2\pi)^{1/2}\|r^\delta u(r)\|_{r^\gamma L^2(\mathbb{R}_r,H)}=(2\pi)^{1/2}\|r^{\delta-\gamma}u(r)\|_{L^2(\mathbb{R}_r,H)},\ \delta\in\overline{\mathbb{R}}_+.$$

1.3 Symbols and kernel cut-off with respect to the Fourier transform

In this subsection we will discuss operator-valued symbols with values in $\mathcal{L}(H,\widetilde{H})$ for Hilbert spaces H and $\widetilde{H}$. The space $\mathcal{L}(H,\widetilde{H})$ denote the space of continuous operator $H\to\widetilde{H}$, equipped with the stronger operator topology. The Hilbert spaces are endowed with group actions in the following sense. A group action $\kappa:=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ in a Hilbert spaces H is a family of isomorphisms

$$\kappa_\delta: H\to H,\quad \delta\in\mathbb{R}_+,$$

such that $\kappa_\delta\kappa_{\delta'}=\kappa_{\delta\delta'}$ for every δ, $\delta'\in\mathbb{R}_+$, and the map $\delta\to\kappa_\delta h$ defines a function in $C(\mathbb{R}_+,H)$ for every $h\in H$.
A Fréchet space E, written as a projective limit of Hilbert spaces $E=\varprojlim_{j\in\mathbb{N}}E^j$ with continuous embeddings $E^j\hookrightarrow E^0$ for all j, is said to be endowed with a group action κ if it acts on E^0 in the above sense, and $\kappa|_{E^j}=\{\kappa_\delta|_{E^j}\}_{\delta\in\mathbb{R}_+}$ is a group action on E^j for every j.

Example 1.8. (i) *Let* $H=L^2(\mathbb{R}_+)$ *and* $(\kappa_\delta u)(r)=\delta^{1/2}u(\delta r)$, $\delta\in\mathbb{R}_+$. *Then* $\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ *is a group action on* $L^2(\mathbb{R}_+)$.

(ii) *Let* $H:=H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, *then*

$$(\kappa_\delta u)(x):=\delta^{n/2}u(\delta x),\ \delta\in\mathbb{R}_+ \tag{1.45}$$

defines a group action on $H^s(\mathbb{R}^n)$ *and* $F\kappa_\delta u(\xi)=\kappa_\delta^{-1}Fu(\xi)$. *More generally,* $\langle x\rangle^g H^s(\mathbb{R}^n)$ *for* $s,g\in\mathbb{R}$ *is a Hilbert space with group action* (1.45) .

Proof. The proof of (i) is very clear for $\delta^{1/2}u(\delta x) \in L^2(\mathbb{R}_+)$, $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}$ for every $\delta,\ \delta' \in \mathbb{R}_+$ and κ_δ is unitary in $L^2(\mathbb{R}_+)$ for every $\delta \in \mathbb{R}_+$ with $\kappa_\delta^{-1}u(y) = \delta^{-1/2}u(\delta y)$.
Next we show (ii) according to the following steps.
(1) If $u \in H^s(\mathbb{R}^n)$ then prove $\kappa_\delta u \in H^s(\mathbb{R}^n)$. Since

$$\begin{aligned} F\kappa_\delta u(\xi) &= \int e^{-ix\xi}\kappa_\delta u(x)dx \\ &= \int e^{-ix\xi}\delta^{\frac{n}{2}}u(\delta x)dx = \int e^{-i\delta^{-1}y\xi}\delta^{\frac{n}{2}}u(y)\delta^{-n}dy \\ &= \delta^{-\frac{n}{2}}\int e^{-i\delta^{-1}y\xi}u(y)dy = \delta^{-\frac{n}{2}}Fu(\delta^{-1}\xi) \end{aligned} \tag{1.46}$$

so

$$\begin{aligned} \|\kappa_\delta u\|_{H^s(\mathbb{R}_n)} &= \int_{\mathbb{R}_n} \langle\xi\rangle^{2s}|F\kappa_\delta u(\xi)|^2 dbar\xi = \int_{\mathbb{R}_n} \langle\xi\rangle^{2s}\delta^{-n}|Fu(\delta^{-1}\xi)|^2 đ\xi \\ &= \int_{\mathbb{R}_n} \langle\delta\eta\rangle^{2s}|Fu(\eta)|^2 d\eta \leq C\|u\|_{H^s(\mathbb{R}_n)} < \infty \end{aligned} \tag{1.47}$$

according to when δ belong to the closed interval there exist constant C such that $\langle\delta\eta\rangle^{2s} \leq C\langle\eta\rangle^{2s}$. Then $\kappa_\delta u \in H^s(\mathbb{R}^n)$.
(2) It is very clearly that

$$\kappa_\delta\kappa_{\delta'}u(x) = \kappa_\delta(\delta')^{n/2}u(\delta' x)) = \delta^{n/2}(\delta')^{n/2}u(\delta\delta' x)) = \kappa_{\delta\delta'}u(x).$$

(3)κ_δ is unitary in $H^s(\mathbb{R}_n)$ for every $\delta \in \mathbb{R}_+$ with $\kappa_\delta^{-1}u(y) = \delta^{-n/2}u(\delta^{-1}y)$.
(4) According to the Fourier transform we have

$$\begin{aligned} F\kappa_\delta u(\xi) &= \int e^{-ix\xi}\delta^{n/2}u(\delta x)dx \\ &= \int e^{-i\delta^{-1}x'\xi}\delta^{-n/2}u(x')dx' \\ &= \delta^{-n/2}Fu(\delta^{-1}\xi) \\ &= \kappa_\delta^{-1}Fu(\xi) \end{aligned} \tag{1.48}$$

□

The following property is well-known.

Proposition 1.9. *Let H be a Hilbert space with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. Then here are constants $C,\ M > 0$ such that*

$$\|\kappa_\delta\|_{\mathcal{L}(H)} \leq C\left(\max\{\delta,\delta^{-1}\}\right)^M.$$

Proof. By assumption for every $h \in H$ the function $\delta \to \kappa_\delta h$ belongs to $C(\mathbb{R}_+, H)$, thus the set $\{\kappa_\delta h : \delta \in [\alpha,\beta]\}$ is a bounded set in H for every compact interval $[\alpha,\beta] \subset \mathbb{R}$. From the Banach-Steinhaus theorem we have for some constant $C \geq 1$ such that

$$\sup_{\delta\in[\alpha,\beta]}\|\kappa_\delta\|_{\mathcal{L}(H)} \leq C.$$

In particular,

$$\sup_{\delta\in[e^{-1},e]}\|\kappa_\delta\|_{\mathcal{L}(H)} \leq C$$

it follows that

$$\|\kappa_{\delta^n}\|_{\mathcal{L}(H)} \leq \|\kappa_\delta\|^n_{\mathcal{L}(H)} \leq C^n,$$

i.e., for all $n \in \mathbb{N}$ and $\delta \in [e^{-n}, e^n]$ we obtain

$$\|\kappa_\delta\|_{\mathcal{L}(H)} \leq C^n.$$

Thus for $\delta_n := e^n > 1$, $n \in \mathbb{N}$ we have

$$\|\kappa_{\delta_n}\|_{\mathcal{L}(H)} \leq C^n = e^{n\log C} = {\delta_n}^{\log C}.$$

Then for $\delta \in [e^{n-1}, e^n]$ there is a constant $\delta_1 \in [1, e]$ such that $\delta = \delta_1 e^{n-1} = \delta_1\delta_{n-1}$ and

$$\begin{aligned}\|\kappa_\delta\|_{\mathcal{L}(H)} &= \|\kappa_{\delta_1\delta_{n-1}}\|_{\mathcal{L}(H)} \\ &\leq \|\kappa_{\delta_1}\|_{\mathcal{L}(H)}\|\kappa_{\delta_{n-1}}\|_{\mathcal{L}(H)} \leq C(\delta_{n-1})^{\log C} \\ &\leq C(\delta_1\delta_{n-1})^{\log C} = C\delta^{\log C}\end{aligned} \tag{1.49}$$

since the right hand side of the latter estimate is independent of n, we proved the asserted estimate for $M = \log C$, for all $\delta \geq 1$. In a similar manner we can argue for $0 < \delta \leq 1$. Set $\delta_{-n} := e^{-n}, n \geq 1$, we have

$$\|\kappa_{\delta_{-n}}\|_{\mathcal{L}(H)} \leq C^n = (\delta^{-1}_{-n})^M.$$

Moreover for $\delta \in [e^{-n}, e^{-(n-1)}]$ there is a constant $\delta_2 \in [e^{-1}, 1]$ such that $\delta = \delta_2\delta_{-(n-1)}$. This gives us

$$\begin{aligned}\|\kappa_\delta\|_{\mathcal{L}(H)} &= \|\kappa_{\delta_2\delta_{-(n-1)}}\|_{\mathcal{L}(H)} \\ &\leq \|\kappa_{\delta_2}\|_{\mathcal{L}(H)}\|\kappa_{\delta_{-(n-1)}}\|_{\mathcal{L}(H)} \leq C(\delta^{-1}_{-(n-1)})^{\log C} \\ &\leq C(\delta_2^{-1}\delta^{-1}_{-(n-1)})^{\log C} = C(\delta^{-1})^M\end{aligned} \tag{1.50}$$

which corresponds to the assertion for $0 < \delta \leq 1$. □

Similar as scalar symbols of Hörmander's classes, for two Hilbert spaces H and $\widetilde{H}$ with group action κ and $\tilde{\kappa}$, respectively, we now formulate operator-valued symbols

$$S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H}) \subset C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$$

for $\mu \in \mathbb{R}$, an open set $\Omega \subseteq \mathbb{R}^p$ and $\mathcal{L}(H, \widetilde{H}))$ equipped with the operator-norm topology, motivated by the following observation.
Let $p(y, \xi, \eta) \in S^\mu(\Omega \times \mathbb{R}^{n+q}_{\xi,\eta})$ for an open set $\Omega \subseteq \mathbb{R}^p$. Then we have a family of continues operators

$$\mathrm{Op}_x(p)(y, \eta) : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n),\ s \in \mathbb{R} \tag{1.51}$$

for all the standard Sobolev spaces in $\mathbb{R}^n$ endowed with the group action

$$\kappa_\delta : u(x) \to \delta^{n/2}u(\delta x),\ \delta \in \mathbb{R}_+$$

where

$$\mathrm{Op}_x(p)(y, \eta)u(x) := \iint e^{i(x-x')\xi}p(y, \xi, \eta)u(x')dx'đ\xi$$

The latter continuity employs the fact that $p(y_0, \xi, \eta_0) \in S^\mu(\mathbb{R}^n_\xi)$ for every fixed $(y_0, \eta_0) \in \Omega \times \mathbb{R}^q$. Then the space $H := H^s(\mathbb{R}^n), \widetilde{H} := H^{s-\mu}(\mathbb{R}^n)$, are endowed with group actions κ and $\tilde{\kappa}$, both defined by the formula (1.45). Then

$$a(y, \eta) := \mathrm{Op}_x(p)(y, \eta)$$

satisfies the estimates

$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y, \eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\, \langle\eta\rangle^{\mu-|\beta|} \tag{1.52}$$

for all $(y, \eta) \in K \times \mathbb{R}^q, K \Subset \Omega,\ \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for some constant $c = c(\alpha, \beta, K) > 0$. Now we generalize the definition of operator-valued symbol as follows:

Definition 1.10.

(i) *Let H and $\widetilde{H}$ be Hilbert space with group action κ and $\tilde{\kappa}$, respectively, the space*

$$S^\mu(\Omega \times \mathbb{R}^q, H, \widetilde{H})$$

for $\mu \in \mathbb{R}$ and open $\Omega \subseteq \mathbb{R}^p$ is defined as the space of all $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ satisfying the estimates (1.52) *for all $(y, \eta) \in K \times \mathbb{R}^q, K \Subset \Omega, \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^q$, for constants $c = c(\alpha, \beta, K) > 0$.*

(ii) *For $\nu \in \mathbb{R}$, we set*

$$\begin{aligned} S^{(\nu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}) := \{ f_{(\nu)}(y, \eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \widetilde{H})) :\\ f_{(\nu)}(y, \delta\eta) = \delta^\nu \tilde{\kappa}_\delta f_{(\nu)}(y, \eta) \kappa^{-1}_\delta \quad \text{for all} \quad \delta > 0\}. \end{aligned} \tag{1.53}$$

(iii) *We define*

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$

the space of classical symbols as the set of all $a(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ such that there are homogeneous components

$$a_{(\mu-j)}(y, \eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}),$$

$j \in \mathbb{N}$, such that for any excision function $\chi(\eta)$ we have

$$r_N(a)(y, \eta) := a(y, \eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y, \eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$

for every $N \in \mathbb{N}$.

Note that Hörmander's symbols in $S^\mu(\Omega \times \mathbb{R}^q)$ correspond to the case $H = \widetilde{H} = \mathbb{C}$ and $\kappa_\delta = \tilde{\kappa}_\delta = \mathrm{id}_{\mathbb{C}}$ for all $\delta \in \mathbb{R}_+$.
It also makes sense to consider the potential symbols correspond to the case $H = \mathbb{C}$ and trace symbols correspond to the case $\widetilde{H} = \mathbb{C}$

$$S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathbb{C}, \widetilde{H}) \text{ and } S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \mathbb{C}),$$

respectively.

Remark 1.11. *Let* $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q; \mathcal{L}(H,\widetilde{H}))$ *have the property* $a(y,\delta\eta) = \delta^\mu \tilde{\kappa}_\delta a(y,\eta)\kappa_\delta^{-1}$ *for all* $|\eta| \geq C,\ \delta \geq 1$, *for some* $C > 0$. *Then* $a(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$. *In particular, for any excision function* $\chi(\eta)$ *we have* $\chi(\eta)S^\mu(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H}) \subset S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$.

Proof. According to the property, we have the identity

$$a(y,\eta) = \delta^{-\mu}\tilde{\kappa}_\delta^{-1} a(y,\delta\eta)\kappa_\delta.$$

It follows that by replaced δ by $\langle\eta\rangle$ and η by $\frac{\eta}{|\eta|}$ for all $|\eta| \geq C,\ \delta \geq 1$, for some $C > 0$

$$a(y,\frac{\eta}{|\eta|}) = \langle\eta\rangle^{-\mu}\tilde{\kappa}_{\langle\eta\rangle}^{-1} a(y,\eta)\kappa_{\langle\eta\rangle}.$$

Since $\|\frac{\eta}{|\eta|}\| = 1$ and $a(y,\frac{\eta}{|\eta|}) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}); \mathcal{L}(H,\widetilde{H}))$ there is a constant $C > 0$ such that $\|a(y,\frac{\eta}{|\eta|})\| \leq C$, therefore

$$\|\tilde{\kappa}_{\langle\eta\rangle}^{-1} a(y,\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq C\langle\eta\rangle^\mu.$$

We get the assertion for the case $\beta = 0$. Next, for arbitrary β, we have

$$D^\beta_\eta a(y,\delta\eta) = \delta^{|\beta|}(D^\beta_\eta a)(y,\delta\eta)$$

and

$$D^\beta_\eta a(y,\delta\eta) = \delta^\mu\tilde{\kappa}_\delta(D^\beta_\eta a(y,\delta\eta))\kappa_\delta^{-1},$$

so

$$\|\tilde{\kappa}_\delta^{-1}(D^\beta_\eta a)(y,\delta\eta)\kappa_\delta\|_{\mathcal{L}(H,\widetilde{H})} = \delta^{\mu-|\beta|}\|(D^\beta_\eta a(y,\eta)\|_{\mathcal{L}(H,\widetilde{H})} \leq C\delta^{\mu-|\beta|}$$

In this similar manner as before, set $\delta = \langle\eta\rangle$ and $\eta = \frac{\eta}{|\eta|}$ we get the estimate of symbol $a(y,\eta)$

$$\|\tilde{\kappa}_{\langle\eta\rangle}^{-1}(D^\beta_\eta a)(y,\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq C\langle\eta\rangle^{\mu-|\beta|}.$$

□

If a consideration is valid both for general and classical symbols, we write subscript (cl). Let us set

$$S^{-\infty}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) = \bigcap_{\mu\in\mathbb{R}} S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$

Observe that the space (1.53) for any fixed ν is isomorphic to a closed subspace of

$$C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H,\widetilde{H}))$$

for $S^{q-1} = \{\eta \in \mathbb{R}^q \setminus \{0\} : |\eta| = 1\}$. In fact, $f_{(\nu)}(y,\eta) \in S^{(\nu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$ implies

$$f_{(\nu)}(y,\frac{\eta}{|\eta|}) \in C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H,\widetilde{H})),$$

and convergence in the Fréchet space $S^{(\nu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); H, \widetilde{H})$ is equivalent to the convergence of the associated restriction to the η -unit sphere. For any $b(y,\eta) \in C^\infty(\Omega \times S^{q-1}, \mathcal{L}(H,\widetilde{H}))$ we have

$$f_{(\nu)}(y,\eta) := |\eta|^\nu\tilde{\kappa}_{|\eta|} b(y,\frac{\eta}{|\eta|})\kappa_{|\eta|}^{-1} \in S^{(\nu)}(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H,\widetilde{H})). \tag{1.54}$$

The function (1.54) is called the extension by homogeneity ν.
By $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})$ we denote the subspace of symbols in $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ with constant coefficients $a(\eta)$.

Remark 1.12. *The space $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is a Fréchet space with the system of semi-norms*

$$\sup_{(y,\eta)\in K\times\mathbb{R}^q}\{\langle\eta\rangle^{-\mu+|\beta|}\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^{\alpha}_y D^{\beta}_{\eta} a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})}\} \tag{1.55}$$

where α, β, K is as in the Definition 1.10.

Observe that $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})$ is a closed subspace of $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$, and we have

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H}) = C^{\infty}(\Omega, S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})) = C^{\infty}(\Omega)\hat{\otimes}_{\pi} S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H}).$$

with $\hat{\otimes}_{\pi}$ indicating the projective tensor product between the involved Fréchet spaces.

There is a general characterization of the projective tensor product $E\hat{\otimes}_{\pi}F$ for Fréchet spaces, namely, that every $g \in E\hat{\otimes}_{\pi}F$ can be written as a convergent sum

$$g = \sum_{j=0}^{\infty} \lambda_j e_j \otimes f_j$$

for coefficients $\lambda_j \in \mathbb{C}$ with $\sum_{j=0}^{\infty}|\lambda_j| < \infty$ and $e_j \in E$, $f_j \in F$ tending to zero in the respective spaces as $j \to \infty$. In the present case every $a(y,\eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ has a representation

$$a(y,\eta) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(y) a_j(\eta), \tag{1.56}$$

for $\sum_{j=0}^{\infty}|\lambda_j| < \infty$ and $\varphi_j \in C^{\infty}(\Omega)$, $\varphi_j \to 0$, and $a_j \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^q; H, \widetilde{H})$, $a_j \to 0$, for $j \to \infty$.
The following result states asymptotic summation in the symbolic calculus.

Theorem 1.13. *Let $a_j(y,\eta) \in S^{\mu_j}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$, $j \in \mathbb{N}$, be an arbitrary sequence with $\mu_j \to -\infty$ as $j \to \infty$* ($\mu_j = \mu - j$ in the classical case). *Then there is an $a(y,\eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ for $\mu = \max_{j\in\mathbb{N}}\{\mu_j\}$ such that*

$$a(y,\eta) - \sum_{j=0}^{N} a_j(y,\eta) \in S^{\nu_N}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$

for every $N \in \mathbb{N}$ and a $\nu_N \in \mathbb{R}$ tending to $-\infty$ as $N \to \infty$. The symbol $a(y,\eta)$ is called an asymptotic sum of the $a_j(y,\eta)$, written

$$a(y,\eta) \sim \sum_{j=0}^{\infty} a_j(y,\eta)$$

If $\tilde{a}(y,\eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ is another asymptotic sum in that sense then we have $a(y,\eta) - \tilde{a}(y,\eta) \in S^{-\infty}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$.

Remark 1.14. *An asymptotic sum is the sense of Theorem* 1.13 *can be obtained in the form*

$$a(y,\eta) \sim \sum_{j=0}^{\infty} \chi(\frac{\eta}{c_j}) a_j(y,\eta)$$

converges in $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ *for a fixed excision functions* $\chi(\eta)$ *and constants* $0 < c_j \to \infty$ *sufficiently fast as* $j \to \infty$. *More precisely, the sum* $\sum_{j=N+1}^{\infty} \chi(\frac{\eta}{c_j}) a_j(y,\eta)$ *converges in* $S^{\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ *for every* $\nu \leq \mu$ *and suitable* $N = N(\nu)$.

A Fréchet spaces E, is said to be endowed with a group action $\kappa := \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ if E is written as a projective limit

$$E = \varprojlim_{j \in \mathbb{N}} E^j$$

of Hilbert spaces E^j, with continuous embedding $E^{j+1} \hookrightarrow E^0$ for all j, where κ is a group action on E^0 which restricts to a group action on E^j for every $j \in \mathbb{N}$.

Example 1.15. *The Schwartz space* $\mathcal{S}(\mathbb{R}^n)$ *can be written as a projective limit*

$$\mathcal{S}(\mathbb{R}^n) = \varprojlim_{j \in \mathbb{N}} \langle x \rangle^{-j} H^j(\mathbb{R}^n).$$

The group actions from Example 1.8 (ii) *turn* $\mathcal{S}(\mathbb{R}^n)$ *to a Fréchet spaces with group action.*

Definition 1.10 can be be extended to the case of Fréchet spaces

$$E = \varprojlim_{j \in \mathbb{N}} E^j \text{ and } \widetilde{E} = \varprojlim_{k \in \mathbb{N}} \widetilde{E}^k$$

with group actions κ and $\widetilde{\kappa}$, respectively. To this end we fix a map $r : \mathbb{N} \to \mathbb{N}$ and form the symbol spaces $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E^{r(k)}, \widetilde{E}^k)$. Then we set

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E, \widetilde{E}) := \bigcup_r \{ \bigcap_{k \in \mathbb{N}} S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; E^{r(k)}, \widetilde{E}^k) \}$$

where the union is taken over all maps $r : \mathbb{N} \to \mathbb{N}$.

An example is the case that E is equal to a Hilbert space H with group action but $\widetilde{E}$ is a Frećhet space with group action. Then

$$S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{E}) = \bigcap_{k \in \mathbb{N}} S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^q; H, \widetilde{E}^k).$$

In the following we consider parameter-dependent symbols

$$a(y,\eta,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{q+l}_{\eta,\lambda}; H, \widetilde{H}).$$

where $(\eta,\lambda) \in \mathbb{R}^{q+l}$ plays the role of covariable. For any

$$\varphi(\theta) \in C^{\infty}_{\mathrm{b}}(\mathbb{R}^q) := \{\varphi \in C^{\infty}(\mathbb{R}^q) : \sup_{\theta \in \mathbb{R}^q} |D^{\alpha}_{\theta} \varphi(\theta)| < \infty\}$$

we form the expression

$$V_\varphi a(y,\eta,\lambda) = \iint e^{-i\theta\tilde{\eta}}\varphi(\theta)a(y,\eta-\tilde{\eta},\lambda)d\theta đ\tilde{\eta}, \tag{1.57}$$

interpreted as an oscillatory integral.

The following kernel cut-off theorem will be formulated for symbols $a(\eta,\lambda)$. The generalisation to the case of symbols $a(y,\eta,\lambda)$ is straight forward and tacitly used below.

Theorem 1.16. *The kernel cut-off operator* (1.57)*,*

$$V:(\varphi,a)\mapsto V_\varphi a\,,$$

defines a bilinear continuous operator

$$V: C_{\mathrm{b}}^\infty(\mathbb{R}^q)\times S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+l}_{\eta,\lambda};H,\widetilde{H}) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^{q+l}_{\eta,\lambda};H,\widetilde{H}).$$

The symbol $V_\varphi a(\eta,\lambda)$ admits an asymptotic expansion

$$V_\varphi a(\eta,\lambda)\sim\sum_{\alpha\epsilon\mathbb{N}^q}\frac{(-1)^{|\alpha|}}{\alpha!}D_\theta^\alpha\varphi(0)\partial_\eta^\alpha a(\eta,\lambda).$$

1.4 Abstract edge spaces

Definition 1.17. *Let H be a Hilbert space with group action $\kappa=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q,H), s\in\mathbb{R}$, is defined as the completion of $C_0^\infty(\mathbb{R}^q,H)$ with respect to the nor-m*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \Big\{\int\langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|^2_H đ\eta\Big\}^{1/2}. \tag{1.58}$$

We get an equivalent norm when we replace $\langle\eta\rangle$ by $[\eta]$, where $\eta\to[\eta]$ denote any positive function in $C^\infty(H)$ such that $[\eta]=|\eta|$ for all $|\eta|\geq C$ for some $C>0$. So

$$u\to\Big\{\int[\eta]^{2s}\|\kappa^{-1}_{[\eta]}\hat{u}(\eta)\|^2_H đ\eta\Big\}^{1/2}$$

is a norm in $\mathcal{W}^s(\mathbb{R}^q,H)$, equivalent to (1.58).
If necessary we write $\mathcal{W}^s(\mathbb{R}^q,H)_\kappa$ instead of $\mathcal{W}^s(\mathbb{R}^q,H)$, in order to indicate the dependence of the abstract edge spaces on κ. We also admit the case $\kappa=\mathrm{id}$ where $\kappa_\delta=\mathrm{id}_H$ for every $\delta\in\mathbb{R}_+$. This gives us the standard Sobolev spaces with values in H, i.e.,

$$\mathcal{W}^s(\mathbb{R}^q,H)_{\mathrm{id}} := H^s(\mathbb{R}^q,H)$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^q,H)} := \Big\{\int\langle\eta\rangle^{2s}\|\hat{u}(\eta)\|^2_H đ\eta\Big\}^{1/2}.$$

So we also called the space $\mathcal{W}^s(\mathbb{R}^q,H)$ edge Sobolev spaces.
Moreover, we have

$$\mathcal{W}^\infty(\mathbb{R}^q,H)_\kappa = H^\infty(\mathbb{R}^q,H), \tag{1.59}$$

i.e., $\mathcal{W}^\infty(\mathbb{R}^q,H)_\kappa$ is independent of the choice of κ.
It can be proved that $\mathcal{W}^s(\mathbb{R}^q,H)$ is a subset of $\mathcal{S}'(\mathbb{R}^q,H)=\mathcal{L}(\mathcal{S}(\mathbb{R}^q),H)$.

Definition 1.18. *We define*

$$\mathcal{W}^s_{\text{comp}}(\Omega, H) \tag{1.60}$$

for open $\Omega \subseteq \mathbb{R}^q$ as the set of all $u \in \mathcal{D}'(\Omega, H)$ with compact support in Ω such that the extension by zero outside Ω belongs to $\mathcal{W}^s(\mathbb{R}^q, H)$.
Moreover,

$$\mathcal{W}^s_{\text{loc}}(\Omega, H)$$

can be defined as the set of all $u \in \mathcal{D}'(\Omega, H)$ such that $\varphi u \in \mathcal{W}^s_{\text{comp}}(\Omega, H)$ for every $\varphi \in C_0^\infty(\Omega)$.

Remark 1.19. *For $s \in \mathbb{R}$ we observe the space $\mathcal{W}^s(\mathbb{R}^q, H)$ also be taken as the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} := \|\langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle} \hat{u}(\eta)\|_{L^2(\mathbb{R}^q,H)} \tag{1.61}$$

where $\langle\eta\rangle = (1 + |\eta|^2)^{1/2}$, and

$$\|v\|_{L^2(\mathbb{R}^q,H)} := \Big\{ \int \|\hat{v}(\eta)\|^2_H đ\eta \Big\}^{1/2}, \; đ\eta = (2\pi)^{-q} d\eta.$$

Remark 1.20. *If for the Hilbert space $H_1 \subset H$, then we have $\mathcal{W}^s(\mathbb{R}^q, H_1) \subset \mathcal{W}^s(\mathbb{R}^q, H)$.*

Proof. we know from (1.58) of the Definition 1.17

$$\begin{aligned} \|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} &= \Big\{ \int \langle\eta\rangle^{2s} \|\hat{u}(\eta)\|^2_H đ\eta \Big\}^{1/2} \\ &\le c \Big\{ \int \langle\eta\rangle^{2s} \|\hat{u}(\eta)\|^2_{H_1} đ\eta \Big\}^{1/2} \\ &= c \|u\|_{\mathcal{W}^s(\mathbb{R}^q,H_1)} \end{aligned} \tag{1.62}$$

since $e : H_1 \to H$ continuous, and $\|eu\|_H \le c\|u\|_{H_1}$. ∎

Observe that the operator

$$K := F^{-1} \kappa_{\langle\eta\rangle} F$$

induces an isomorphism

$$K : H^s(\mathbb{R}^q, H) \to \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$$

for every $s \in \mathbb{R}$. In fact, we can write

$$\begin{aligned} \|v\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)_\kappa} &= \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle} Fv(\eta)\|^2_H đ\eta \\ &= \int \langle\eta\rangle^{2s} \|FF^{-1}\kappa^{-1}_{\langle\eta\rangle} Fv(\eta)\|^2_H đ\eta \\ &= \int \langle\eta\rangle^{2s} \|(\widehat{K^{-1}v})(\eta)\|^2_H đ\eta \\ &= \|K^{-1}v\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)_{\text{id}}} \\ &= \|K^{-1}v\|^2_{H^s(\mathbb{R}^q,H)} \end{aligned}$$

Note that for $H = \mathbb{C}$ and $\kappa_\delta = \text{id}_{\mathbb{C}}, \delta \in \mathbb{R}$, we have

$$\mathcal{W}^s(\mathbb{R}^q, \mathbb{C}) = H^s(\mathbb{R}^q).$$

Example 1.21.

(i) *For the space* $H^s(\mathbb{R}^n)$ *with group action* κ, *cf.* (1.45) *we have*

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^n)) = H^s(\mathbb{R}^n \times \mathbb{R}^q) = H^s(\mathbb{R}^{n+q}).$$

(ii) *The space* $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ *for* H *equipped with a group action* κ *is again a Hilbert space with group action* χ, *namely,*

$$(\chi_\delta v)(y) := \delta^{q/2}\kappa_\delta v(\delta y),\ \ \delta \in \mathbb{R}_+,$$

where κ_δ *is acting on the values of* v *in* H, *and then we have*

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi = \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa,$$

cf. [55, Proposition 1.3.44].

Remark 1.22. *Definition* 1.17 *admits a generalization to the case of a Fréchet space* $E = \varprojlim_{j\in\mathbb{N}} E^j$ *with group action. According to Definition* 1.17 *we form* $\mathcal{W}^s(\mathbb{R}^q, E^j)$ *and then set*

$$\mathcal{W}^s(\mathbb{R}^q, E) := \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j) \tag{1.63}$$

which is a Fréchet space.

Let Z be a smooth closed manifold of dimension n. Choose an open covering $\{U_1, \dots, U_N\}$ of Z by coordinate neighborhoods $\chi_j : U_j \to \mathbb{R}^n$ charts, and let $\{\varphi_1, \dots, \varphi_N\}$ be a subordinate partition of unity. We define global edge spaces

$$\begin{aligned}\mathcal{W}^s(\mathbb{R}^q \times Z, H) := \Big\{ \sum_{l=1}^{N} (\varphi_l u_l) \circ (\mathrm{id}_{\mathbb{R}^q} \times \chi_l^{-1}) :& \\ u_l \in \mathcal{W}^s(\mathbb{R}^{q+n}, H), l = 1, \dots N \Big\}&.\end{aligned} \tag{1.64}$$

Moreover, if $E = \varprojlim_{j\in\mathbb{N}} E^j$ is a Fréchet space we form

$$\mathcal{W}^s(\mathbb{R}^q \times Z, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q \times Z, E^j). \tag{1.65}$$

In the case $q = 0$ we simply write

$$\mathcal{W}^s(Z, H) \ \text{ and } \ \mathcal{W}^s(Z, E), \tag{1.66}$$

respectively.

For any symbol $a(y, y', \eta) \in S^\mu(\Omega_y \times \Omega_{y'} \times \mathbb{R}^q_\eta; H, \tilde{H})$ for an open set $\Omega \subseteq \mathbb{R}^q$ we form the associated operator $\mathrm{Op}_y(a) =: \mathrm{Op}(a)$, defined as

$$\mathrm{Op}_y(a)u(y) = \iint e^{i(y-y')\eta} a(y, y', \eta) u(y') dy' đ\eta, \tag{1.67}$$

first for $u \in C_0^\infty(\Omega, H)$. In this connection $a := a_{\mathrm{D}}$ with its dependence on $(y, y') \in \Omega \times \Omega$ is also called a double symbol. If a only depends on y or y' we talk about left or right symbols, occasionally denoted by $a_{\mathrm{L}}(y, \eta)$ and $a_{\mathrm{R}}(y', \eta)$, respectively. The expression (1.67) is interpreted as an oscillatory integral. It is well-known that $\mathrm{Op}(a)$ induces a continuous operator

$$\mathrm{Op}(a) : C_0^\infty(\Omega, H) \to C^\infty(\Omega, \widetilde{H})$$

for every $s \in \mathbb{R}$.

Definition 1.23. *Let H and $\widetilde{H}$ be Hilbert spaces with group action κ and $\widetilde{\kappa}$, respectively. Then we set*

$$L^\mu_{(\mathrm{cl})}(\Omega; H, \widetilde{H}) := \Big\{\mathrm{Op}(a) : a(y, y', \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; H, \widetilde{H})\Big\}.$$

An elementary observation is the following result.

Theorem 1.24. *For every $a(\eta) \in S^\mu(\mathbb{R}^q; H, \widetilde{H})$, the operator*

$$\mathrm{Op}(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})$$

is continuous for every $s \in \mathbb{R}$, and we have

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))} := \sup_{\eta\in\mathbb{R}^q}\Big\{\langle\eta\rangle^{-\mu}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}a(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})}\Big\}. \tag{1.68}$$

In other words, $\mathrm{Op}(\cdot)$ *induces a continuous operator*

$$\mathrm{Op}(\cdot) : S^\mu(\mathbb{R}^q; H, \widetilde{H}) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})). \tag{1.69}$$

Proof. We have from (1.58)

$$\begin{aligned}\|\mathrm{Op}(a)u\|^2_{\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H})} :&= \int \langle\eta\rangle^{2(s-\mu)}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}F(\mathrm{Op}(a)u)(\eta)\|^2_{\widetilde{H}}đ\eta \\ &= \int \langle\eta\rangle^{2(s-\mu)}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}a(\eta)(Fu)(\eta)\|^2_{\widetilde{H}}đ\eta,\end{aligned}$$

using $\mathrm{Op}(a)u = F^{-1}aFu$. The right hand side can be estimated by

$$\begin{aligned}&\int \langle\eta\rangle^{2(s-\mu)}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}a(\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(H,\widetilde{H})}\|\kappa^{-1}_{\langle\eta\rangle}(Fu)(\eta)\|^2_H đ\eta \\ \le \sup_{\eta\in\mathbb{R}^q}&\Big\{\langle\eta\rangle^{-2\mu}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}a(\eta)\kappa_{\langle\eta\rangle}\|^2_{\mathcal{L}(H,\widetilde{H})}\int\langle\eta\rangle^{2s}\|\kappa^{-1}_{\langle\eta\rangle}(Fu)(\eta)\|^2_H đ\eta\Big\}.\end{aligned}$$

The second facts on the right hand side just equals $\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)}$ and hence we obtain

$$\|\mathrm{Op}(a)u\|^2_{\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H})} \le c_a^2\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)} \tag{1.70}$$

for

$$c_a := \sup_{\eta\in\mathbb{R}^q}\Big\{\langle\eta\rangle^{-\mu}\|\widetilde{\kappa}^{-1}_{\langle\eta\rangle}a(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})}\Big\}.$$

This gives us immediately (1.68). The number c_a is a semi-norm in the symbol space $S^\mu(\mathbb{R}^q; H, \widetilde{H})$, see the formula (1.55). Thus, because of

$$\|\mathrm{Op}(a)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))} \le c_a$$

we proved the continuity of the mapping (1.69). $\square$

Let $\mathcal{M}_\varphi$ for $\varphi \in \mathcal{S}(\mathbb{R}^q)$ denote the operator of multiplication by φ, i.e., $\mathcal{M}_\varphi = \varphi u$. In the following proposition is from Hirschmann [23]. The proof will be given for completeness.

Proposition 1.25. *For every $\varphi \in \mathcal{S}(\mathbb{R}^q)$ the correspondence $\varphi \to \mathcal{M}_\varphi$ defines a continuous operator*

$$\mathcal{S}(\mathbb{R}^q) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, H), \mathcal{W}^s(\mathbb{R}^q, H)), \tag{1.71}$$

for every $s \in \mathbb{R}$.

Proof. In this proof c will denote different constants. The space $\mathcal{S}(\mathbb{R}^q, H)$ is dense in $\mathcal{W}^s(\mathbb{R}^q, H)$. We first show

$$\|\mathcal{M}_\varphi u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} \leq c\|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} \tag{1.72}$$

for all $u \in \mathcal{S}(\mathbb{R}^q, H)$, for some $c = c_\varphi > 0$. We have (up to equivalence of norms)

$$\|\mathcal{M}_\varphi u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)} = \int [\eta]^{2s} \|\kappa^{-1}_{[\eta]} F_{y\to\eta}(\varphi u)(\eta)\|^2_H đ\eta.$$

From $F(\varphi u) = F\varphi * Fu$, let

$$m(\eta) := \|[\eta]^s \kappa^{-1}_{[\eta]} \int (F\varphi)(\eta - \xi) Fu(\xi) đ\xi\|_H,$$

hence we have

$$\|\mathcal{M}_\varphi u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} = \|m\|_{L^2(\mathbb{R}^q_\eta)}. \tag{1.73}$$

From Peetre's inequality

$$[\eta]^s \leq c[\eta - \xi]^{|s|}[\xi]^s$$

and Proposition 1.9 we obtain the estimate

$$\|\kappa^{-1}_{[\eta]/[\xi]}\|_{\mathcal{L}(H)} \leq c[\eta - \xi]^M \text{ for some } M > 0.$$

This gives us

$$\begin{aligned}
m(\eta) &= \| \int [\eta]^s \kappa^{-1}_{[\eta]} \hat{\varphi}(\eta - \xi) \hat{u}(\xi) đ\xi \|_H \\
&= \| \int [\eta]^s \kappa^{-1}_{[\eta]/[\xi]} \hat{\varphi}(\eta - \xi) \kappa^{-1}_{[\xi]} \hat{u}(\xi) đ\xi \|_H \\
&\leq c \int [\xi]^s [\eta - \xi]^{|s|} \|\kappa^{-1}_{[\eta]/[\xi]} \hat{\varphi}(\eta - \xi) \kappa^{-1}_{[\xi]} \hat{u}(\xi)\|_H đ\xi \\
&\leq c \int [\eta - \xi]^{M+|s|-N} [\eta - \xi]^N |\hat{\varphi}(\eta - \xi)| \|[\xi]^s \kappa^{-1}_{[\xi]} \hat{u}(\xi)\|_H đ\xi \\
&\leq cc_\varphi \int [\eta - \xi]^{M+|s|-N} \|[\xi]^s \kappa^{-1}_{[\xi]} \hat{u}(\xi)\|_H đ\xi
\end{aligned}$$

for $c_\varphi = \sup_{\xi \in \mathbb{R}^q}\{[\xi]^N |\hat{\varphi}(\xi)| < \infty\}$. We have for N large enough

$$g(\eta) := [\eta - \xi]^{M+|s|-N} \in L^1(\mathbb{R}^q),$$

and

$$h(\xi) := \|[\xi]^s \kappa^{-1}_{[\xi]} \hat{u}(\xi)\|_H \in L^2(\mathbb{R}^q),$$

we proved that
$$m(\eta) \leq cc_\varphi (g * h)(\eta).$$
Next we employ Young's inequality:

For every $f \in L^1(\mathbb{R}^n),\ u \in L^p(\mathbb{R}^n),\ 1 \leq p \leq \infty$, we have
$$f * u(x) = \int f(x-y)u(y)dy \in L^p(\mathbb{R}^n)$$
and
$$\|f * u\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)}.$$
Thus from (1.73) we obtain
$$\begin{aligned}\|\mathcal{M}_\varphi u\|_{\mathcal{W}^s(\mathbb{R}^q,H)} &= \|m\|_{L^2(\mathbb{R}^q)} \\ &\leq cc_\varphi \|g * h\|_{L^2(\mathbb{R}^q)} \\ &\leq cc_\varphi \|g\|_{L^1(\mathbb{R}^q)} \|h\|_{L^2(\mathbb{R}^q)} \\ &= cc_\varphi \|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)}\end{aligned}$$
since $\|h\|_{L^2(\mathbb{R}^q)} = \|u\|_{\mathcal{W}^s(\mathbb{R}^q,H)}$. This gives us the estimate (1.72). At the same time we obtain the continuity (1.71).

□

Theorem 1.26. *For every $a \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H}),\ \Omega \in \mathbb{R}^q$, the operator* $\mathrm{Op}(a)$ *extends to a continuous map*
$$\mathrm{Op}(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H})$$
for every $s \in \mathbb{R}$.

Proof. Using the expansion (1.56) we can write
$$\mathrm{Op}(a)u(y) = \sum_{j=0}^{\infty} \lambda_j \varphi_j(y) \mathrm{Op}(a_j)u.$$
Thus for any $u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ and any $\psi \in C_0^\infty(\Omega)$ we obtain
$$(\psi \mathrm{Op}(a)u)(y) = \sum_{j=0}^{\infty} \lambda_j \psi(y) \varphi_j(y) \mathrm{Op}(a_j)u$$
which entails
$$\|\mathrm{Op}(a)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))}$$
$$\leq \sum_{j=0}^{\infty} |\lambda_j| \|\mathcal{M}_{\psi\varphi_j}\|_{\mathcal{L}(\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))} \|\mathrm{Op}(a_j)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))}. \tag{1.74}$$
From Theorem 1.24 we know that
$$\mathrm{Op}(a_j) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H})$$

is continuous and

$$\|\mathrm{Op}(a_j)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,H),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))} \to 0 \text{ as } j \to \infty.$$

Moreover, by virtue of Proposition 1.25 we have

$$\|\mathcal{M}_{\psi\varphi_j}\|_{\mathcal{L}(\mathcal{W}^{s-\mu}(\mathbb{R}^q,\widetilde{H}))} \to 0 \text{ as } j \to \infty.$$

Thus (1.74) converges, and we obtain the assertion. □

1.5 Symbols and kernel cut-off with respect to the Mellin transform

Similarly as (1.10) with a symbol $f(r,r',w) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$, we associate a pseudo-differential operator based on the weighted Mellin transform $M_\gamma, \gamma \in \mathbb{R}$, cf. the formula (1.35), first acting on $u \in C_0^\infty(\mathbb{R}_+)$. According the formula (1.38) We have

$$\mathrm{Op}^{\gamma}_{M}(f)u(r) := \int_{-\infty}^{+\infty}\int_{0}^{+\infty} (\frac{r}{r'})^{-(\frac{1}{2}-\gamma+i\rho)} f(r,r',\frac{1}{2}-\gamma+i\rho)u(r')\frac{dr'}{r'}đ\rho, \tag{1.75}$$

where $đ\rho = (2\pi)^{-1}d\rho$. For $\gamma = 0$ we also write $\mathrm{Op}_M = \mathrm{Op}^0_M$. The expression (1.75) for a symbol $f(r,r',w)$ can also be written in the form

$$\mathrm{Op}^{\gamma}_{M}(f)u = M^{-1}_{\gamma} f M_\gamma u = r^\gamma \mathrm{Op}_M(T^{-\gamma}f)r^{-\gamma}u \tag{1.76}$$

for $(T^{-\gamma})f(r,w) := f(r,w-\gamma)$. In fact, the integral on the right hand side of (1.75) is interpreted as a Mellin oscillatory integral. By rearranging the order of integration we obtain

$$\int_{-\infty}^{+\infty} r^{-(\frac{1}{2}-\gamma+i\rho)}\Big\{\int_{0}^{+\infty}(r')^{\frac{1}{2}-\gamma+i\rho}f(r,r',\frac{1}{2}-\gamma+i\rho)u(r')\frac{dr'}{r'}\Big\}đ\rho \tag{1.77}$$

which is just the first relation of (1.76). The second equality follows from passing to

$$r^\gamma\int_{-\infty}^{+\infty} r^{-(\frac{1}{2}+i\rho)}\Big\{\int_{0}^{+\infty}(r')^{\frac{1}{2}+i\rho}(T^{-\gamma}f)(r,r',\frac{1}{2}+i\rho)(r')^{-\gamma}u(r')\frac{dr'}{r'}\Big\}đ\rho$$

Another consequence of (1.76) is the relation

$$r^\beta \mathrm{Op}^{\gamma}_{M}(f)r^{-\beta} = \mathrm{Op}^{\gamma+\beta}_{M}(T^\beta f).$$

In fact,

$$\begin{aligned} r^\beta \mathrm{Op}^{\gamma}_{M}(f)r^{-\beta} &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty}(\frac{r}{r'})^{\beta}(\frac{r}{r'})^{-(\frac{1}{2}-\gamma+i\rho)}f(r,r',\frac{1}{2}-\gamma+i\rho)u(r')\frac{dr'}{r'}đ\rho \\ &= \int_{-\infty}^{+\infty}\int_{0}^{+\infty}(\frac{r}{r'})^{-(\frac{1}{2}-\gamma-\beta+i\rho)}f(r,r',\frac{1}{2}-\gamma-\beta+\beta+i\rho)u(r')\frac{dr'}{r'}đ\rho \\ &= \mathrm{Op}^{\gamma+\beta}_{M}(T^\beta f) \end{aligned} \tag{1.78}$$

Remark 1.27. *For $(\kappa_\delta u)(r) := u(\delta r)$, $\delta > 0$ and $f \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ we have*

$$\kappa_\delta \mathrm{Op}^{\gamma}_{M}(f)\kappa^{-1}_{\delta} = \mathrm{Op}^{\gamma}_{M}(f_\delta)$$

for $f_\delta(r,r',z) = f(\delta r,\delta r',z)$.

In fact,

$$\begin{aligned}
\kappa_\delta \mathrm{Op}_M^\gamma(f)\kappa_\delta^{-1} &= \kappa_\delta \int_{-\infty}^{+\infty}\int_0^{+\infty} (\frac{r}{r'})^{-z} f(r,r',z)u(\delta^{-1}r')\frac{dr'}{r'}đ\rho \\
&= \int_{-\infty}^{+\infty}\int_0^{+\infty} (\frac{\delta r}{\delta r'})^{-z} f(\delta r,\delta r',z)u(\delta\delta^{-1}r')\frac{dr'}{r'}đ\rho \\
&= \mathrm{Op}_M^\gamma(f_\delta)
\end{aligned} \tag{1.79}$$

Let us first consider the case $\gamma = \frac{1}{2}$ and scalar symbols $f = f(w) \in S^\mu_{\mathrm{cl}}(\Gamma_0)$ with constant coefficients.

$$l_f(s) := \int_0^\infty s^{-i\rho} f(i\rho)đ\rho. \tag{1.80}$$

Then

$$\mathrm{Op}_M^{\frac{1}{2}}(f)u(r) = \int_0^\infty l_f(\frac{r'}{r})u(r')\frac{dr'}{r'},$$

which is a convolution with respect to the Mellin transform . By definition we have $l_f(s) = (M_{\frac{1}{2}}^{-1}f)(s)$. From (1.80) it follows that $(\log s)^j l_f(s) = l_{D_\rho^j f}(s)$ for every $j \in \mathbb{N}$. In fact, the integration by parts gives us

$$\begin{aligned}
l_{D_\rho f}(s) &= \int_0^\infty s^{-i\rho}(D_\rho^j f)(i\rho)đ\rho \\
&= \int_0^\infty (-1)^j(D_\rho^j s^{-i\rho})f(i\rho)đ\rho = \int_0^\infty \partial_\rho^j(s^{-i\rho})f(i\rho)đ\rho \\
&= \int_0^\infty s^{-i\rho}\ (\log s)^j\ f(i\rho)đ\rho = (\log s)^j\ l_f(s).
\end{aligned}$$

Setting $s = e^{-\theta}, \theta \in \mathbb{R}$, it follows that

$$l_f(e^{-\theta}) = \int_{-\infty}^\infty e^{i\rho\theta} f(i\rho)đ\rho.$$

Analogously as (5.22) we have for any excision function $\kappa(\theta)$, the relation

$$\kappa(\theta)l_f(e^{-\theta}) \in \mathcal{S}(\mathbb{R}_\theta).$$

In particular ,

$$\mathrm{sing\ supp}\, l_f \subsetneqq \{1\}.$$

let us write

$$l_f(s) = \kappa(-\log s)l_f(s) + (1-\kappa(-\log s))l_f(s)$$

Then

$$\kappa(-\log s)l_f(s) \in \mathcal{T}^0(\mathbb{R}_+).$$

Here $\mathcal{T}^0(\mathbb{R}_+)$ is the Mellin analogue (for weight 0) of the Schwartz space (i.e., $\mathcal{T}^0(\mathbb{R}_+) = M_{\frac{1}{2}}^{-1}\mathcal{S}(\Gamma_0)$). For any excision function $\psi(s)$ with respect to 1 (i.e., $\psi \in C_0^\infty(\mathbb{R}_+)), \psi(s) \equiv 1$ in a neighborhood of $s = 1$) we now write

$$(W_\psi f)(i\rho) := M_{\frac{1}{2}}(\psi(s)l_f(s))(i\rho).$$

By virtue of

$$M_{\frac{1}{2}}(1-\psi(s))l_f(s)) \in S^{-\infty}(\Gamma_0)$$

it follows that W_ψ defines a linear and continuous operator

$$W_\psi : S^\mu_{\mathrm{cl}}(\Gamma_0) \to S^\mu_{\mathrm{cl}}(\Gamma_0)$$

which is the Mellin kernel cut-off operator with the cut-off function ψ. The kernel cut-off operator also makes sense for more general functions ψ. Define the space

$$C^\infty_{\mathrm{B}}(\mathbb{R}_+) := \big\{\varphi \in C^\infty(\mathbb{R}_+) : \sup_{s\in\mathbb{R}_+} |(s\partial_s)^j \varphi(s)| < \infty \text{ for all } j \in \mathbb{N}\big\}.$$

Note that for $\chi : \mathbb{R} \to \mathbb{R}_+$, $\chi(\theta) = e^{-\theta} = s$, the function pull back χ^* induces an isomorphism

$$\chi^* : C^\infty_{\mathrm{B}}(\mathbb{R}_+) \to C^\infty_{\mathrm{b}}(\mathbb{R})$$

we interpret

$$\begin{aligned}(W_\varphi f)(i\rho) &= M_{\frac{1}{2}, s\to i\rho}\varphi(s) \int_{\mathbb{R}} s^{-i\rho'} f(i\rho') đ\rho' \\ &= \int_0^\infty s^{i\rho}\{\varphi(s) \int_{\mathbb{R}} s^{-i\rho'} f(i\rho') đ\rho'\} \frac{ds}{s} \\ &= \iint s^{i\tilde{\rho}} \varphi(s) f(i(\rho - \tilde{\rho})) \frac{ds}{s} đ\tilde{\rho}.\end{aligned}$$

We apply the Mellin kernel cut-off process to operator-valued symbols

$$f(r, r', z, \lambda) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_0; H, \widetilde{H}). \tag{1.81}$$

For convenience the general kernel cut-off theorem will be formulated for symbols with constant coefficients. The case of symbols (1.81) is completely analogous.

Theorem 1.28. *The kernel cut-off operator $W : (\varphi, f) \to W_\varphi f$ defines a bilinear continuous mapping*

$$W_\varphi : C^\infty_{\mathrm{B}}(\mathbb{R}_+) \times S^\mu_{(\mathrm{cl})}(\Gamma_0 \times \mathbb{R}^q; H, \widetilde{H}) \to S^\mu_{(\mathrm{cl})}(\Gamma_0 \times \mathbb{R}^q; H, \widetilde{H}).$$

The symbol $W_\varphi f(i\rho, \lambda)$ admits an asymptotic expansion

$$W_\varphi f(i\rho, \lambda) \sim \sum_{k=0}^\infty \frac{1}{k!}(s\partial_s)^k \varphi(1) \partial^k_\rho f(i\rho, \lambda).$$

1.6 Weighted cone spaces

Asymptotics are of interest also on manifolds with edge, locally near the edge modeled on a wedge $X^\Delta \times \mathbb{R}^q$ for a smooth closed manifold X where X^Δ is a straight cone with base X. We set $X^\wedge := \mathbb{R}_+ \times X$ which is the open stretched cone. In the corner calculus we employ spaces

$$\mathcal{H}^{s,\gamma}(X^\wedge),\ \mathcal{K}^{s,\gamma}(X^\wedge) \tag{1.82}$$

where it is convenient to normalize the weight convention in such a way that $\Gamma_{\frac{1}{2}-\gamma}$ from the one-dimensional case is replaced by $\Gamma_{\frac{n+1}{2}-\gamma}$ for $\dim X = n$.

By $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)$ for a Hilbert space H with group action we denote the completion of $C_0^\infty(\mathbb{R}_+, H)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,H)} := \Big\{ \int_{\Gamma_{\frac{h+1}{2}-\gamma}} \|\langle w\rangle^s \kappa^{-1}_{\langle w\rangle} Mu(w)\|^2_H đw \Big\}^{1/2}, \tag{1.83}$$

for a number $h \in \mathbb{R}$ that is given in connection with the space H. We then have

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) = r^\gamma \mathcal{H}^{s,0}(\mathbb{R}_+, H) \tag{1.84}$$

for every $s, \gamma \in \mathbb{R}$ according to formula (1.33). Note that the transformation

$$(S_{\gamma-h/2}u)(\boldsymbol{r}, \cdot) = e^{-(\frac{h+1}{2}-\gamma)\boldsymbol{r}} u(e^{-\boldsymbol{r}}, \cdot)$$

gives rise to an isomorphism

$$S_{\gamma-h/2} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+, H) \to \mathcal{W}^s(\mathbb{R}, H) \tag{1.85}$$

for every $s \in \mathbb{R}$. In fact,

$$\begin{aligned}
&\|S_{\gamma-h/2}u(\rho)\|^2_{\mathcal{W}^s(\mathbb{R},H)} \\
&= \int \langle\rho\rangle^{2s} \|\kappa^{-1}_{\langle\rho\rangle} F S_{\gamma-h/2} u(\rho)\|_H đ\rho \\
&= \int \langle\rho\rangle^{2s} \|\kappa^{-1}_{\langle\rho\rangle} M_{\gamma-h/2} u(\frac{h+1}{2} - \gamma + i\rho)\|^2_H đ\rho \\
&\leq \int_{\Gamma_{\frac{h+1}{2}-\gamma}} \langle\rho\rangle^{2s} \|\kappa^{-1}_{\langle w\rangle} Mu(w)\|^2_H đw \\
&\leq c\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+,H)}
\end{aligned} \tag{1.86}$$

Moreover, for a covering $\{U_1, \ldots, U_N\}$ of X by coordinate neighborhoods, a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_N\}$ and charts $\chi_j : U_j \to \mathbb{R}^n$, we form the space $\mathcal{H}^{s,\gamma}(X^\wedge)$ as the completion of $C_0^\infty(\mathbb{R}_+, C^\infty(X))$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} := \Big\{ \sum_{j=1}^N \|(\varphi_j u) \circ (\mathrm{id}_{\mathbb{R}_+} \times \chi_j^{-1})\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \Big\}^{1/2}. \tag{1.87}$$

From the definition it follows that

$$\mathcal{H}^{s,\gamma+\beta}(X^\wedge) = r^\beta \mathcal{H}^{s,\gamma}(X^\wedge) \tag{1.88}$$

for every $s, \gamma, \beta \in \mathbb{R}$. Note that the considerations on the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ extend in a natural way to $\mathcal{H}^{s,\gamma}(X^\wedge)$. In particular, the operator

$$S_{\gamma-n/2} : C_0^\infty(\mathbb{R}_+ \times X) \to C_0^\infty(\mathbb{R} \times X)$$

$$S_{\gamma-n/2} : u(r,x) \mapsto e^{-(\frac{n+1}{2}-\gamma)t} u(e^{-t}, x)$$

with the inverse

$$S^{-1}_{\gamma-n/2} v(r,x) = r^{\gamma - \frac{n+1}{2}} v(\log r, x)$$

extends to an isomorphism

$$S_{\gamma-n/2} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times X) \to H^s(\mathbb{R} \times X). \tag{1.89}$$

Here $H^s(\mathbb{R} \times X)$ is the cylindrical Sobolev space which can be defined as the completion of $C_0^\infty(\mathbb{R} \times X)$ with respect to the norm

$$\|v\|_{H^s(\mathbb{R}\times X)} = \Big\{ \sum_{j=1}^{N} \|\varphi_j v \circ (\mathrm{id}_\mathbb{R} \times {\chi_j}^{-1})\|^2_{H^s(\mathbb{R}^{1+n})} \Big\}^{\frac{1}{2}}$$

where

$$\mathrm{id}_\mathbb{R} \times {\chi_j}^{-1} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times U_j.$$

Corollary 1.29. *The multiplication by r^β induces an isomorphism*

$$r^\beta : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times X) \to \mathcal{H}^{s,\gamma+\beta}(\mathbb{R}_+ \times X)$$

for every $s, \gamma, \beta \in \mathbb{R}$.

In fact, from the following mapping

$$S_{\gamma-n/2} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times X) \to H^s(\mathbb{R} \times X)$$

$$S^{-1}_{\gamma+\beta-n/2} : H^s(\mathbb{R} \times X) \to \mathcal{H}^{s,\gamma+\beta}(\mathbb{R}_+ \times X)$$

we have

$$S^{-1}_{\gamma+\beta-n/2} S_{\gamma-n/2} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times X) \to \mathcal{H}^{s,\gamma+\beta}(\mathbb{R}_+ \times X)$$

is an isomorphism which just corresponds to the multiplication by $r^\beta = e^{-t\beta}$.
Later on we also employ cylindrical Sobolev spaces

$$H^s(\mathbb{R}^m \times X) \tag{1.90}$$

for a closed smooth manifold X, defined as the completion of $C_0^\infty(\mathbb{R}^m \times X)$ with respect to the norm

$$\|u\|_{H^s(\mathbb{R}^m\times X)} := \Big\{ \sum_{j=1}^{N} \|(\varphi_j u) \circ (\mathrm{id}_{\mathbb{R}^m} \times \chi_j^{-1})\|^2_{H^s(\mathbb{R}^{m+n})} \Big\}^{1/2},$$

for the standard Sobolev space $H^s(\mathbb{R}^{m+n})$ in $\mathbb{R}^{m+n}$, charts $\chi_j : U_j \to \mathbb{R}^n$ and φ_j as before.

Remark 1.30. *We have*

$$\mathcal{H}^{s,\gamma}(X^\wedge) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times X)$$

for every $s,\ \gamma \in \mathbb{R}$.

In fact, it suffices to use (1.89) which shows that

$$S^{-1}_{\gamma-n/2} H^s(\mathbb{R} \times X) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+ \times X).$$

In this paper a cut-off function ω on the half-axis is any $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega(r) = 1$ in a neighbourhood of $r = 0$.
We also consider spaces in terms both of the Fourier and the Mellin transform, namely,

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, H) \tag{1.91}$$

defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^q,H)} := \Big\{ \int_{\Gamma_{\frac{q+h+1}{2}-\gamma}} \int_{\mathbb{R}^q} \langle w,\eta\rangle^{2s} \|\kappa^{-1}_{\langle w,\eta\rangle}(MFu)(w,\eta)\|^2_H đw đ\eta \Big\}^{1/2}, \tag{1.92}$$

where $đw = (2\pi i)^{-1}dw$, $đ\eta = (2\pi)^{-q}d\eta$. Also here, if necessary, we write $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, H)_\kappa$, in order to indicate the dependence on κ. Clearly, the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ in the Definition 1.3 equal to the formula (1.92) in the case $H = \mathbb{C}$, $\kappa = \mathrm{id}$, $q = n$. Similar notation is used for Fréchet spaces $E = \varprojlim_{j\in\mathbb{N}} E^j$ with group action as follows

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, E^j).$$

Analogously as (1.84) we have

$$\mathcal{H}^{s,\gamma+\beta}(\mathbb{R}_+ \times \mathbb{R}^q, H) = r^\beta \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, H)$$

for every $s, \gamma, \beta \in \mathbb{R}$. Next we also need spaces

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q \times Z, H) := \Big\{ \sum_{l=1}^{N} (\varphi_l u_l) \circ (\mathrm{id}_{\mathbb{R}_+\times\mathbb{R}^q} \times \chi_l^{-1}) : \\ u_l \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{q+d}, H), l = 1, \dots N \Big\} \tag{1.93}$$

for a smooth compact manifold Z of dimension d, charts $\chi_j : U_j \to \mathbb{R}^d$ and a subordinate partition of unity. Similarly, if $E = \varprojlim_{j\in\mathbb{N}} E^j$ is a Fréchet space with group action we set

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q \times Z, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{q+d}, E^j). \tag{1.94}$$

Let us also recall a standard notation for local Sobolev spaces $H^s_{\mathrm{loc}}(\widetilde{X})$ on a smooth manifold $\widetilde{X}$ (not necessarily compact), $n = \dim \widetilde{X}$. These are defined as the space of all $u \in \mathcal{D}'(\widetilde{X})$, such that for any chart $\chi : U \to \mathbb{R}^n$ on $\widetilde{X}$ and any $\varphi \in C_0^\infty(U)$ we have $(\varphi u) \circ \chi^{-1} \in H^s(\mathbb{R}^n)$. Moreover, $H^s_{\mathrm{comp}}(\widetilde{X})$ is the subspace of all $u \in H^s_{\mathrm{loc}}(\widetilde{X})$ with compact support.
Next, in order to define the space $\mathcal{K}^{s,\gamma}(X^\wedge)$, we employ the space $H^s_{\mathrm{cone}}(X^\wedge)$, $s \in \mathbb{R}$, defined as the set of all $u(r, x) \in H^s_{\mathrm{loc}}(\mathbb{R} \times X)|_{\mathbb{R}_+\times X}$ such that for any coordinate neighborhood $U \subset X$ and a diffeomorphism $\vartheta : U \to V$ to an open subset V of S^n (the unit sphere in $\mathbb{R}^n$) we have $(1 - \omega)\varphi u \circ \beta^{-1} \in H^s(\mathbb{R}^{1+n}_{\tilde{x}})$ for every $\varphi \in C_0^\infty(U)$, where the map $\beta : \mathbb{R}_+ \times U \to \mathbb{R}^{1+n}$ is defined by $\beta(r, x) := r\vartheta(x)$.
We set

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{\omega u + (1 - \omega)v : u \in \mathcal{H}^{s,\gamma}(X^\wedge),\ v \in H^s_{\mathrm{cone}}(X^\wedge)\} \tag{1.95}$$

for some fixed cut-off function ω.

Remark 1.31. *The spaces* (1.95) *for* $s, \gamma \in \mathbb{R}$, *are Hilbert spaces with group action* $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ *given by*

$$(\kappa_\delta u)(r, x) := \delta^{(n+1)/2} u(\delta r, x), \tag{1.96}$$

$n = \dim X$. *Observe that*

$$\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge) = r^{-n/2} L^2(\mathbb{R}_+ \times X) \tag{1.97}$$

with respect to the measure $drdx$ *and* dx *referring to a Riemannian metric on* X. *Then the operators* (1.96) *are unitary in* $\mathcal{K}^{0,0}(X^\wedge)$.

Note 1.32. *The space* $\mathcal{K}^{s,\gamma}(X^\wedge)$ *cf.* (1.95) *is independent of the choice of* ω.

In fact, we have

$$\mathcal{H}^{s,\gamma}(X^\wedge)|_{(c,c')\times X} = H^s_{\mathrm{cone}}(X^\wedge)|_{(c,c')\times X}$$

for any $0 < c < c'$. We can endow $\mathcal{K}^{s,\gamma}(X^\wedge)$ with the structure of a Hilbert space via the non-direct sum

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega] H^s_{\mathrm{cone}}(X^\wedge). \tag{1.98}$$

Here, for (in general) Fréchet spaces E and F, embedded in a Hausdorff topological vector space, we set

$$E + F := E \oplus F / \bigtriangleup \quad \text{for} \quad \bigtriangleup := \{(e, -e) : e \in E \cap F\}.$$

For purposes below we set

$$\mathcal{K}^{s,\gamma;e}(X^\wedge) := \langle r \rangle^{-e} \mathcal{K}^{s,\gamma}(X^\wedge)$$

for any $e \in \mathbb{R}$. Equivalently we can define

$$\mathcal{K}^{s,\gamma;e}(X^\wedge) := [r]^{-e} \mathcal{K}^{s,\gamma}(X^\wedge)$$

or

$$\mathcal{K}^{s,\gamma;e}(X^\wedge) := \mathrm{w}^{-e} \mathcal{K}^{s,\gamma}(X^\wedge) \tag{1.99}$$

where w is a strictly positive function in $C^\infty(\mathbb{R}_+)$ such that $\mathrm{w}(r) = 1$ for $0 \le r \le c_0$, $\mathrm{w}(r) = r$ for $r \ge c_1$, for some constant $c_0 < c_1$.
In our notation we also admit the cases $s = \infty$, $\gamma = \infty$, $e = \infty$. For instance,

$$\mathcal{K}^{\infty,\gamma;e}(X^\wedge) := \bigcap_{s \in \mathbb{R}} \mathcal{K}^{s,\gamma;e}(X^\wedge).$$

This space is contained in $C^\infty(X^\wedge)$. Moreover,

$$\mathcal{K}^{s,\infty;e}(X^\wedge) := \bigcap_{\gamma \in \mathbb{R}} \mathcal{K}^{s,\gamma;e}(X^\wedge)$$

consists of functions of infinite flatness at $r = 0$, while

$$\mathcal{K}^{s,\gamma;\infty}(X^\wedge) := \bigcap_{e \in \mathbb{R}} \mathcal{K}^{s,\gamma;e}(X^\wedge)$$

consists of functions of infinite flatness at $r = \infty$. This allows us to consider $\mathcal{K}^{\infty,\infty;e}(X^\wedge)$, etc., including

$$\mathcal{K}^{\infty,\infty;\infty}(X^\wedge) = \{u \in \mathcal{S}(\mathbb{R}, C^\infty(X)) : \operatorname{supp} u \subseteq \overline{\mathbb{R}}_+\}.$$

Proposition 1.33. *We have continuous embedding*

$$\mathcal{K}^{s',\gamma';e'}(X^\wedge) \hookrightarrow \mathcal{K}^{s,\gamma;e}(X^\wedge) \tag{1.100}$$

for every $s' \geq s, \gamma' \geq \gamma$, $e' \geq e$. For $s' > s$, $\gamma' > \gamma$, $e' > e$ the embeddings(1.100) *are compact.*

Remark 1.34. *Let $\mathrm{k}(r) \in C^\infty(\overline{\mathbb{R}}_+)$ be a strictly positive function such that $\mathrm{k}(r) = r$ for $0 \leq r \leq \varepsilon_0$, $\mathrm{k}(r) = 1$ for $r \geq \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$. Then for every $\beta \in \mathbb{R}$ we have*

$$\mathrm{k}^\beta \mathcal{K}^{s,\gamma;e}(X^\wedge) = \mathcal{K}^{s,\gamma+\beta;e}(X^\wedge) \text{ for all } s, \gamma, e \in \mathbb{R}.$$

Proposition 1.35. *For every $\varphi \in C_0^\infty(\overline{\mathbb{R}}_+)$ and $\beta \in \mathbb{R}$ the operator $\mathcal{M}_{r^\beta\varphi}$ of multiplication by $r^\beta\varphi$ defines continuous operators*

$$\mathcal{M}_{r^\beta\varphi} : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma+\beta}(X^\wedge) \text{ for all } s \in \mathbb{R}.$$

Proof. We can write

$$\mathcal{M}_{r^\beta\varphi} = \mathcal{M}_\omega \mathcal{M}_{r^\beta} \mathcal{M}_\varphi$$

for a cut-off function ω such that $\omega \succ \varphi$. Then $\mathcal{M}_{r^\beta\varphi}$ be regarded as a composition of

$$\mathcal{M}_\varphi : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge), \tag{1.101}$$

$$\mathcal{M}_{r^\beta} : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma+\beta}(X^\wedge), \tag{1.102}$$

$$\mathcal{M}_\omega : \mathcal{H}^{s,\gamma+\beta}(X^\wedge) \to \mathcal{K}^{s,\gamma+\beta}(X^\wedge). \tag{1.103}$$

The continuity of (1.102) was observed in (1.88).The operator (1.103) is continuous because of relation (1.95). The continuity of (1.101) follows from the continuity of

$$\mathcal{M}_\varphi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge), \tag{1.104}$$

since $\mathcal{M}_\varphi = \mathcal{M}_\varphi \mathcal{M}_\omega$ for a cut-off function ω on the half-axis such that $\varphi \prec \omega$ and

$$\mathcal{M}_\omega \mathcal{K}^{s,\gamma}(X^\wedge) = \mathcal{M}_\omega \mathcal{H}^{s,\gamma}(X^\wedge),$$

cf. the relation (1.95) . In order to show (1.104) according to (5.4) we form the function

$$\varphi_1(r) := \sum_{j=0}^{\infty} \frac{1}{j!} c_j \omega(r d_j) r^j$$

in $C^\infty(\overline{\mathbb{R}}_+)$ where $c_j := (\frac{d}{dr})^j \varphi|_{r=0}$, and $d_j \to \infty$ sufficiently fast. Then

$$\varphi = \varphi_0 + \varphi_1$$

where $\varphi_0 = 0$ at $r = 0$ of infinite order, because $c_j = (\frac{d}{dr})^j \varphi_1|_{r=0}$, for all j. We have

$$\mathcal{M}_{\varphi_0} : \mathcal{H}^{s,\gamma}(X^\wedge) \to \tilde{\omega}\mathcal{H}^{s,\infty}(X^\wedge)$$

for any cut-off function $\tilde{\omega} \succ \varphi_0$, but $\tilde{\omega}\mathcal{H}^{s,\infty}(X^\wedge)$ is continuously embedded in $\mathcal{H}^{s,\gamma}(X^\wedge)$, i.e.,

$$\mathcal{M}_{\varphi_0} : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

is continuous. Next observe that the multiplication by

$$\tau_j(r) := \frac{1}{j} c_j \omega(r d_j) r^j$$

gives us a continuous operator

$$\mathcal{H}^{s,\gamma}(X^\wedge) \to \tilde{\omega}\mathcal{H}^{s,\gamma+j}(X^\wedge)$$

for any other cut-off function $\tilde{\omega} \succ \omega$. Moreover, we have a continuous embedding

$$\tilde{\omega}\mathcal{H}^{s,\gamma+j}(X^\wedge) \hookrightarrow \mathcal{H}^{s,\gamma}(X^\wedge).$$

This shows the continuity

$$\mathcal{M}_{\tau_j} : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge)$$

for every j. A final observation is that

$$\|\mathcal{M}_{\tau_j}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(x^\wedge))} \leq 2^{-j}$$

for sufficiently large $d_j > 0$ and $j + \gamma > 0$. This gives us

$$\mathcal{M}_{\varphi_1} = \sum_{j=0}^{\infty} \mathcal{M}_{\tau_j}$$

and

$$\begin{aligned}
\|\mathcal{M}_{\varphi_1}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge))} &\leq \sum_{j=0}^{\infty} \|\mathcal{M}_{\tau_j}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}X^\wedge))} \\
&= \sum_{j\in\mathbb{N},\, j+\gamma\leq 0} \|\mathcal{M}_{\tau_j}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge))} + \sum_{j\in\mathbb{N},\, j+\gamma>0} \|\mathcal{M}_{\tau_j}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge))} \\
&\leq \sum_{j\in\mathbb{N},\, j+\gamma\leq 0} \|\mathcal{M}_{\tau_j}\|_{\mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge))} + 2 < \infty.
\end{aligned}$$

□

1.7 Holomorphic operator-valued symbols

Definition 1.36. *Let H and $\widetilde{H}$ be Hilbert spaces with group action κ and $\widetilde{\kappa}$, respectively. Then $S^{\mu}_{\mathcal{O}}(\mathbb{R}^q; H, \widetilde{H})$ is defined as the space of all holomorphic function acting on $\mathbb{C}$ with parameter $\eta \in \mathbb{R}^q$, i.e. $a(w,\eta) \in \mathcal{A}(\mathbb{C}, S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; H, \widetilde{H}))$, such that $a(\beta + i\rho, \eta) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{1+q}_{\rho,\eta}; H, \widetilde{H})$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.*

Here if E is a fréchet space and $U \subseteq \mathbb{C}$ an open set, $\mathcal{A}(U, E)$ means the space of E-valued holomorphic functions in U.

Definition 1.37. *The holomorphic function $a(w,\eta) \in S^{\mu}_{\mathcal{O}}(\mathbb{R}^q; H, \widetilde{H})$ is called elliptic if there is a $\beta \in \mathbb{R}$ such that*

$$a(\beta + i\rho, \eta) : H \to \widetilde{H}$$

is invertible for all $|\rho, \eta| \geq C$ for some $C = C(\beta) > 0$ and

$$\|\kappa_{\langle\rho,\eta\rangle} a^{-1}(\beta + i\rho, \eta) \tilde{\kappa}^{-1}_{\langle\rho,\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq C\langle\rho,\eta\rangle^{-\mu}$$

for all $|\rho, \eta| \geq C(\beta)$ and $C = C(\beta) > 0$, where $|\rho, \eta| := (|\rho|^2 + |\eta|^2)^{\frac{1}{2}}$.

This definition of ellipticity does not depend on the choice of β, i.e., the above β is arbitrary.

Theorem 1.38. *Let $h(w,\eta,\lambda) \in S^{\mu}_{\mathcal{O}}(\mathbb{R}^{q+l}; H, \widetilde{H})$ and let $h(i\rho,\eta,\lambda)$ be elliptic with respect to ρ,η,λ rather than w,η. Then $h^{-1}(i\rho,\eta,\lambda)$ exists for large $|\rho,\eta,\lambda|$ ($|\rho,\eta,\lambda| := (|\rho|^2 + |\eta|^2 + |\lambda|^2)^{\frac{1}{2}}$) and*

$$\langle \rho,\eta,\lambda\rangle^{\mu} \|\kappa_{\langle\rho,\eta,\lambda\rangle} h^{-1}(i\rho,\eta,\lambda)\tilde{\kappa}^{-1}_{\langle\rho,\eta,\lambda\rangle}\|_{\mathcal{L}(\widetilde{H},H)}$$

is bounded as $|\rho,\eta,\lambda| \to \infty$. Then for every $B > 0$ there exists a $C > 0$ such that for

$$h_{\beta}(\rho,\eta,\lambda) := h(\beta + i\rho,\eta,\lambda)$$

the operator

$$\mathrm{Op}_y \mathrm{Op}_M^{1/2}(h_\beta)(\lambda) : \mathcal{H}^{s,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H) \to \mathcal{H}^{s-\mu,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, \widetilde{H})$$

is invertible for all $\beta \in [-B, B]$ and $|\lambda| \geq C$.

Proof. By virtue of the ellipticity condition on the symbol $h(w,\eta,\lambda)$. Here is an excision function $\chi(\rho,\eta,\lambda)$ such that

$$h_0^{(-1)}(\rho,\eta,\lambda) := \chi(\rho,\eta,\lambda)h^{(-1)}(i\rho,\eta,\lambda).$$

belongs to $S^{-\mu}(\mathbb{R}^{1+q+l}; \widetilde{H}, H)$. The symbol $h_\beta(\rho,\eta,\lambda)$ belongs to $C(\mathbb{R}_\beta, S^{\mu}(\mathbb{R}^{1+q+l}; H, \widetilde{H}))$. Then

$$r_\beta := h_0^{(-1)} h_\beta - 1 \in C(\mathbb{R}_\beta, S^{-1}(\mathbb{R}^{1+q+l}; H, H)),$$

and we have

$$h_\beta - h_0 \in C(\mathbb{R}_\beta, S^{\mu-1}(\mathbb{R}^{1+q+l}; H, \widetilde{H})).$$

By computing the Leibniz inverse of h_β we obtain a

$$p_\beta \in C(\mathbb{R}_\beta, S^{\mu-1}(\mathbb{R}^{1+q+l}; H, H)$$

such that

$$c_\beta := \rho_\beta \# h_\beta - 1 \in C(\mathbb{R}_\beta, S^{-\infty}(\mathbb{R}^{1+q+l}; H, H)).$$

Then

$$\mathrm{Op}_y \mathrm{Op}_{M_r}^{1/2}(\rho_\beta)(\lambda) \mathrm{Op}_y \mathrm{Op}_{M_r}^{1/2}(h_\beta)(\lambda) = \mathrm{Op}_y \mathrm{Op}_{M_r}^{1/2}(\rho_\beta \# h_\beta)(\lambda) = 1 + \mathrm{Op}_y \mathrm{Op}_{M_r}^{1/2}(c_\beta)(\lambda) \tag{1.105}$$

where 1 denote the identity operator in $\mathcal{H}^{s,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H)$. Since

$$\|\mathrm{Op}_y \mathrm{Op}_M^{1/2}(c_\beta)(\lambda)\|_{\mathcal{H}^{s,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H)} \leq D\langle\lambda\rangle^{-N}$$

for every $N \in \mathbb{N}$ for some $D = D(N, B), \beta \in [-B, B]$, the observation follows from (1.105),by applying Neumann series .

We now Leibniz invert $1 + r_\beta$, i.e., obtain a $p \in C(\mathbb{R}_\beta, S^{-1}(\mathbb{R}^{1+q+l}; \widetilde{H}, H))$ such that

$$c_\beta := \rho_\beta h_\beta - 1 \in C(\mathbb{R}_\beta, S^{-\infty}(\mathbb{R}^{1+q+l}; H, H)).$$

Thus

$$\mathrm{Op}_y \mathrm{Op}_M^{1/2}(\rho_\beta)(\lambda) \mathrm{Op}_y \mathrm{Op}_M^{1/2}(h_\beta)(\lambda) = \mathrm{Op}_y \mathrm{Op}_M^{1/2}(\rho_\beta h_\beta)(\lambda) = 1 + c_\beta(\lambda)$$

for

$$c_\beta(\lambda) = \mathrm{Op}(c_\beta)(\lambda) \in C(\mathbb{R}_\beta, \mathcal{S}(\mathbb{R}^l, \mathcal{L}(\mathcal{H}^{s,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H), \mathcal{H}^{\infty,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H))))$$

where 1 denotes the identity operator in $\mathcal{H}^{s,\frac{b+1}{2}}(\mathbb{R}_+ \times \mathbb{R}^q, H)$. $\square$

2 Asymptotics

2.1 Weighted spaces with asymptotics

We study weighted Sobolev spaces on a manifold with singularities and subspaces with asymptotics. A simple special case is the half-space $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$ in the variables (r, y). Then smoothness of a function $u(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q)$ up to the boundary is connected with the Taylor expansion at $r = 0$,

$$u(r, y) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^j}{\partial r^j} u(0, y) r^j.$$

Smooth solutions to elliptic equations in the open half-space are not necessarily smooth up to $r = 0$. In this case, there are more general asymptotics, for instance, the singular function of asymptotic

$$u(r, y) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} c_{jk}(y) r^{-p_j} \log^k r \tag{2.1}$$

as $r \to 0$, for coefficients $c_{jk} \in C^\infty(\mathbb{R}^q)$, with exponents $-p_j \in \mathbb{C}$ and logarithmic powers in the variable r normal to the boundary. Such a behavior can appear for solutions u to edge-degenerate equations of order $\mu \in \mathbb{N}$,

$$Au(r, y) := r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(r, y) \big(- r \frac{\partial}{\partial r} \big)^j (r D_y)^\alpha u(r, y). \tag{2.2}$$

for coefficients $a_{j\alpha} \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q)$. Ellipticity of A means that the homogeneous principal symbol of A of order μ, namely,

$$\sigma_\psi(A)(r, y, \rho, \eta) = r^{-\mu} \sum_{j+|\alpha| = \mu} a_{j\alpha}(r, y)(-ir\rho)^j (r\eta)^\alpha, \tag{2.3}$$

does not vanish when $(\rho, \eta) \neq 0$, for all $(r, y) \in \mathbb{R}_+ \times \mathbb{R}^q$. Edge-degenerate means that the derivatives in r and y are multiplied by the factor $r \in \mathbb{R}_+$. Then the ellipticity degenerates for $r \to 0$. In the analysis of such operators we assume in addition that

$$\tilde{\sigma}_\psi(A)(r, y, \rho, \eta) = r^\mu \sigma_\psi(A)(r, y, r^{-1}\rho, r^{-1}\eta) \tag{2.4}$$

does not vanish for $(\rho, \eta) \neq 0$, up to $r = 0$. The asymptotic data in (2.1) depend on the conormal symbol

$$\sigma_c(A)(y, w) = \sum_{j=0}^{\mu} a_{j0}(0, y) w^j, \quad w \in \mathbb{C},$$

subordinate to the principal edge symbol

$$\sigma_\wedge(A)(y, \eta) = r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j\alpha}(0, y)(-r \frac{\partial}{\partial r})^j (r\eta)^\alpha, \ \eta \neq 0, \tag{2.5}$$

the edge symbol (2.5) is a (y, η)-depending family of differential operators over $\mathbb{R}_+$. The ellipticity implies that there is a (y -depending) discrete set $D(y) \subset \mathbb{C}$ with finite intersection

$$D(y) \cap \{c \leq \operatorname{Re} w \leq c'\}$$

for every $c \leq c'$, such that $\sigma_c(A)(y,w) \neq 0$ for all $w \in \mathbb{C}\backslash D(y)$. The inverse $(\sigma_c(A))^{-1}(y,w)$ extends to a meromorphic function with poles at the points $s(y) \in D(y)$. Let $n(y)+1$ denote the multiplicity of $s(y)$. We obtain a sequence of pairs

$$R(y) := \{(s_j(y), n_j(y))\}_{j\in\mathbb{I}} \subset \mathbb{C}\times\mathbb{N},$$

a so-called discrete (y-depending) Mellin asymptotic type where

$$\Pi_{\mathbb{C}} R(y) = \{s_j(y)\}_{j\in\mathbb{I}} = D(y).$$

In the characterization of regularity of solutions u we can expect a similar sequence

$$P(y) := \{(p_j(y), m_j(y))\}_{j\in\mathbb{N}} \subset \mathbb{C}\times\mathbb{N},$$

called a (y-depending) discrete asymptotic type, in this case for distributions on $\mathbb{R}_+$, where $P(y)$ is derived from $R(y)$ in the process of constructing a parametrix, according to the rules of the edge pseudo-differential calculus, In general we also have a dependence of $P(y)$ on a prescribed weight $\gamma \in \mathbb{R}$, such that

$$\{\mathrm{Re}\, w = \frac{1}{2} - \gamma\} \cap D(y) = \emptyset.$$

We mainly study here the case that $D(y)$ is independent of y. Otherwise, if $D(y)$ is not constant in y we employ the concept of continuous and variable discrete asymptotics.
We now consider subspaces of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ i.e. the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+, H)$ for $H = \mathbb{C}$ with discrete asymptotics as $r \to 0$. Because of heavy technicalities we focus on the case of $\mathbb{R}_+$. Similar consideration are valid on more general stretched cones $X^\wedge$. We define various types of asymptotics, motivated by an analogue of Taylor asymptotics, i.e., the Taylor expansion. A sequence

$$P := \{(p_j, m_j)\}_{j=0,\dots,N} \subset \mathbb{C}\times\mathbb{N} \tag{2.6}$$

for an $N \in \mathbb{N}\cup\{\infty\}$ is called a discrete asymptotic type associated with the weight data (γ, Θ) for $\Theta = (\theta, 0],\ -\infty \leq \theta \leq 0$, if

$$\Pi_{\mathbb{C}} P := \{p_j\}_{j=0,\dots,N} \subset \{\frac{1}{2} - \gamma + \theta < \mathrm{Re}\, w < \frac{1}{2} - \gamma\},$$

and $\mathrm{Re}\, p_j \to -\infty$, as $j \to \infty$, when $\theta = -\infty$ and $N = +\infty$. In this sense the Taylor asymptotic type $T = \{(-j, 0)\}_{j\in\mathbb{N}}$ is associated with the weight data $(0, (-\infty, 0])$. Without loss of generality we assume $\mathrm{Re}\, p_{j+1} \leq \mathrm{Re}\, p_j$ for all j. We say that P satisfies the shadow condition if $p \in \Pi_{\mathbb{C}} P$ implies $p - k \in \Pi_{\mathbb{C}} P$ for all $k \in \mathbb{N}$ with

$$\mathrm{Re}\, p - k > \frac{1}{2} - \gamma + \theta.$$

In future we facility assume that our asymptotic types satisfy the shadow condition .
In this paper, a cut-off function is a real-valued $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$ such that $\omega(r) = 1$ in a neighborhood of $r = 0$. For references below we introduce translated asymptotic types, namely,

$$T^\beta P := \{(p+\beta, m) : (p,m) \in P\}. \tag{2.7}$$

Definition 2.1. *Let P be an asymptotic type associated with the weight data (γ, Θ).*
(i) *For finite Θ we form spaces of singular functions of type P on the half-axis*

$$\mathcal{E}_P(\mathbb{R}_+) := \Big\{\omega(r)\sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r : c_{jk} \in \mathbb{C}\Big\},$$

for a fixed cut-off function ω and

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) := \mathcal{E}_P(\mathbb{R}_+) + (1-\omega)\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$$

as a non-direct sum.
(ii) *For P associated with $(\gamma, (-\infty, 0])$ we form*

$$P_l := \{(p, m) \in P : \operatorname{Re} p > -(l+1)\}$$

for $l \in \mathbb{N}$ which is associated with $(\gamma, (-(l+1), 0])$. Then, using (i) *we form the space $\mathcal{H}_{P_l}^{s,\gamma}(\mathbb{R}_+)$ and then*

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) := \bigcap_{l \in \mathbb{N}} \mathcal{H}_{P_l}^{s,\gamma}(\mathbb{R}_+).$$

Let us set

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = \{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+),\ v \in H^s(\mathbb{R}_+)\}, \tag{2.8}$$

for some fixed cut-off function ω and $H^s(\mathbb{R}_+) := H^s(\mathbb{R})|_{\mathbb{R}_+}$. Moreover, define

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) := \{\omega u + (1-\omega)v : u \in \mathcal{H}_P^{s,\gamma}(\mathbb{R}_+),\ v \in \mathcal{H}^s(\mathbb{R}_+)\}. \tag{2.9}$$

For purpose below we also form spaces with weight $e \in \mathbb{R}$ at infinity, namely,

$$\mathcal{K}^{s,\gamma;e}(\mathbb{R}_+) := \big\{\omega u + (1-\omega)v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+), v \in r^{-e}\mathcal{H}^s(\mathbb{R}_+)\big\}$$

and similarly

$$\mathcal{K}_P^{s,\gamma;e}(\mathbb{R}_+) := \big\{\omega u + (1-\omega)v : u \in \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+), v \in r^{-e}\mathcal{H}^s(\mathbb{R}_+)\big\}.$$

Remark 2.2. *Setting*

$$\mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\theta-\varepsilon}(\mathbb{R}_+)$$

which is a Fréchet subspace of $\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+)$, we have

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{K}_P^{\infty,\gamma}(\mathbb{R}_+)$$

as a non-direct sum of Fréchet spaces, or alternatively, for finite Θ and $\Pi_\mathbb{C} P \subset \{\frac{1}{2}-\gamma+\theta <$ $\operatorname{Re} w < \frac{1}{2}-\gamma\}$

$$\mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{E}_P(\mathbb{R}_+)$$

which is a direct sum. For infinite Θ we have

$$\mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+) = \mathcal{K}^{s,\infty}(\mathbb{R}_+) = \bigcap_{N \in \mathbb{N}} \mathcal{K}^{s,N}(\mathbb{R}_+),$$

independent of γ. Similarly, setting

$$\mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+) = \varprojlim_{\varepsilon>0}(\omega\mathcal{H}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+) + (1-\omega)\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$$

we have

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{H}_P^{\infty,\gamma}(\mathbb{R}_+).$$

Let us fix a discrete asymptotic type $P = \{(p_j, m_j)\}_{j=0,\dots,N}$ associated with the weight data (γ, Θ). By a P-excision function we understand a $\chi_P \in C^\infty(\mathbb{C})$ such that $\chi_P(z) = 0$ for $\mathrm{dist}(z, \Pi_{\mathbb{C}} P) < \varepsilon_0$, $\chi_P(z) = 1$ for $\mathrm{dist}(z, \Pi_{\mathbb{C}} P) > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$.
Define $\mathcal{A}_P^{\infty,\gamma}$ as the space of all meromorphic functions f in the strip $\{\frac{1}{2} - \gamma + \vartheta < \mathrm{Re}\, z < \frac{1}{2} - \gamma\}$ with poles at the points p_j of multiplicity $m_j + 1$, such that $\chi_P f \in C^\infty\big(\{\frac{1}{2} - \gamma + \vartheta < \mathrm{Re}\, z \le \frac{1}{2} - \gamma\}\big)$ for every P-excision function χ_P and $\chi_P f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for all $\frac{1}{2} - \gamma + \vartheta < \beta \le \frac{1}{2} - \gamma$, uniformly in compact β intervals.
Moreover, let $\mathcal{A}_\Theta^{s,\gamma}$ be the space of all

$$f \in \mathcal{A}\big(\{\frac{1}{2} - \gamma + \theta < \mathrm{Re}\, w < \frac{1}{2} - \gamma\}\big) \bigcap C\big(\{\frac{1}{2} - \gamma + \theta < \mathrm{Re}\, w \le \frac{1}{2} - \gamma\}\big)$$

such that $f|_{\Gamma_\beta} \in \hat{H}^s(\Gamma_\beta)$ for every $\frac{1}{2} - \gamma + \theta < \beta < \frac{1}{2} - \gamma$ uniformly in compact β-subintervals. Both $\mathcal{A}_P^{\infty,\gamma}$ and $\mathcal{A}_\Theta^{s,\gamma}$ are Fréchet spaces in a natural way and we set

$$\mathcal{A}_P^{s,\gamma} := \mathcal{A}_\Theta^{s,\gamma} + \mathcal{A}_P^{\infty,\gamma} \tag{2.10}$$

in the Fréchet topology of the non-direct sum.

Proposition 2.3. *Let ω be a cut-off function on the half-axis.*
(i) *The weighted Mellin transform induces a map*

$$M_\gamma \omega : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{A}_P^{s,\gamma}.$$

(ii) *The inverse of the weighted Mellin transform induces a map*

$$\omega M_\gamma^{-1} : \mathcal{A}_P^{s,\gamma} \to \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+).$$

Proof. (i) We first assume that the weight interval Θ is finite and P associated with (γ, Θ). As before we identify functions f with M_f. Then we can write $M_\gamma \omega = M_\gamma M_\omega$. From (2.9) we see that

$$\mathcal{M}_\omega : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}_P^{s,\gamma}(\mathbb{R}_+)$$

is a continuous operator. Let us write

$$\mathcal{H}_P^{s,\gamma}(\mathbb{R}_+) = \mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+) + \mathcal{E}_P(\mathbb{R}_+).$$

We now employ the fact that M_γ induces continuous operators

$$M_\gamma : \mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+) \to \mathcal{A}_\Theta^{s,\gamma}, \tag{2.11}$$

$$M_\gamma : \mathcal{E}_P(\mathbb{R}_+) \to \mathcal{A}_P^{\infty,\gamma}. \tag{2.12}$$

In fact, the first map is continuous because of the Paley-Wiener theorem for the Mellin transform. Next we show the continuity of the second map. To this end we first observe the relation

$$M\omega(w) = w^{-1} M(-r\partial_r \omega)(w). \tag{2.13}$$

Because of $\partial_r \omega(r) \in C_0^\infty(\mathbb{R}_+)$ we obtain $M(-r\partial_r\omega) \in \mathcal{A}(\mathbb{C})$ and $M(-r\partial_r\omega)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. Thus, by virtue of (2.13) we obtain that $M\omega(w)$ is meromorphic with a simple pole at $w = 0$, and $\chi(w)M\omega(w)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ as before, for any 0-excision function χ (i.e., $\chi(w) = 0$ for $|w| < \varepsilon_0,\ \chi(w) = 1$ for $|w| > \varepsilon_1$). Now differentiating (2.13) with respect to w gives us

$$\begin{aligned}\partial_w M\omega(w) = \int_0^\infty (\partial_w r^w)\omega(r)\frac{dr}{r} &= \int_0^\infty (r^w \log r)\omega(r)\frac{dr}{r}\\ &= M(\omega(r)\log r)(w)\\ &= w^{-1}M(-r\partial_r(\omega(r)\log r))(w)\\ &= -w^{-2}M(-r\partial_r\omega(r)) + w^{-1}\partial_w M(-r\partial_r\omega(r))\\ &= -w^{-2}f(w) + w^{-1}g(w)\end{aligned}$$

for

$$f(w) = M(-r\partial_r\omega(r)) \in \mathcal{A}^{\infty,0}_\Theta,\ g(w) = \partial_w M(-r\partial_r\omega(r)) \in \mathcal{A}^{\infty,0}_\Theta.$$

Thus

$$M(\omega(r)\log r) \in \mathcal{A}^{\infty,0}_{P_1}$$

for the asymptotic type $P_1 = \{(0,1)\}$ associated with $(0,\Theta)$, $\Theta = (-\infty, 0]$. By induction it follows that

$$M(\omega(r)\log^k r) \in \mathcal{A}^{\infty,0}_{P_2}$$

for $P_2 = \{(0,k)\}$ associated with $(0,\Theta)$, $\Theta = (-\infty,0]$. Thus $M\omega(w) \in \mathcal{A}^{\infty,0}_{P_0}$ for any the asymptotic type $P_0 = \{(0,0)\}$ associated with $(0,\Theta)$, $\Theta = (-\infty,0]$. Finally, using the identify

$$M(r^{-p}u)(z) = Mu(w-p)$$

for any $p \in \mathbb{C}$. Thus from the latter computation it follows that

$$M(\omega(r)r^{-p}\log^k r) \in \mathcal{A}^{\infty,\gamma}_P$$

for every $p \in \mathbb{C},\ \mathrm{Re}\, p < \frac{1}{2} - \gamma$, and $p = \{(p,k)\}$ associated with $(\gamma,\Theta),\ \Theta = (-\infty,0]$. For a finite weight interval Θ the space $\mathcal{E}_P(\mathbb{R}_+)$ is of finite dimension since the weighted Mellin transform is an injective map, say, on $r^\gamma L^2(\mathbb{R}_+)$, the image of (2.12) is of finite dimension as well. By the computation before we have $M_\gamma\mathcal{E}_P(\mathbb{R}_+) \subset \mathcal{A}^{\infty,\gamma}_P$ and we immediately see the continuity of (2.12).

For the case of infinite Θ we can form

$$P_l := \{(p,n) \in P : \mathrm{Re}\, p > \frac{1}{2} - \gamma - (l+1)\}$$

for every $l \in \mathbb{N}$. Then P_l is associated with (γ, Θ_l) and we have the continuity of the map

$$M_\gamma : \mathcal{E}_{P_l}(\mathbb{R}_+) \to \mathcal{A}^{\infty,\gamma}_{P_l}$$

for every l. Because of the first part of the proof this implies the continuity of

$$M_\gamma\omega : \mathcal{K}^{s,\gamma}_{P_l}(\mathbb{R}_+) \to \mathcal{A}^{s,\gamma}_{P_l}$$

for every l. Since

$$\mathcal{K}^{s,\gamma}_{P_{l+1}}(\mathbb{R}_+) \hookrightarrow \mathcal{K}^{s,\gamma}_{P_l}(\mathbb{R}_+),\ \mathcal{A}^{s,\gamma}_{P_{l+1}} \hookrightarrow \mathcal{A}^{s,\gamma}_{P_l}$$

are continuous and

$$\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+) = \varprojlim_{l\in N} \mathcal{K}^{s,\gamma}_{P_l}(\mathbb{R}_+),\ \mathcal{A}^{s,\gamma}_P = \varprojlim_{l\in N} \mathcal{A}^{s,\gamma}_{P_l}$$

we obtain the continuity of (2.12) also for $\Theta = (-\infty, 0]$.
(ii) First we employ that M_γ^{-1} induces an isomorphism

$$M_\gamma^{-1} : \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}) \to \mathcal{H}^{s,\gamma}(\mathbb{R}_+),$$

cf. the relation (1.37). Thus, applying $\mathcal{M}_\omega$, we have a continuous operator

$$\omega M_\gamma^{-1} : \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$$

Let us now consider the case of a finite weight interval $\Theta = (-(l+1), 0]$, and P is associated with the weight data (γ, Θ). From the first part of the proof we know that $M_\gamma \mathcal{E}_P(\mathbb{R}_+)$ is a finite-dimensional subspace of $\mathcal{A}_P^{\infty,\gamma}$. Moreover,

$$\mathcal{A}_P^{s,\gamma} = \mathcal{A}_\Theta^{s,\gamma} + M_\gamma \mathcal{E}_P(\mathbb{R}_+)$$

is a direct sum for every $s \in \mathbb{R}$. Because of the definition of $\mathcal{A}_\Theta^{s,\gamma}$ for $\Theta = (-(l+1), 0]$ and from the isomorphism (1.37) we obtain that for every $f \in \mathcal{A}_\Theta^{s,\gamma}$ we have

$$(M_\gamma^{-1} f)(r) \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+)$$

but also

$$(M_{\gamma+m}^{-1} f)(r) \in \mathcal{H}^{s,\gamma+m}(\mathbb{R}_+) \tag{2.14}$$

for $0 \le m < l+1$. By Cauchy's theorem relation (2.14) can also be interpreted as

$$(M_\gamma^{-1} f) \in \bigcap_{0 \le m < l+1} \mathcal{H}^{s,\gamma+m}(\mathbb{R}_+)$$

because (2.14) refers to the extension of $f(w)$ into the strip $\frac{1}{2} - \gamma - (l+1) - \varepsilon < \mathrm{Re}\, w \le \frac{1}{2} - \gamma$ as a holomorphic function for every $\varepsilon > 0$, it follows that

$$\omega (M_\gamma^{-1} f)(r) \in \omega \mathcal{H}_\Theta^{s,\gamma}(\mathbb{R}_+) = \omega \mathcal{K}_\Theta^{s,\gamma}(\mathbb{R}_+).$$

The arguments for the case of infinite Θ are similar as at the end of (i). □

Let us now turn to Mellin symbols with discrete asymptotics. Similarly as (2.6) we consider sequences

$$R = \{(r_j, n_j)\}_{j \in \mathbb{I}} \subset \mathbb{C} \times \mathbb{N} \tag{2.15}$$

for an index set $\mathbb{I} \subseteq \mathbb{Z}$ where we assume $\mathrm{Re}\, r_j \le \mathrm{Re}\, r_k$ whenever $j \le k$ for $j, k \in \mathbb{I}$, and $\Pi_\mathbb{C} R := \{r_j\}_{j \in \mathbb{I}}$ intersects every strip $\{c \le \mathrm{Re}\, w \le c'\}$ in a finite set. Moreover, if $\Pi_\mathbb{C} R$ is infinite, we assume $\mathrm{Re}\, r_j \to \infty$ as $|j| \to \infty$.
Let $M_R^{-\infty}$ be the subspace of all meromorphic functions $f(w)$ in $\mathbb{C}$ with poles at the points $r_j \in \Pi_\mathbb{C} R$ of multiplicity $n_j + 1$, for all $j \in \mathbb{I}$, and $\chi_R f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for any R-excision function χ_R and every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.
Moreover, let $M_\mathcal{O}^\mu$ for $\mu \in \mathbb{R}$ be the space of all $h \in \mathcal{A}(\mathbb{C})$ such that $h|_{\Gamma_\beta} \in S_{\mathrm{cl}}^\mu(\Gamma_\beta)$ for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.
Both $M_R^{-\infty}$ and $M_\mathcal{O}^\mu$ are Fréchet spaces in a natural way. Set

$$M_R^\mu := M_\mathcal{O}^\mu + M_R^{-\infty} \tag{2.16}$$

in the Fréchet topology of the non-direct sum . Let $f \in M_R^\mu$ and $\gamma \in \mathbb{R}$ such that

$$\Gamma_{\frac{1}{2}-\gamma} \bigcap \pi_{\mathbb{C}} R = \emptyset.$$

Then

$$f|_{\Gamma_{\frac{1}{2}-\gamma}} \in S_{\mathrm{cl}}^\mu(\Gamma_{\frac{1}{2}-\gamma})$$

and it makes sense to form the associated weighted Mellin operators

$$\mathrm{Op}_M^\gamma(f) = M_\gamma^{-1} f M_\gamma = r^\gamma \mathrm{Op}_M(T^{-\gamma} f) r^{-\gamma}, \tag{2.17}$$

where $(T^{-\gamma}f)(w) := f(w-\gamma)$. Let ω, ω' be cut-off functions on the half-axis.

Theorem 2.4. *We have continuous operators*

$$\omega \mathrm{Op}_M^\gamma(f)\omega' : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s-\mu,\gamma}(\mathbb{R}_+) \tag{2.18}$$

and

$$\omega \mathrm{Op}_M^\gamma(f)\omega' : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}_Q^{s-\mu,\gamma}(\mathbb{R}_+) \tag{2.19}$$

for every discrete asymptotic type P and some resulting Q, both associated with (γ, Θ).

Proof. The operator $\omega \mathrm{Op}_M^\gamma(f)\omega'$ is a composition $\omega M_\gamma^{-1} \mathcal{M}_f M_\gamma \omega'$, and we have continuities of

$$M_\gamma \omega' : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}),$$
$$\mathcal{M}_f : \hat{H}^s(\Gamma_{\frac{1}{2}-\gamma}) \to \hat{H}^{s-\mu}(\Gamma_{\frac{1}{2}-\gamma}),$$
$$\omega M_\gamma^{-1} : \hat{H}^{s-\mu}(\Gamma_{\frac{1}{2}-\gamma}) \to \mathcal{K}^{s-\mu,\gamma}(\mathbb{R}_+),$$

This shows that (2.18) is continuous. Concerning (2.19) we have continuous operators

$$M_\gamma \omega' : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{A}_P^{s,\gamma}, \tag{2.20}$$

cf. Proposition 2.3 (i). Moreover, Proposition 2.3 (ii) gives us

$$\omega M_\gamma^{-1} : \mathcal{A}_Q^{s-\mu,\gamma} \to \mathcal{K}_Q^{s-\mu,\gamma}(\mathbb{R}_+). \tag{2.21}$$

Next we show the continuity of

$$\mathcal{M}_f : \mathcal{A}_P^{s,\gamma} \to \mathcal{A}_Q^{s-\mu,\gamma}. \tag{2.22}$$

To this end we write

$$\mathcal{A}_P^{s,\gamma} = \mathcal{A}_\Theta^{s,\gamma} + \mathcal{A}_P^{\infty,\gamma}.$$

Then

$$\mathcal{M}_f : \mathcal{A}_\Theta^{s,\gamma} \to \mathcal{A}_{Q_1}^{s-\mu,\gamma} \quad \text{and} \quad \mathcal{M}_f : \mathcal{A}_P^{\infty,\gamma} \to \mathcal{A}_{Q_2}^{\infty,\gamma}$$

give rise to

$$\mathcal{M}_f : \mathcal{A}_P^{s,\gamma} \to \mathcal{A}_{Q_1}^{s-\mu,\gamma} + \mathcal{A}_{Q_2}^{\infty,\gamma} \subseteq \mathcal{A}_Q^{s-\mu,\gamma}$$

for a discrete asymptotic type Q. Using (2.20), (2.21) and (2.22) yields the continuity of (2.19). $\square$

Definition 2.5. *An operator*

$$G : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+). \tag{2.23}$$

continuous for all $s \in \mathbb{R}$, *is called a Green operator, associated with the weight data* $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\Theta = (\theta, 0]$, $-\infty \le \theta < 0$, *if* (2.23) *induces continuous operators*

$$G : \mathcal{K}^{s,\gamma;e}(\mathbb{R}_+) \to \mathcal{K}_P^{s',\gamma-\mu;e'}(\mathbb{R}_+),$$

$$G^* : \mathcal{K}^{s,-\gamma+\mu;e}(\mathbb{R}_+) \to \mathcal{K}_Q^{s',-\gamma;e'}(\mathbb{R}_+),$$

for every s, e, s', $e' \in \mathbb{R}$, *where* P *and* Q *are* G*-dependent discrete asymptotic types, associated with* $(\gamma - \mu, \Theta)$ *and* $(-\gamma, \Theta)$, *respectively.*

Let $L_G(\mathbb{R}_+, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ denote the space of all Green operators associated with the weight data $\boldsymbol{g}$.

Example 2.6. *Let* $P = \{(p_j, m_j)\}_{j=0,\dots,N}$ *and* $Q = \{(q_l, n_l)\}_{l=0,\dots,L}$ *be fixed asymptotic types as in the Definition* 2.5, *choose functions*

$$f(r) \in \mathcal{E}_P(\mathbb{R}_+),\ \ f'(r') \in \mathcal{E}_Q(\mathbb{R}_+).$$

Then the operator G *defined by*

$$Gu(r) := \int_0^\infty f(r)\overline{f'(r')}u(r')dr'$$

is a Green operator associated with $\boldsymbol{g}$ *which makes sense since* $\int \overline{f'(r')}u(r')dr'$ *is finite for every* $u \in \mathcal{K}^{s,\gamma;e}(\mathbb{R}_+)$.

Proposition 2.7. *The operator*

$$M := r^{-\mu} \sum_{j=0}^{k} r^j \omega \mathrm{Op}_M^{\gamma_j}(f_j)\omega'$$

for symbols $f_j \in M_{R_j}^{-\infty}$, *with asymptotic types* R_j *and weights* γ_j *satisfying the conditions*

$$\Gamma_{\frac{1}{2}-\gamma_j} \bigcap \Pi_{\mathbb{C}} R_j = \emptyset,\ \gamma - j \le \gamma_j \le \gamma,$$

for $j = 0, \dots, k, k \in \mathbb{N}$ *induce continuous operator*

$$M : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+)$$

and

$$M : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}_Q^{\infty,\gamma-\mu}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$ *and every asymptotic type* P *associated with* $(\gamma, \Theta), \Theta = (-(k+1), 0]$, *for some resulting* Q *associated with* $(\gamma - \mu, \Theta)$.

Proof. We have

$$M = \sum_{j=0}^{k} M_j, \quad M_j := r^{-\mu+j}\omega \mathrm{Op}_M^{\gamma_j}(f_j)\omega'.$$

Let us show that

$$M_j : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+)$$

is continuous for every $s \in \mathbb{R}$, $j = 0, \dots, k$. In fact, the operator can be written as a composition

$$r^{-\mu+j} r^{\gamma_j}\omega(r) M^{-1}(T^{-\gamma_j} f_j) M \omega'(r) r^{-\gamma_j}$$

of continuous operators.

$$\omega'(r) r^{-\gamma_j} : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \omega'\mathcal{H}^{s,\gamma-\gamma_j}(\mathbb{R}_+) \hookrightarrow \mathcal{H}^{s,0}(\mathbb{R}_+),$$
$$M : \mathcal{H}^{s,0}(\mathbb{R}_+) \to \hat{H}^s(\Gamma_{\frac{1}{2}}),$$
$$T^{-\gamma_j} f_j : \hat{H}^s(\Gamma_{\frac{1}{2}}) \to \hat{H}^\infty(\Gamma_{\frac{1}{2}}),$$

since

$$f_j(\frac{1}{2} - \gamma_j + i\rho) = (T^{-\gamma_j} f_j)(\frac{1}{2} + i\rho) \in S^{-\infty}(\Gamma_{\frac{1}{2}}),$$

Moreover

$$\omega(r) M^{-1} : \hat{H}^\infty(\Gamma_{\frac{1}{2}}) \to \omega\mathcal{H}^{\infty,0}(\mathbb{R}_+),$$
$$r^{-\mu+j+\gamma_j} : \omega\mathcal{H}^{\infty,0}(\mathbb{R}_+) \to \omega\mathcal{H}^{\infty,j+\gamma_j-\mu}(\mathbb{R}_+) \hookrightarrow \mathcal{K}^{\infty,\gamma-\mu}(\mathbb{R}_+).$$

because $\gamma_j + j \geq \gamma$. Moreover, M_j defines continuous operators

$$M_j : \mathcal{K}_P^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}_Q^{\infty,\gamma-\mu}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$, $j = 0, \dots, K$, and every asymptotic type P associated with (γ, Θ) for some resulting Q associated with $(\gamma-\mu, \Theta)$. However, this is a easily reduced to Theorem 2.4. $\square$

Definition 2.8. *Let $L_{M+G}(\mathbb{R}_+, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta = (-(k+1), 0]$, $k \in \mathbb{N}$, be the space of all operators of the form $M + G$ for arbitrary $G \in L_G(\mathbb{R}_+, \boldsymbol{g})$ and M as in Proposition 2.7. For $\Theta = (-\infty, 0]$ we set*

$$L_{M+G}(\mathbb{R}_+, (-\infty, 0]) := \bigcap_{k \in \mathbb{N}} L_{M+G}(\mathbb{R}_+, (-(k+1), 0]).$$

The operators in $L_{M+G}(\mathbb{R}_+, \boldsymbol{g})$ are called smoothing Mellin plus Green operators in the cone algebra on $\mathbb{R}_+$, associated with the weight data $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$.

Let us set

$$\boldsymbol{\sigma}_{\mathrm{c}}^{\mu-j}(M)(w) := f_j(w),\ j = 0, \dots, k,$$

and $\boldsymbol{\sigma}_{\mathrm{c}}(M) := (\boldsymbol{\sigma}_{\mathrm{c}}^{\mu-j}(M))_{j=0,\dots,k}$.

Theorem 2.9. *Let*

$$M = r^{-\mu} \sum_{j=0}^{k} r^j \omega \mathrm{Op}_M^{\gamma_j}(f_j)\omega',\ \widetilde{M} = r^{-\mu} \sum_{j=0}^{k} r^j \tilde{\omega} \mathrm{Op}_M^{\tilde{\gamma}_j}(\tilde{f}_j)\tilde{\omega}'.$$

Then the following properties are equivalent:
(i) $M - \widetilde{M}$ *is a Green operator*
(ii) $\boldsymbol{\sigma}_{\mathrm{c}}(M) = \boldsymbol{\sigma}_{\mathrm{c}}(\widetilde{M})$.

2.2 Edge asymptotics

Considering a discrete asymptotic type $P = \{(p_j, m_j)\}_{j=0,\dots,N}$, $N \in \mathbb{N} \cup \{\infty\}$, for distributions in $\mathcal{K}^{s,\gamma}(X^\wedge)$, $n = \dim X$, we say that P is associated with the weight data (γ, Θ), $\Theta = (\theta, 0]$, $-\infty \leq \theta < 0$, if

$$\Pi_{\mathbb{C}} P \subset \{\frac{n+1}{2} - \gamma + \theta < \operatorname{Re} w < \frac{n+1}{2} - \gamma\}.$$

Set for fixed a cut-off function ω and finite Θ

$$\mathcal{E}_P(X^\wedge) := \omega(r)\Big\{\sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk}(x) r^{-p_j} \log^k r : c_{jk} \in C^\infty(X)\Big\},$$

for all j, k and

$$\mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) := \varprojlim_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge).$$

We then have $\mathcal{E}_P(X^\wedge) \bigcap \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge)$

$$\mathcal{K}^{s,\gamma}_{P}(X^\wedge) := \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) + \mathcal{E}_P(X^\wedge) \tag{2.24}$$

which is a direct sum. Analogously as in the case $\mathbb{R}_+$ we can introduce spaces $\mathcal{K}^{s,\gamma}_{P}(X^\wedge)$ also for infinite Θ. Both for finite and infinite Θ we have non-direct decompositions

$$\mathcal{K}^{s,\gamma}_{P}(X^\wedge) := \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) + \mathcal{K}^{\infty,\gamma}_{P}(X^\wedge)$$

for $\mathcal{K}^{\infty,\gamma}_{P}(X^\wedge) = \varprojlim_{s\in\mathbb{R}} \mathcal{K}^{s,\gamma}_{P}(X^\wedge)$. The spaces (1.95) are Hilbert spaces where $\mathcal{K}^{s,\gamma}(X^\wedge)$ is endowed with the topology of the non-direct sum, while the space (2.24) are Fréchet.

Remark 2.10. (i) $\mathcal{K}^{s,\gamma}(X^\wedge), s, \gamma \in \mathbb{R}$, *is a Hilbert space with group action* $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ *given by*

$$(\kappa_\delta u)(r, x) := \delta^{(n+1)/2} u(\delta r, x), \tag{2.25}$$

$n = \dim X$. *Observe that*

$$\mathcal{H}^{0,0}(X^\wedge) = \mathcal{K}^{0,0}(X^\wedge) = r^{-n/2} L^2(\mathbb{R}_+ \times X) \tag{2.26}$$

with respective to the measure drdx and dx referring to a Riemannian metric on X. Then the operators (2.25) *are unitary in* $\mathcal{K}^{0,0}(X^\wedge)$.
(ii) *The space* $\mathcal{K}^{s,\gamma}_{P}(X^\wedge)$ *for any discrete asymptotic type is a Fréchet space with group action* κ *and it can be written as a projective limit*

$$\mathcal{K}^{s,\gamma}_{P}(X^\wedge) = \varprojlim_{j\in\mathbb{N}} E^j \tag{2.27}$$

of Hilbert spaces for $E^0 := \mathcal{K}^{s,\gamma}(X^\wedge)$ *and continuous embedding* $E^j \hookrightarrow E^0$ *for all* j, *where* E^0 *is endowed with a group action* $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ *of the form* (2.25) *that restricts to a group action on* E^j *for every* $j \in \mathbb{N}$.

Remark 2.11. *There is an immediate analogue of Theorem* 2.4, *namely, the continuity of*

$$\omega \mathrm{Op}_M^{\gamma-n/2}(f)\omega' : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge) \tag{2.28}$$

for a Mellin symbol $f(w)$ taking values in $L^\mu(X;\Gamma_{\frac{n+1}{2}-\gamma})$, $n = \dim X$, and

$$\omega \mathrm{Op}_M^{\gamma-n/2}(f)\omega' : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma}(X^\wedge) \tag{2.29}$$

for a Mellin symbol $f(w) \in M_R^\mu(X)$ with a meromorphic extension to the complex w-plane, analogously as (2.16).

The proof is based on an extension of the spaces occurring in (2.10). Since we elaborate similar things below for the case of iterated asymptotics we drop the details.
In this case we can form $\mathcal{W}^s(\mathbb{R}^q, E^j)$ which are Hilbert spaces in a natural way, with continuous embeddings $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^0)$ for all j, as in the Remark 1.22. Applying Definition 1.17 to $H = \mathcal{K}^{s,\gamma}(X^\wedge)$ cf. (1.95) we can form the spaces

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)). \tag{2.30}$$

based on Remark 1.31.

Lemma 2.12. *We have*

$$\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge))_\kappa = L^2(\mathbb{R}^q, r^{-n/2}L^2(\mathbb{R}_+ \times X)). \tag{2.31}$$

Proof. The norm in $\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge))_\kappa$ is equal to

$$\Big\{ \int_{\mathbb{R}^q_\eta} \|\kappa^{-1}_{\langle\eta\rangle}(F_{y\to\eta}u)(\eta)\|^2_{\mathcal{K}^{0,0}(X^\wedge)} đ\eta \Big\}^{1/2} = \Big\{ \int_{\mathbb{R}^q} \|(F_{y\to\eta}u)(\eta)\|^2_{\mathcal{K}^{0,0}(X^\wedge)} đ\eta \Big\}^{1/2} \tag{2.32}$$

since $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, cf. formula (2.25), is acting as a group of unitary operators in $\mathcal{K}^{0,0}(X^\wedge)$. By using Parseval's formula, applied to functions with values in a Hilbert space we have

$$L^2(\mathbb{R}^q_\eta, H) = L^2(\mathbb{R}^q_y, H), \tag{2.33}$$

more precisely,

$$\|(Fu)(\eta)\|^2_{L^2(\mathbb{R}^q_\eta, H)} = (2\pi)^q \|u(y)\|^2_{L^2(\mathbb{R}^q_y, H)}.$$

For a $v(r,x) \in \mathcal{K}^{0,0}(X^\wedge)$ we have

$$\|v\|^2_{\mathcal{K}^{0,0}(X^\wedge)} = \int_{\mathbb{R}_+}\int_X |v(r,x)|^2 r^n dr dx$$

for $n = \dim X$, which corresponding to (2.26). Thus from (2.32), (2.33) it follows that

$$\|u\|^2_{\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge))} = \int_{\mathbb{R}^q}\int_{\mathbb{R}_+} \|u(y,r,\cdot)\|^2_{L^2(X)} r^n dr dy, \tag{2.34}$$

i.e., we obtain (2.31). ∎

For references below we fix in (2.31) a corresponding Hilbert scalar product, namely

$$(u,v)_{\mathcal{W}^0(\mathbb{R}^q,\mathcal{K}^{0,0}(X^\wedge))} = \iint (u(y,r,\cdot), v(y,r,\cdot))_{L^2(X)} r^n dr dy \tag{2.35}$$

In order to understand the nature of discrete edge asymptotic we briefly return to the general distribution (1.58) and (1.63) of abstract edge spaces. Let us first look at (1.58). From the norm expression (1.58) we obtain

$$\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)_\kappa} = \int \langle\eta\rangle^{2s} \|FF^{-1}\kappa^{-1}_{\langle\eta\rangle} Fu(\eta)\|^2_H đ\eta$$

$$= \int \langle\eta\rangle^{2s} \|F(K^{-1}u)\|^2_H đ\eta = \|(K^{-1}u)\|^2_{H^s(\mathbb{R}^q,H)} = \|(K^{-1}u)\|^2_{\mathcal{W}^s(\mathbb{R}^q,H)_{\mathrm{id}}}$$

for the operator $K := F^{-1}\kappa_{\langle\eta\rangle}F$. In other words, K induces an isomorphism

$$K : H^s(\mathbb{R}^q, H) \to \mathcal{W}^s(\mathbb{R}^q, H)_\kappa. \tag{2.36}$$

From the definition of edge spaces it follows that

$$\mathcal{W}^\infty(\mathbb{R}^q, H)_\kappa = H^\infty(\mathbb{R}^q, H) \tag{2.37}$$

is independent of the choice of κ.

Proposition 2.13. *Let $H := H^s(\mathbb{R}^{n+1})$, $s \in \mathbb{R}$, be endowed with the group action*

$$\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}, \ (\kappa_\delta u)(x) := \delta^{\frac{n+1}{2}} u(\delta x), \ \delta \in \mathbb{R}_+.$$

Then we have

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^{n+1})) = H^s(\mathbb{R}^{n+1} \times \mathbb{R}^q).$$

Proof. By definition for $u(\tilde{x}, y) \in \mathcal{W}^s(\mathbb{R}^q_y, H^s(\mathbb{R}^{n+1}_{\tilde{x}}))$ we have (up to equivalence of norms)

$$\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,H^s(\mathbb{R}^{n+1}))} = \int \langle\eta\rangle^{2s} \|\kappa^{-1}_{\langle\eta\rangle}(F_{y\to\eta}u)(\tilde{x},\eta)\|^2_{H^s(\mathbb{R}^{n+1}_{\tilde{x}})} d\eta \tag{2.38}$$

Moreover, using the substitution $\tilde{\zeta} = \tilde{\xi}/[\eta]$, i.e., $d\tilde{\xi} = [\eta]^{n+1} d\tilde{\zeta}$, for $F = F_{\tilde{x}\to\tilde{\zeta}} F_{y\to\tilde{\eta}}$ we obtain

$$\begin{aligned} \|u\|^2_{H^s(\mathbb{R}^{n+1+q})} &= \iint (|\tilde{\xi}|^2 + [\eta]^2)^s |(Fu)(\tilde{\xi},\eta)|^2 d\tilde{\xi} d\eta \\ &= \iint [\eta]^{2s} (1 + (\frac{|\tilde{\xi}|}{[\eta]})^2)^s |(Fu)(\tilde{\xi},\eta)|^2 d\tilde{\xi} d\eta \\ &= \iint [\eta]^{2s} (1 + |\tilde{\zeta}|^2)^s |[\eta]^{(n+1)/2} (Fu)([\eta]\tilde{\zeta},\eta)|^2 d\tilde{\zeta} d\eta \\ &= \iint [\eta]^{2s} \langle\tilde{\zeta}\rangle^{2s} |\kappa_{\langle\eta\rangle}(Fu)(\tilde{\zeta},\eta)|^2 d\tilde{\zeta} d\eta \\ &= \iint [\eta]^{2s} \langle\tilde{\xi}\rangle^{2s} |F_{\tilde{x}\to\tilde{\xi}} \kappa^{-1}_{\langle\eta\rangle}(F_{y\to\eta}u)(\tilde{x},\eta)|^2 d\tilde{\xi} d\eta \end{aligned} \tag{2.39}$$

which is equal to (2.38), using the identity

$$\kappa_\lambda(F_{\tilde{x}\to\tilde{\zeta}} f)(\tilde{\zeta}) = F_{\tilde{x}\to\tilde{\zeta}}(\kappa^{-1}_\lambda f)(\tilde{\zeta}).$$

$\square$

Observe, cf. [53] or [58], that

$$(\chi_\delta f)(y) := \delta^{\frac{q}{2}}(\kappa_\delta f)(\delta y)$$

with κ_δ operating on the values of f in the corresponding spaces defines a group action $\chi = \{\chi_\delta\}_{\delta\in\mathbb{R}_+}$ on $\mathcal{W}^s(\mathbb{R}^q, H)_\kappa$. Then we have on the space $\mathcal{W}^s(\mathbb{R}^q, H)$ the relation

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi = \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa. \tag{2.40}$$

Another useful variant of (2.40) has been obtained Proposition 2.2 in [48]. Let us formulate this relation here in a more general form, namely, an H-valued variant of (1.40) for a Hilbert space H with group action κ, defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, H)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^q,H)} := \Big\{ \int_{\Gamma_{\frac{b+1}{2}-\gamma}} \int_{\mathbb{R}^q} \langle w,\eta\rangle^{2s} \|\kappa^{-1}_{\langle w,\eta\rangle}(MFu)(w,\eta)\|^2_H đw đ\eta \Big\}^{1/2}. \tag{2.41}$$

The number b is specified in connection with H, e.g., $b = n$ for $H = \mathcal{K}^{s,\gamma}(X^\wedge),\ n = \dim X$.

Proposition 2.14. *We have*

$$\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi = \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^{p+q}, H)_\kappa \tag{2.42}$$

for every $s, \gamma \in \mathbb{R}$.

Proof. The result follows from the identification of $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^p, \tilde{H})_\chi$ by applying the isomorphism

$$S_{\gamma-\frac{b}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^p, \tilde{H})_\chi \to \mathcal{W}^s(\mathbb{R} \times \mathbb{R}^p, \tilde{H})_\chi \tag{2.43}$$

for a Hilbert spaces $\widetilde{H}$ with group action χ, defined by

$$(S_{\gamma-\frac{b}{2}}u)(\boldsymbol{t}, z) = e^{-((b+1)/2-\gamma)\boldsymbol{t}}u(e^{-\boldsymbol{t}}, z), \tag{2.44}$$

and

$$(S^{-1}_{\gamma-\frac{b}{2}}v)(t, z) = t^{((b+1)/2-\gamma)}v(-\log t, z),$$

$t \in \mathbb{R}_+,\ \boldsymbol{t} \in \mathbb{R},\ z \in \mathbb{R}^p$. Using the isomorphism,

$$\mathcal{I}_p : \mathcal{W}^s(\mathbb{R} \times \mathbb{R}^p, \tilde{H})_\chi \to \mathcal{W}^s(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^p, \tilde{H}))_{\tilde{\chi}}$$

coming from (2.40) for $\widetilde{H} = \mathcal{W}^s(\mathbb{R}^q, H)_\kappa$ and $\tilde{\chi}_\delta f(z) = \delta^{p/2}(\chi_\delta f)(\delta z)$ we have

$$\mathcal{I}_p S_{\gamma-\frac{b}{2}} : \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_\chi \to \mathcal{W}^s(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_{\tilde{\chi}}.$$

Using the relation (2.40) we obtain

$$\tilde{\mathcal{I}} : \mathcal{W}^s(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, H)_\kappa)_{\tilde{\chi}} \to \mathcal{W}^s(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa)_{\tilde{\tilde{\chi}}}$$

for $\tilde{\tilde{\chi}}_\delta g(z, y) = \delta^{(p+q)/2}(\kappa_\delta g)(\delta z, \delta y)$. According to the isomorphism

$$\mathcal{I}^{-1}_{p+q} : \mathcal{W}^s(\mathbb{R}, \mathcal{W}^s(\mathbb{R}^{p+q}, H)_\kappa)_{\tilde{\tilde{\chi}}} \to \mathcal{W}^s(\mathbb{R} \times \mathbb{R}^{p+q}, H)_\kappa,$$

obtained in a similar manner as (2.40), and

$$S^{-1}_{\gamma-\frac{b}{2}} : \mathcal{W}^s(\mathbb{R}\times\mathbb{R}^{p+q}, H)_\kappa \to \mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^{p+q}, H)_\kappa$$

the claimed identification (2.42) follows using (2.43) for $\widetilde{H} = \mathcal{W}^s(\mathbb{R}^q, H)$ as the composition of isomorphisms

$$S^{-1}_{\gamma-\frac{b}{2}}\mathcal{I}^{-1}_{p+q}\tilde{\mathcal{I}}\mathcal{I}_p S_{\gamma-\frac{b}{2}}.$$

□

Corollary 2.15. *In the case* $H = \mathcal{K}^{s,\gamma_1}(X^\wedge)$ *or* $H = \mathcal{K}^{s,\gamma_1}_P(X^\wedge)$ *for an asymptotic type* P *and the group action* κ *defined by* (2.25) *we have*

$$\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma_1}(X^\wedge))_\kappa)_\chi = \mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^{p+q}, \mathcal{K}^{s,\gamma_1}(X^\wedge))_\kappa,$$

and

$$\mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^p, \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma_1}_P(X^\wedge))_\kappa)_\chi = \mathcal{H}^{s,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^{p+q}, \mathcal{K}^{s,\gamma_1}_P(X^\wedge))_\kappa,$$

for $s, \gamma_1, \gamma_2 \in \mathbb{R}$.

The following result is well-known, cf. Proposition 3.1.21 in [55], but the idea of proof will be employed later on; therefore we briefly recall some details.

Proposition 2.16. *Let* X *be a smooth closed manifold. For every* $s, \gamma \in \mathbb{R}$ *we have*

$$H^s_{\mathrm{comp}}(\mathbb{R}_+\times X\times\mathbb{R}^q) \subset \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \subset H^s_{\mathrm{loc}}(\mathbb{R}_+\times X\times\mathbb{R}^q). \qquad (2.45)$$

In order to give an idea let us sketch a part of the arguments. We first choose an open covering of X by coordinate neighbourhoods $\{U_1, \dots, U_N\}$ and a subordinate partition of unity $\{\varphi_1, \dots, \varphi_N\}$; then

$$X^\wedge = \bigcup_{j=1}^N U_j^\wedge,\ U_j^\wedge = \mathbb{R}_+\times U_j$$

let us form the spaces

$$H_j := [\varphi_j]\mathcal{K}^{s,\gamma}(X^\wedge),\ j = 1, \dots, N,$$

which are also Hilbert spaces with group action . So we can form the spaces

$$\mathcal{W}^s(\mathbb{R}^q, [\varphi_j]\mathcal{K}^{s,\gamma}(X^\wedge))$$

and $\mathcal{K}^{s,\gamma}(X^\wedge) = \sum_{j=1}^N([\varphi_j]\mathcal{K}^{s,\gamma}(X^\wedge))$ as a non-direct sum as well as

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) = \sum_{j=1}^N \mathcal{W}^s(\mathbb{R}^q, H_j).$$

Let $\varepsilon > 0$, and let $[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)_{r>\varepsilon}$ denote the set of all elements in $[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)$ supported by $r > \varepsilon$, and let $[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})_{|\tilde{x}|>\varepsilon}$ be the set of all elements in $[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})$ supported by $|\tilde{x}| > \varepsilon$. From Proposition 2.13 and the identification between $[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)_{r\geq\varepsilon}$ and $[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})_{|\tilde{x}|>\varepsilon}$ for $\varphi = \vartheta^*\tilde{\varphi}$, we have the equivalence of norms

$$c_1(\varepsilon)\|g\|^2_{[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})} \leq \|f\|^2_{[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)} \leq c_2(\varepsilon)\|g\|^2_{[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})} \tag{2.46}$$

for all

$$g \in [\tilde{\varphi}]H^s(\mathbb{R}^{n+1})_{|\tilde{x}|>\varepsilon},\ f \in [\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)_{r>\varepsilon},$$

and

$$f = \beta^* g,\ \beta : U^\wedge \to \mathbb{R}^{n+1},$$

for some constants $c_j(\varepsilon) > 0$, $i = 1, 2$. Applying (2.46) to η-dependent families of functions $\hat{v}(\langle\eta\rangle^{-1}\tilde{x}, \eta)$ and $\hat{u}(\langle\eta\rangle^{-1}r, x, \eta)$ it follows that

$$\begin{aligned} &c_1(\varepsilon)\langle\eta\rangle^{2s}\big\|\langle\eta\rangle^{-(n+1)/2}\hat{v}(\langle\eta\rangle^{-1}\tilde{x}, \eta)\big\|^2_{[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})} \\ &\leq \langle\eta\rangle^{2s}\big\|\langle\eta\rangle^{-(n+1)/2}\hat{u}(\langle\eta\rangle^{-1}r, x, \eta)\big\|^2_{[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge)} \\ &\leq c_2(\varepsilon)\langle\eta\rangle^{2s}\big\|\langle\eta\rangle^{-(n+1)/2}\hat{v}(\langle\eta\rangle^{-1}\tilde{x}, \eta)\big\|^2_{[\tilde{\varphi}]H^s(\mathbb{R}^{n+1})}. \end{aligned}$$

The support condition in (2.46) for $\hat{v}$ in $|\tilde{x}| > \varepsilon$ and $\hat{u}$ in $r > \varepsilon$ is satisfied for all $\eta \in \mathbb{R}^q$. Thus

$$c_1(\varepsilon)\|v\|^2_{\mathcal{W}^s(\mathbb{R}^q,[\tilde{\varphi}]H^s(\mathbb{R}^{n+1}))} \leq \|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,[\varphi]\mathcal{K}^{s,\gamma}(X^\wedge))} \leq c_2(\varepsilon)\|v\|^2_{\mathcal{W}^s(\mathbb{R}^q,[\tilde{\varphi}]H^s(\mathbb{R}^{n+1}))}$$

holds for all elements in the respective spaces supported by $r > \varepsilon$ and $|\tilde{x}| > \varepsilon$, respectively. By virtue of Proposition 2.13 we can identify the norm in $\mathcal{W}^s(\mathbb{R}^q, [\tilde{\varphi}]H^s(\mathbb{R}^{n+1}))$ with the norm of $H^s(\mathbb{R}^{n+1} \times \mathbb{R}^q)$ on those functions which are supported in $|\tilde{x}| > \varepsilon$ for all y and localized in $\tilde{x}$ in the coordinate neighborhood U (the latter just shows the role of the factor $\tilde{\varphi}$). On those functions the norms in the spaces $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ and $H^s(\mathbb{R}^{n+1} \times \mathbb{R}^q)$ are equivalent. This implies the inclusions (2.45).
Edge asymptotics in local form, i.e., in the variables $(r, x, y) \in X^\wedge \times \mathbb{R}^q$, is expressed by spaces

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_P(X^\wedge)) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j), \tag{2.47}$$

analogously as (2.30) and Remark 2.10. Note that as a simple consequence of (2.45) we have the following relation. For $s, \gamma \in \mathbb{R}$ and every asymptotic type P we have

$$H^s_{\text{comp}}(\mathbb{R}_+ \times X \times \mathbb{R}^q) \subset \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_P(X^\wedge)) \subset H^s_{\text{loc}}(\mathbb{R}_+ \times X \times \mathbb{R}^q). \tag{2.48}$$

In order to express the singular functions of edge asymptotics we slightly modify the notion of edge spaces (1.58) by admitting also parameter spaces that are not preserved under the group action κ, cf. [53]. Let E be a Hilbert (or Fréchet) space with group action $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, and represent E as a non-direct sum

$$E = E_0 + E_1 \tag{2.49}$$

In addition if E is a left module over an algebra A by $[a]E$ for $a \in A$ we denote the closure of $\{ae : e \in E\}$ in E. We have

$$H^s(\mathbb{R}^q, E) = H^s(\mathbb{R}^q, E_0) + H^s(\mathbb{R}^q, E_1) \tag{2.50}$$

for any $s \in \mathbb{R}$. According to (2.36) we can form

$$\mathcal{W}^s(\mathbb{R}^q, E)_\kappa = KH^s(\mathbb{R}^q, E). \tag{2.51}$$

The subspaces $KH^s(\mathbb{R}^q, E_j), j = 0, 1$, are closed in (2.51), and the non-direct sum

$$KH^s(\mathbb{R}^q, E_0) + KH^s(\mathbb{R}^q, E_1) \tag{2.52}$$

makes sense where

$$\mathcal{W}^s(\mathbb{R}^q, E)_\kappa = KH^s(\mathbb{R}^q, E_0) + KH^s(\mathbb{R}^q, E_1). \tag{2.53}$$

Remark 2.17. *Assuming that* (2.51) *is direct, also the sums* (2.50), (2.52) *and* (2.53) *are direct.*

Let us apply this to the case

$$E = \mathcal{K}_P^{s,\gamma}(X^\wedge) = E_0 + E_1 \tag{2.54}$$

for a discrete asymptotic type P associated with (γ, Θ), Θ finite, and

$$E_0 := \mathcal{K}_\Theta^{s,\gamma}(X^\wedge), \quad E_1 := \mathcal{E}_P(X^\wedge).$$

The Fréchet space E_0 is also endowed with the group action, the restriction of the one over $\mathcal{K}^{s,\gamma}(X^\wedge)$ to $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$. The space $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge))$ represents weighted edge distributions of flatness Θ and is a subspace of $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge))$. However, κ is not acting on $\mathcal{E}_P(X^\wedge)$.
From (2.54) we conclude a direct decomposition

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)) + KH^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge)).$$

Remark 2.18. *The summand* $KH^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge))$ *is contained in*

$$KH^s(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge)) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge))$$

and is generated by

$$F^{-1}_{\eta \to y}\Big\{ \sum_{j=0}^{N} \sum_{k=0}^{m_j} \omega(r[\eta])[\eta]^{\frac{n+1}{2}} c_{jk}(x)(r[\eta])^{-p_j} \log^k(r[\eta])\hat{v}_{jk}(\eta) \Big\}, \tag{2.55}$$

where $c_{jk} \in C^\infty(X)$, $v_{jk} \in H^s(\mathbb{R}^q)$.

In (2.55) we took the function $\eta \to [\eta]$ rather than $\langle \eta \rangle$; this makes the expressions more transparent, but the above observations do remain in force under replacing $\langle \eta \rangle$ by $[\eta]$, or changing the cut-off function ω. In fact, the remainders belong to

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)) + \mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge)).$$

Our next objective is to study cones and wedges where the base is a manifold B with edge Y. Such a B belongs to $\mathfrak{M}_1$, the system of (pseudo-) manifolds of singularity order 1, containing a smooth submanifold $s_1(B) \in \mathfrak{M}_0$ (subscript 0 means smoothness and $Y = s_1(B)$ where $s_0(B) := B \setminus s_1(B) \in \mathfrak{M}_0$.

In addition we require that $s_1(B)$ has a neighborhood V in B with the structure of a locally trivial X^Δ- bundle over $s_1(B)$, for an $X \in \mathfrak{M}_0$ and $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$. Concerning more details on the nature of such spaces, see, for instance, [63].
For notational convenience we assume that this bundle is trivial, i.e. isomorphic to the Cartesian product

$$s_1(B) \times (([0,1) \times X)/(\{0\} \times X)).$$

Locally we represent the bundle as $X^\wedge \times \mathbb{R}^q$ for $q = \dim s_1(B)$. Then, as before, concentrating on the open stretched cone $X^\wedge = \mathbb{R}_+ \times X$ with the splitting of variables (r, x), the manifold $s_0(B) = B \setminus Y$ for $Y = s_1(B)$ is locally in any neighborhood of Y represented by $\mathbb{R}_+ \times X \times \mathbb{R}^q$, with the corresponding splitting of variables (r, x, y).
In the following we employ the stretched version $\mathbb{B}$ of $B \in \mathfrak{M}_1$. It is obtained by attaching an X-bundle at $B \setminus s_1(B)$, i.e., we first replace $X^\Delta \times s_1(B)$ by $(\mathbb{R}_+ \times X) \times s_1(B)$ and then attach the X-bundle $X \times s_1(B)$ which yields $(\overline{\mathbb{R}}_+ \times X) \times s_1(B)$. Later on we also form the double $2\mathbb{B}$ obtained by gluing together two copies of $\mathbb{B}$ along $X \times s_1(B)$. Locally near $s_1(B)$ the double is of the form $(\mathbb{R} \times X) \times s_1(B)$, and we then have $2\mathbb{B} \in \mathfrak{M}_0$.
Let B be a compact manifold with edge Y. Then we formulate the spaces

$$H^{s,\gamma}(B) \text{ and } H_P^{s,\gamma}(B)$$

for $s, \gamma \in \mathbb{R}$ and an asymptotic type P.
For $B \in \mathfrak{M}_1$ we have the strata $s_0(B), s_1(B)$. Assuming first $\dim s_1(B) = 0$ we can identify B locally near $s_1(B)$ with $X^\Delta, X \in \mathfrak{M}_0$, and then $H^{s,\gamma}(B)$ is the subspace of $H^s_{\text{loc}}(B \setminus s_1(B))$ which is close to $s_1(B)$ identified with $\mathcal{K}^{s,\gamma}(X^\wedge)$. For $\dim s_1(B) = q > 0$ locally close to $s_1(B)$ we can identify B with $\mathbb{R}^q \times X^\Delta$ for an $X \in \mathfrak{M}_0$. Then $\mathbb{B}$ is locally modeled on $(\overline{\mathbb{R}}_+ \times X) \times \mathbb{R}^q$ with the splitting of variables (r, x, y). Choose an open covering $\{G_1, \ldots, G_L\}$ of $s_1(B)$ by coordinate neighborhoods, let $\{\psi_1, \ldots, \psi_L\}$ be a subordinate partition of unity, and let $\alpha_l : G_l \to \mathbb{R}^q$ be charts. In addition let $\psi_0 \in C_0^\infty(\text{int}\,\mathbb{B})$ be a function and ω a cut-off function on the r half-axis such that $\psi_0 + \sum_{l=1}^{L} \omega\psi_l \equiv 1$. Then $H^{s,\gamma}(B)$ is defined as the completion of $C_0^\infty(B \setminus s_1(B))$ with respect to the norm

$$\|u\|_{H^{s,\gamma}(B)} := \Big\{ \|\psi_0 u\|^2_{H^s(2\mathbb{B})} + \sum_{l=1}^{L} \|(\omega\psi_l u) \circ \alpha_l^{-1}\|^2_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))} \Big\}^{1/2}. \tag{2.56}$$

Here the variables in $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ refer to the above-mentioned identification of a neighborhood of $s_1(B)$ with $X^\Delta \times \mathbb{R}^q$ in the splitting (r, x, y) of $s_1(B)$.
Moreover, we define $H_P^{s,\gamma}(B)$ as the subspace of all $u \in H^{s,\gamma}(B)$ such that

$$\|(\omega\psi_l u) \circ \alpha_l^{-1}\|_{\mathcal{W}^s(\mathbb{R}^q, E^j)} \tag{2.57}$$

is finite for all j where $\varprojlim_{j \in \mathbb{N}} E^j = \mathcal{K}_P^{s,\gamma}(X^\wedge)$, cf. formula (2.27). The space $H_P^{s,\gamma}(B)$ is Fréchet in the semi-norm system (2.57), $j \in \mathbb{N}$, together with $\|\psi_0 u)\|_{H^s(2\mathbb{B})}$ in (2.56).
In the following we mainly focus on the case $s = \gamma = 0$. We endow $H^{0,0}(B)$ with a Hilbert space scalar product which can be chosen as

$$\begin{aligned}(u, v)_{H^{0,0}(B)} :=& (\psi_0 u, \psi_0 v)_{L^2(2\mathbb{B})} \\ &+ \sum_{l=1}^{L} ((\omega\psi_l u) \circ \alpha_l^{-1}, (\omega\psi_l v) \circ \alpha_l^{-1})_{\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge))},\end{aligned} \tag{2.58}$$

cf. (2.35). The space $H^0(2\mathbb{B})$ is identified with $L^2(2\mathbb{B})$, the space of square integrable functions on $2\mathbb{B}$ with its scalar product, based on a Riemannian metric on $2\mathbb{B}$.
The space $\mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge))$ will be referred below to as the local version of $H^{0,0}(B)$ close to the edge Y. In abuse of notation for $B = X^\Delta \times Y$ we write

$$H^{0,0}(B) = \mathcal{W}^0(\mathbb{R}^q, \mathcal{K}^{0,0}(X^\wedge)), \tag{2.59}$$

cf. (2.31). Let $L^{-\infty}(B, \boldsymbol{g}_0)$ for weight data $\boldsymbol{g}_0 = (0, 0, (-\infty, 0])$ be defined as the space of all continuous operators $G : H^{s,0}(B) \to H^{\infty,0}(B)$, $s \in \mathbb{R}$, which induce continuous operator

$$G : H^{s,0}(B) \to H^{\infty,0}_P(B),\ G^* : H^{s,0}(B) \to H^{\infty,0}_Q(B) \tag{2.60}$$

for all $s \in \mathbb{R}$ and G-dependent asymptotic types P and Q, where G^* is the formal adjoint, with respect to (2.58).

2.3 Iterated edge asymptotics

We now pass to edge spaces and subspaces with asymptotics on a wedge, or, more generally, on a compact manifold with second order corner. For convenience we content ourselves with the case $s = \gamma = 0$. Spaces with arbitrary weights and smoothness can be reached by reductions of weights and orders, cf. [48].
We iterate asymptotics starting from $\mathcal{K}^{0,0}(X^\wedge) = r^{-n/2}L^2(\mathbb{R}_+ \times X)$ where $\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ is unitary. In that case in (2.55) we have $v_{jk} \in L^2(\mathbb{R}^q)$.
Although the operator K in formula (2.36) is the identity map for $s = 0$ and $H = \mathcal{K}^{0,0}(X^\wedge)$, in the asymptotic terms of (2.55) we keep writing $r[\eta]$ rather than r, since κ is acting on $\mathcal{E}_P(X^\wedge)$ which is endowed with a stronger topology than that induced by $\mathcal{K}^{0,0}(X^\wedge)$.
In order to study iterated asymptotics on a manifold $M \in \mathfrak{M}_2$ we first look at the structure of elements in $\mathfrak{M}_2$. First there is a singular stratum $s_2(M) \in \mathfrak{M}_0$ such that $M \setminus s_2(M) \in \mathfrak{M}_1$, and $s_2(M) =: Z$ has a neighborhood W in M with the structure of a locally trivial B^Δ- bundle over Z for a $B \in \mathfrak{M}_1$. Similarly as for first order singularity we consider the case that this bundle is trivial.
By virtue of $M \setminus s_2(M) \in \mathfrak{M}_1$, there is a singular stratum $s_1(M) := s_1(M \setminus s_2(M)) \in \mathfrak{M}_0$ and $s_1(M) =: Y$ has a neighborhood V in $M \setminus s_2(M)$ with the structure of an X^Δ -bundle over Y for an $X \in \mathfrak{M}_0$.
Let us define the space $H^{0,0,0}(M)$; the upper subscript $0, 0, 0$ stands for s, γ_2, γ_1 for s-moothness $s = 0$ and weights $\gamma_2 = \gamma_1 = 0$. Recall that we considered the spaces $H^{0,0}(B)$ which is given in local form for $B := X^\Delta \times \mathbb{R}^q$ by relation (2.58) referring to the splitting of variables $(r, x, y) \in X^\wedge \times \mathbb{R}^q$. In a similar manner, for $M := B^\Delta \times \mathbb{R}^p$ and identifying $M \setminus s_1(M)$ locally with $B^\wedge \times \mathbb{R}^p \ni (t, b, z)$ for compact B we define

$$H^{0,0,0}(M) = L^2(\mathbb{R}^p, \mathcal{K}^{0,0,0}(B^\wedge)),$$

where

$$\mathcal{K}^{0,0,0}(B^\wedge) := t^{-\frac{\dim B}{2}} L^2(\mathbb{R}_+, H^{0,0}(B)), \tag{2.61}$$

cf. also (2.26).

Proposition 2.19. *Let $B \in \mathfrak{M}_1$ be a compact manifold with edge.*
(i) *$\mathcal{K}^{0,0,0}(B^\wedge)$ is a Hilbert space with unitary group action ${}^2\kappa = \{{}^2\kappa_\delta\}_{\delta\in\mathbb{R}_+}$ given by*

$$({}^2\kappa_\delta f)(t, b) = \delta^{\frac{\dim B+1}{2}} f(\delta t, b), \delta \in \mathbb{R}_+.$$

(ii) *We have*

$$\mathcal{W}^0(\mathbb{R}^p, \mathcal{K}^{0,0,0}(B^\wedge))_{2\kappa} = L^2(\mathbb{R}^p, \mathcal{K}^{0,0,0}(B^\wedge))$$

which is locally on B the same as

$$t^{-\frac{\dim B}{2}} r^{-n/2} L^2(\mathbb{R}^p_z \times \mathbb{R}_{+,t}, L^2(\mathbb{R}^q_y \times \mathbb{R}_{+,r}, L^2(X))).$$

Let us now fix asymptotic types $P = \{(p_j, m_j)\}_{j=0,\dots,N}$ with respect to $r \to 0$, associated with the weight data $(0, \Theta)$ for $\Theta = (\theta, 0]$, $-\infty \le \vartheta \le 0$, and $Q = \{(q_l, d_l)\}_{l=0,\dots,L}$ with respect to $t \to 0$, associated with the weight data $(0, \Lambda)$ for $\Lambda = (\lambda, 0]$, $-\infty \le \lambda < 0$, (i.e., Q is first assumed to be finite). Then we form the space

$$\mathcal{E}_{Q,P}(B^\wedge) := \Big\{\sigma(t) \sum_{l=0}^{L} \sum_{i=0}^{d_l} c_{li} t^{-q_l} \log^i t : c_{li} \in H_P^{\infty,0}(B), l = 0, \dots, L, 0 \le i \le d_l\Big\} \tag{2.62}$$

for a cut-off function σ on the t half axis. The space (2.62) represents singular functions of corner asymptotics for $r \to 0$ and $t \to 0$.
Using the space $H_P^{0,0}(B)$ defined at the end of the preceding section we form the subspace

$$\mathcal{K}_P^{0,0,0}(B^\wedge) := t^{-\frac{\dim B}{2}} L^2(\mathbb{R}_+, H_P^{0,0}(B)) \tag{2.63}$$

of (2.61) of elements with asymptotics of type P close to $s_1(B)$. Moreover, for any cut-off function σ on the t half axis we form

$$\begin{aligned}\mathcal{K}_{\Lambda,P}^{0,0,0}(B^\wedge) := \varprojlim_{\varepsilon>0} \sigma(t) t^{-\frac{\dim B}{2} - \lambda - \varepsilon} L^2(\mathbb{R}_+, H_P^{0,0}(B)) \\ + (1 - \sigma(t)) t^{-\frac{\dim B}{2}} L^2(\mathbb{R}_+, H_P^{0,0}(B)).\end{aligned} \tag{2.64}$$

This space encodes flatness for $t \to 0$ of order Λ.

Definition 2.20. *We set*

$$\mathcal{K}_{Q,P}^{0,0,0}(B^\wedge) = \mathcal{K}_{\Lambda,P}^{0,0,0}(B^\wedge) + \mathcal{E}_{Q,P}(B^\wedge) \tag{2.65}$$

which is a direct decomposition.

Proposition 2.21. *The space* (2.65) *is a Fréchet space with group action from Proposition* 2.19*, i.e. a projective limit*

$$\mathcal{K}_{Q,P}^{0,0,0}(B^\wedge) = \varprojlim_{j \in N} F^j$$

for Hilbert spaces F^j with group action ${}^2\kappa$, with $F^{j+1} \hookrightarrow F^j$ being continuous for all j, and $F^0 = \mathcal{K}^{0,0,0}(B^\wedge)$.

Proposition 2.21 and the general procedure (1.63) allow us to generate edge spaces

$$\mathcal{W}^s(\mathbb{R}^p, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge)),\ s \in \mathbb{R}, \tag{2.66}$$

with iterated asymptotics of type P for $r \to 0$ and Q for $t \to 0$. Here we assume $P > 0$; The case $P = 0$ is simple and treated in [56]. Moreover, a compact space $M \in \mathfrak{M}_2$ is locally near $s_2(M)$ modelled on $\mathbb{R}^p \times B^\Delta$. We then have $\mathbb{M}$, the stretched manifold to M, and then $2\mathbb{M} \in \mathfrak{M}_1$. This gives us the space $H^{0,0}(2\mathbb{M})$ according to (2.56) and subspaces

$H_P^{0,0}(2\mathbb{M})$. We choose a function $\chi \in C^\infty(\mathbb{R}_t)$ such that $\chi(t) = 0$ for $|t| < \varepsilon_0$, $\chi(t) = 1$ for $|t| > \varepsilon_1$, for $0 < \varepsilon_0 < \varepsilon_1$, and ε_1 sufficiently small. Then we can interpret χ as an excision function on $2\mathbb{M}$, and $1-\chi$ as a cut-off function with respect to $s_2(M)$. Set $\chi_+ := \chi|_{\overline{\mathbb{R}}_+}$, $(1-\chi)_+ := (1-\chi)|_{\overline{\mathbb{R}}_+}$. Then we have the space

$$H_{Q,P}^{0,0,0}(M) := \Big\{\chi_+ u_0 + \sum_{l=1}^{L}(1-\chi)_+\psi_l(v_l \circ \alpha_l) : \\ u_0 \in H_P^{0,0}(2\mathbb{M}), v_l \in \mathcal{W}^0(\mathbb{R}^p, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge))\Big\}. \tag{2.67}$$

Here $\{G_1, \ldots, G_L\}$ is an open covering of $s_2(M)$ by coordinate neighbourhoods, $\{\psi_1, \ldots, \psi_L\}$ a subordinate partition of unity, and $\alpha_l : G_l \to \mathbb{R}^q$ are charts.
The new ingredients are coming from the space (2.66). So we investigate these terms, especially, the singular functions of edge asymptotics.
Let us consider an analogous of the isomorphism (2.36), namely,

$$K = F^{-1}\kappa_{[\zeta]}F : H^s(\mathbb{R}^p, E) \to \mathcal{W}^s(\mathbb{R}^p, E), \tag{2.68}$$

now with the Fourier transform $F = F_{z\to\zeta}$ in $\mathbb{R}^p$ and the function $\zeta \to [\zeta]$. Let us write

$$\mathcal{K}_{Q,P}^{0,0,0}(B^\wedge) = E_0 + E_1 \tag{2.69}$$

for

$$E_0 := \mathcal{K}_{\Lambda,P}^{0,0,0}(B^\wedge), \quad E_1 := \mathcal{E}_{Q,P}(B^\wedge).$$

We can restrict (2.68) to

$$H^s(\mathbb{R}^p, E_1) \subseteq H^s(\mathbb{R}^p, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge))$$

and obtain an operator

$$H^s(\mathbb{R}^p, E_1) \to \mathcal{W}^s(\mathbb{R}^p, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge)).$$

Although we often focus on smoothness zero this makes sense for arbitrary s which is useful for higher corner asymptotics. In any case we have the following remark.

Remark 2.22. *The singular functions of iterated edge asymptotics in*

$$\mathcal{W}^s(\mathbb{R}_z^p, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge))$$

are of the form

$$F_{\zeta\to z}^{-1}\Big\{\sum_{l=0}^{L}\sum_{i=0}^{d_l}\sigma(t[\zeta])[\zeta]^{\frac{\dim B+1}{2}} f_{li}(b)(t[\zeta])^{-q_l}\log^i(t[\zeta])\hat{w}_{li}(\zeta)\Big\}, \tag{2.70}$$

where $f_{li} \in H_P^{\infty,0}(B)$, $w_{li} \in H^s(\mathbb{R}^p)$.

The singular functions in the space $\mathcal{W}^\infty(\mathbb{R}^q, \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge))$ can be written as

$$\sum_{l=0}^{L}\sum_{i=0}^{d_l}\sigma(t)f_{li}(b)t^{-q_l}\log^i t\, w_{li}(z)$$

for $f_{li} \in H_P^{\infty,0}(B)$, $w(z) \in H^\infty(\mathbb{R}^p)$. Moreover, $H_P^{\infty,0}(B)$ locally near $Y = s_1(B)$ in the local representation of Remark 2.18 gives rise to a linear combination of coefficients $f_{li}(b)$ of the form

$$\sum_{j=0}^{N}\sum_{k=0}^{m_j} \omega(r) c_{li,jk}(x) r^{-p_j} \log^k r v_{jk}(y)$$

for $c_{li,jk} \in C^\infty(X)$, $v_{jk}(y) \in H^\infty(\mathbb{R}^q)$. In other words for the singular functions of iterated asymptotics for $s = \infty$ (i.e., smoothness) we obtain linear combinations of the kind

$$\sum_{l=0}^{L}\sum_{i=0}^{d_l}\sum_{j=0}^{N}\sum_{k=0}^{m_j} \sigma(t)\omega(r) c_{li,jk}(x) t^{-q_l} \log^i t\, r^{-p_j} \log^k r\, w_{li}(z) v_{jk}(y).$$

Let B be a compact manifold with smooth edge Y. Then $M_{\mathcal{O}}^0(B)$ is defined as the space of all $h(w) \in \mathcal{A}(\mathbb{C}, L^0(B, \boldsymbol{g}_0))$ for weight data $\boldsymbol{g}_0 = (0, 0, (-\infty, 0])$ with $L^0(B, \boldsymbol{g}_0)$ being the space of all edge pseudo-differential operators of order 0 and with constant discrete asymptotics such that

$$h(\beta + i\tau) \in L^0(B, \boldsymbol{g}_0; \Gamma_\beta) \tag{2.71}$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. In this definition we use the fact that $L^0(B, \boldsymbol{g}_0)$ as well as $L^0(B, \boldsymbol{g}_0; \Gamma_\beta)$ are unions of Fréchet space.
Moreover, for a Mellin asymptotic type R in the complex w-plane $\mathbb{C}$, cf. formula (2.15), let $M_R^{-\infty}(B)$ be the the space of all meromorphic functions $f(w)$ in $\mathbb{C}$ with values in $L^{-\infty}(B, \boldsymbol{g}_0)$ and poles at $r_j \in \Pi_{\mathbb{C}} R$ of multiplicity $n_j + 1$ for $j \in \mathbb{I}$ notation as in (2.15) and $\chi_R f|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, L^{-\infty}(B, \boldsymbol{g}_0))$ for any $\Pi_{\mathbb{C}} R$-excision function χ_R, uniformly in compact β-intervals. In addition we ask the Laurent coefficients of f at $(w - r^j)^{-(k+1)}$, $0 \le k \le n_j$, to be of finite rank. Recall that the operators in $L^{-\infty}(B, \boldsymbol{g}_0)$ contain asymptotic information; in fact, they are smoothing Green operators of the edge calculus. We now set

$$M_R^0(B) = M_{\mathcal{O}}^0(B) + M_R^{-\infty}(B),$$

cf. also formula (2.10) for X rather than B.

Theorem 2.23. *For every $f \in M_R^0(B)$ the operator $\mathrm{Op}_{M_t}^{-\dim B/2}(f)$ induces continuous operators*

$$\sigma \,\mathrm{Op}_{M_t}^{-\dim B/2}(f)\, \sigma' : \mathcal{K}^{0,0,0}(B^\wedge) \to \mathcal{K}^{0,0,0}(B^\wedge) \tag{2.72}$$

and

$$\sigma \,\mathrm{Op}_{M_t}^{-\dim B/2}(f)\, \sigma' : \mathcal{K}_{Q,P}^{0,0,0}(B^\wedge) \to \mathcal{K}_{S,T}^{0,0,0}(B^\wedge) \tag{2.73}$$

for every pair Q, P of asymptotic types and some resulting S, T.

Proof. First we show the continuity (2.72). Let $m := \dim B$, then, by assumption, we have

$$f \in L^0(B, \boldsymbol{g}_0, \Gamma_{\frac{m+1}{2}}).$$

Because of (2.17) we can write

$$\mathrm{Op}_M^{-m/2}(f) = t^{-m/2} \mathrm{Op}_M(l) t^{m/2}$$

where

$$l(w) := f(w + m/2) \in L^0(B, \boldsymbol{g}_0; \Gamma_{\frac{1}{2}}).$$

We have

$$\mathcal{K}^{0,0,0}(B^\wedge) = t^{-m/2}L^2(\mathbb{R}_+, H^{0,0}(B)), \tag{2.74}$$

cf. formula (2.61). Thus it suffices to show the continuity of

$$\mathrm{Op}_M(l) : L^2(\mathbb{R}_+, H^{0,0}(B)) \to L^2(\mathbb{R}_+, H^{0,0}(B)).$$

By virtue of Proposition 1.4 in [64] we have

$$\sup_{w\in\Gamma_{\frac{1}{2}}} \|l(w)\|_{\mathcal{L}(H^{0,0}(B))} \leq c \tag{2.75}$$

for some $c > 0$. Thus, for $H := H^{0,0}(B)$ it follows that $\mathrm{Op}_t(l) = M^{-1}\mathcal{M}_l M$ for the Mellin transform M is a composition of continuous operators

$$M : L^2(\mathbb{R}_+, H) \to L^2(\Gamma_{\frac{1}{2}}, H),$$

$$\mathcal{M}_l : L^2(\Gamma_{\frac{1}{2}}, H) \to L^2(\Gamma_{\frac{1}{2}}, H),$$

$$M^{-1} : L^2(\Gamma_{\frac{1}{2}}, H) \to L^2(\mathbb{R}_+, H).$$

Also the operator of multiplication by cut-off functions σ, σ' is continuous in $L^2(\mathbb{R}_+, H)$. This completes the proof of (2.72).
For (2.73) we first define some necessary spaces of holomorphic and meromorphic functions in the complex w-plane with w being the covariable to the corner axis variable $t \in \mathbb{R}_+$. According to (2.74) we consider pairs G, P of asymptotic types where $G := \{(g_l, d_l)\}_{l=0,\dots,L}$ refers to the weight interval $\Lambda = (\lambda, 0]$ and discretes asymptotics for $t \to 0$ while P an in (2.6) refers to $\Theta = (\vartheta, 0]$ and discrete asymptotics for $r \to 0$.
Then we set

$$\begin{aligned}\mathcal{A}^{0,0,0}_{\Lambda,P}(B) : &= \mathcal{A}(\frac{1}{2} + \lambda < \mathrm{Re}\, w < \frac{1}{2}, H^{0,0}_P(B)) \\ &\bigcap C(\frac{1}{2} + \lambda < \mathrm{Re}\, w \leq \frac{1}{2}, H^{0,0}_P(B)),\end{aligned} \tag{2.76}$$

cf. Subsection 3.2, and let $\mathcal{A}^{\infty,0,0}_{G,P}(B)$ be the space of all meromorphic functions $k(w)$ in $\frac{1}{2} + \lambda < \mathrm{Re}\, w < \frac{1}{2}$ with values in $H^{\infty,0}_P(B)$ with poles at all $g_l \in \Pi_{\mathbb{C}} G$ of multiplicity $d_l + 1$ such that for every G-excision function χ_G we have

$$\chi_G k(w) \in \mathcal{S}(\Gamma_\beta, H^{\infty,0}_P(B))$$

for all $\frac{1}{2} + \lambda < \beta < \frac{1}{2}$, uniformly in compact β-intervals.
Both $\mathcal{A}^{0,0,0}_{\Lambda,P}(B)$ and $\mathcal{A}^{\infty,0,0}_{G,P}(B)$ are Fréchet spaces in a natural way. We then define

$$\mathcal{A}^{0,0,0}_{G,P}(B) = \mathcal{A}^{0,0,0}_{\Lambda,P}(B) + \mathcal{A}^{\infty,0,0}_{G,P}(B)$$

in the Fréchet topology of the non-direct sum.
In order to complete the proof we show that for any pair Q_0, P of asymptotics types there is a pair S_0, T such that

$$\mathcal{M}_l : \mathcal{A}^{0,0,0}_{Q_0,P}(B) \to \mathcal{A}^{0,0,0}_{S_0,T}(B) \tag{2.77}$$

is continuous and that also the operators

$$M\sigma' : t^{-\frac{m}{2}}\mathcal{K}^{0,0,0}_{Q,P}(B^\wedge) \to \mathcal{A}^{0,0,0}_{Q,P}(B), \tag{2.78}$$

$$\sigma M^{-1} : \mathcal{A}^{0,0,0}_{S_1,T}(B) \to t^{-\frac{m}{2}}\mathcal{K}^{0,0,0}_{S_1,T}(B^\wedge), \tag{2.79}$$

are continuous, with $M = M_{t\to w}$ being the Mellin transform. The arguments for (2.78) and (2.79) are analogous to the proof of Proposition 2.3. It remains to note that we have continuous operators

$$\mathcal{M}_l : \mathcal{A}^{0,0,0}_{\Lambda,P}(B) \to \mathcal{A}^{0,0,0}_{S_1,P}(B),$$

$$\mathcal{M}_l : \mathcal{A}^{\infty,0,0}_{Q_0,P}(B) \to \mathcal{A}^{\infty,0,0}_{S_2,P}(B),$$

which gives us continuity of

$$\mathcal{M}_l : \mathcal{A}^{0,0,0}_{Q_0,P}(B) \to \mathcal{A}^{0,0,0}_{S_0,T}(B)$$

for an asymptotic type S_0 containing S_1, S_2. □

2.4 Singular functions and edge potential operators

Trace and potential symbols occur in boundary value problems as well as in edge problems. Products

$$g(y,\eta) = k(y,\eta)t(y,\eta)$$

for a trace symbol $t(y,\eta)$ and a potential symbol $k(y,\eta)$ are"abstract" prototypes of Green symbols, cf. Boutet de Monvel [3] or Egorov and Schulze [11].

Proposition 2.24. *The singular functions of corner asymptotics occurring in* (2.70) *contain potential symbols*

$$k_{li}(\zeta) := \sigma(t[\zeta])[\zeta]^{\frac{m+1}{2}} f_{li}(b)(t[\zeta])^{-q_l}\log^i(t[\zeta]),$$

and

$$k_{li}(\zeta) \in S^0_{\mathrm{cl}}(\mathbb{R}^p; \mathbb{C}, \mathcal{K}^{0,0,0}_{Q,P}(B^\wedge)) \tag{2.80}$$

Proof. We have

$$k_{li}(\zeta) \in C^\infty(\mathbb{R}^p_\zeta, \mathcal{L}(\mathbb{C}, \mathcal{K}^{0,0,0}_{Q,P}(B^\wedge))$$

and

$$k_{li}(\delta\zeta) = {}^2\kappa_\delta^{-1} k_{li}(\zeta)$$

for every $\delta \geq 1$, $|\zeta| \geq$ const for a constant > 0. This yields relation (2.80). □

Corollary 2.25. *The singular functions* (2.70) *just have the form*

$$\sum_{l=0}^{L}\sum_{i=0}^{d_i} \mathrm{Op}_z(k_{li})w_{li}$$

where $\mathrm{Op}_z(k_{li}) = F^{-1}k_{li}F$ *is regarded as a continuous operator*

$$\mathrm{Op}_z(k_{li}) : H^s(\mathbb{R}^p) \to \mathcal{W}^s(\mathbb{R}^p, \mathcal{K}^{0,0,0}_{Q,P}(B^\wedge)), \tag{2.81}$$

cf. Theorem 1.26.

Summing up the singular functions of second order edge asymptotics are in the image under the action of a specific potential operator (2.81).

The role of this final subsection is to outline some elements of the edge psuedo-differential calculus on a compact manifold B with smooth edge Y, $q = \dim Y$. By definition B contains Y and both Y and $B \setminus Y$ are smooth manifolds locally near Y the space B is identified with a Cartesian product $\mathbb{R}^q \times X^\triangle$ for a smooth compact manifold X. Modulo smoothing operators G defined by the mapping properties (2.60) for all s the edge space $L^0(B, \boldsymbol{g}_0)$ for weight data $\boldsymbol{g}_0 = (0, 0, (-\infty, 0])$ is the subspace of all $A \in L^0(B \setminus Y)$ (with L^0_{cl} being the space of all classical zero order pseudo-differential operators on the respective open manifold) such that locally near Y, $A = \mathrm{Op}_y(a)$ for an edge symbol $a(y, \eta)$ belonging to $S^0(\mathbb{R}^q_y \times \mathbb{R}^p_\eta; \mathcal{K}^{0,0}(X^\wedge), \mathcal{K}^{0,0}(X^\wedge))$. More precisely, $a(y, \eta)$ is a family of operators in the cone calculus, depending on (y, η). The precise form may be found, for instance, the Definition 3.3.30 in [55] where only the summand

$$a(y, \eta) := \tilde{\omega}(r)\{a_0(y, \eta) + a_1(y, \eta)\}\tilde{\omega}_0(r) + (m + g)(y, \eta)$$

is important (the other summand may be ignored, since off the edge it is absorbed by $L^0_{\mathrm{cl}}(B \setminus Y)$. All this refers to the case of constant (in y) asymptotic types. The space is then a union of Fréchet spaces. Thus it makes sense to talk about holomorphic functions with values in $L^0(B, \boldsymbol{g}_0)$. Moreover, parameter-dependent edge operators, occurring in (2.71), are obtained by replacing the edge covariable $\eta \in \mathbb{R}^q$ in the above definition by $(\rho, \eta) \in \mathbb{R}^{1+q}$. Clearly outside Y the parameter-dependent operators belong to $L^0_{\mathrm{cl}}(B \setminus Y; \mathbb{R}_\rho)$ (or $L^0_{\mathrm{cl}}(B \setminus Y; \Gamma_\beta)$).

3 A new characterization of Kegel Space

3.1 The exit behavior of edge-degenerate operators

The operator

$$A = r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j (rD_y)^\alpha \tag{3.1}$$

can be represented as a mixture between an operator based on the Fourier and on the Mellin transform. In fact, from the subsection 1.2 we know that the weighted Mellin transform cf.(1.35) and the operator

$$\begin{aligned} \mathrm{Op}^\gamma_M(f)u(r) &= M^{-1}_\gamma f(w) M_\gamma u(r) \\ &= \iint \left(\frac{r}{r'}\right)^{-(1/2-\gamma+i\rho)} f(1/2 - \gamma + i\rho) u(r') \frac{dr'}{r'} đ\rho \end{aligned}$$

cf. (1.38) for a symbol $f(w) \in S^\mu(\Gamma_{1/2-\gamma})$. Here $S^\mu(\mathbb{R})$ is the space of symbols of order $\mu \in \mathbb{R}$ in a covariable $\in \mathbb{R}$. When we replace $\mathbb{R}$ by Γ_β for some β we write $S^\mu(\Gamma_\beta)$ where $\mathrm{Im}\, w$ for $w \in \Gamma_\beta$ has the meaning of covatiable. In the following we also employ Mellin symbols depending on r, r' (i.e., double symbols) and we admit symbols taking values in some operator spaces.
In particular, the operator (3.1) can be written in the form

$$A = \mathrm{Op}_y(a)$$

for

$$a(y,\eta) := r^{-\mu}\mathrm{Op}_M^\gamma\Big(\sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)w^j(r\eta)^\alpha\Big) \tag{3.2}$$

for any $\gamma \in \mathbb{R}$. Later on we write $a_{j\alpha}(r,y)$ in local coordinates $x \in \Sigma, \Sigma \subseteq \mathbb{R}^n$ open, $n = \dim X$, in the form

$$a_{j\alpha}(r,y) = \sum_{|\beta|\le\mu-(j+|\alpha|)} a_{j\alpha,\beta}(r,x,y)D_x^\beta$$

for coefficients $a_{j\alpha,\beta} \in C^\infty(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega)$. At the same time we have

$$a(y,\eta) = r^{-\mu}\mathrm{Op}_r(p)(y,\eta) \tag{3.3}$$

for

$$p(r,y,\rho,\eta) := \tilde{p}(r,y,r\rho,r\eta),\ \ \tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) = \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)(i\tilde{\rho})^j\tilde{\eta}^\alpha.$$

In expressions for the principal edge symbol we write

$$p_0(r,y,\rho,\eta) := \tilde{p}(0,y,r\rho,r\eta). \tag{3.4}$$

In the considerations below we write

$$\tilde{f}(r,y,w,\tilde{\eta}) = \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)w^j\tilde{\eta}^\alpha,\ \ f(r,y,w,\eta) = \tilde{f}(r,y,w,r\eta).$$

Then

$$a(y,\eta) = r^{-\mu}\mathrm{Op}_M^\gamma(f)(y,\eta).$$

Set

$$f_0(r,y,w,\eta) := \tilde{f}(0,y,w,r\eta).$$

In terms of the Fourier transform we have the representation (3.3). Edge-degenerate operators A as elements of $\mathrm{Diff}^\mu(\mathbb{R}_+ \times X \times \Omega)$ have a homogeneous principal symbol

$$\sigma_0(A)(r,x,y,\rho,\xi,\eta) \tag{3.5}$$

which refers to local coordinates x on X. By definition (3.5) has the form

$$\sigma_0(A)(r,x,y,\rho,\xi,\eta) = r^{-\mu}p_{(\mu)}(r,x,y,\rho,\xi,\eta),$$

where $p_{(\mu)}$ is the parameter-dependent homogeneous principal symbol of order μ of $p(r,y,\rho,\eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\rho,\eta}))$. More explicitly, from (3.1) we have

$$\begin{aligned}&\sigma_0(A)(r,x,y,\rho,\xi,\eta)\\ &= r^{-\mu}\sum_{j+|\alpha|\le\mu}\Big(\sum_{|\beta|=\mu-(j+|\alpha|)} a_{j\alpha,\beta}(r,x,y)\xi^\beta\Big)(-ir\rho)^j(r\eta)^\alpha.\end{aligned} \tag{3.6}$$

In the description of edge-degenerate ellipticity we also refer to

$$\tilde{\sigma}_0(A)(r,x,y,\rho,\xi,\eta) = r^\mu\sigma_0(A)(r,x,y,r^{-1}\rho,\xi,r^{-1}\eta) \tag{3.7}$$

which is homogeneous in $(\rho,\xi,\eta) \neq 0$ of order μ but smooth in r up to $r = 0$. From (3.6) we see that

$$\tilde{\sigma}_0(A)(r,x,y,\rho,\xi,\eta) = \sum_{j+|\alpha|\le\mu}\Big(\sum_{|\beta|=\mu-(j+|\alpha|)} a_{j\alpha,\beta}(r,x,y)\xi^\beta\Big)(-i\rho)^j\eta^\alpha. \tag{3.8}$$

Definition 3.1. *The operator* (3.1) *is called* σ_0*-elliptic if* A *is elliptic over* $X^\wedge \times \Omega$ *in the standard sense, i.e.,* (3.6) *does not vanish for all* $(\xi, \rho, \eta) \neq 0$. *Moreover,* A *is called* $\tilde{\sigma}_0$*-elliptic if* (3.8) *does not vanish for all* $(\xi, \rho, \eta) \neq 0$, *up to* $r = 0$.

Another essential symbolic object is the principal edge symbol, namely,

$$\sigma_1(A)(y,\eta) := r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0,y)(-r\frac{\partial}{\partial_r})^j (r\eta)^\alpha = r^{-\mu}\mathrm{Op}_r(p_0)(y,\eta) \tag{3.9}$$

which is considered for $\eta \neq 0$. An alternative expression for $\sigma_1(A)$ is

$$\sigma_1(A)(y,\eta) := r^{-\mu}\mathrm{Op}_M^\gamma(f_0)(y,\eta). \tag{3.10}$$

For the calculus of edge operators it is important to realize the spaces for the action of (3.9). Those are denoted here by $\mathcal{K}^{s,\gamma}(X^\wedge)$ for $s, \gamma \in \mathbb{R}$ and X compact. Let us postpone for the moment the precise definition. We also could consider (3.9) as a family of operators $C_0^\infty(X^\wedge) \longrightarrow C^\infty(X^\wedge)$.
Assuming, for instance, $X = S^n$ for the unit sphere in $\mathbb{R}^{n+1}_{\tilde{x}}$, then, if ω is a cut-off function, i.e., $\omega \in C_0^\infty(\mathbb{R}^{n+1})$, $\omega \equiv 1$ close to $\tilde{x} = 0$, we have

$$(1-\omega)\mathcal{K}^{s,\gamma}((S^n)^\wedge) = (1-\omega)H^s(\mathbb{R}^{n+1})$$

for the standard Sobolev space $H^s(\mathbb{R}^{n+1})$ of smoothness $s \in \mathbb{R}$.
The notation refers to differential operators in $\mathbb{R}^{n+1}_{\tilde{x}}$

$$\tilde{A} = \sum_{|\delta|\leq\mu} a_\delta(\tilde{x}) D^\delta_{\tilde{x}} \tag{3.11}$$

with coefficients $a_\delta(\tilde{x})$ which are symbols in $S^0_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})$ with $\tilde{x}$ being treated as a covariable. Let $a_{\delta,(0)}(\tilde{x})$ be the homogeneous principal part of $a_\delta(\tilde{x})$ in $\tilde{x} \neq 0$ of order 0. Then exit ellipticity means

$$\sigma_\psi(\tilde{A})(\tilde{x},\tilde{\xi}) := \sum_{|\delta|=\mu} a_\delta(\tilde{x})\tilde{\xi}^\delta \neq 0, \text{ for } (\tilde{x},\tilde{\xi}) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \tag{3.12}$$

$$\sigma_{\mathrm{e}}(\tilde{A})(\tilde{x},\tilde{\xi}) := \sum_{|\delta|\leq\mu} a_{\delta,(0)}(\tilde{x})\tilde{\xi}^\delta \neq 0, \text{ for } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1} \setminus \{0\}) \times \mathbb{R}^{n+1} \tag{3.13}$$

$$\sigma_{\psi,\mathrm{e}}(\tilde{A})(\tilde{x},\tilde{\xi}) := \sum_{|\delta|=\mu} a_{\delta,(0)}(\tilde{x})\tilde{\xi}^\delta \neq 0, \text{ for } (\tilde{x},\tilde{\xi}) \in (\mathbb{R}^{n+1} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}), \tag{3.14}$$

cf. also [11]. As is known, non-vanishing of $\sigma_\psi(\tilde{A}), \sigma_{\mathrm{e}}(\tilde{A}), \sigma_{\psi,\mathrm{e}}(\tilde{A})$ is necessary and sufficient for the Fredholm property of

$$\tilde{A} : H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1}) \tag{3.15}$$

for any fixed $s = s_0 \in \mathbb{R}$. Then the Fredholm property holds for all $s \in \mathbb{R}$. The result is a consequence of the fact that there is a pseudo-differential parametrix $\tilde{P}$ in the exit pseudo-differential calculus. The typical effects for $|\tilde{x}| \to \infty$ come from non-vanishing of $\sigma_{\mathrm{e}}(\tilde{A})$ and $\sigma_{\psi,\mathrm{e}}(\tilde{A})$ for large $|\tilde{x}|$. This guarantees that remainders G_{L} and G_{R} in $\tilde{P}\tilde{A} = 1 - G_{\mathrm{L}}$, $\tilde{A}\tilde{P} = 1 - G_{\mathrm{R}}$ have kernels in $\mathcal{S}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$.

Theorem 3.2. *Let A be an edge-degenerate operator* (3.1) *which is σ_0-and $\tilde{\sigma}_0$-elliptic. Then, the edge symbol $\sigma_1(A)(y,\eta)$ in the variables $(\tilde{x},\tilde{\xi},\eta)$ for fixed $\eta \neq 0$ is elliptic with respect to the subordinate symbols of the exit calculus, which means in this case for all $|\tilde{x}| \neq 0$,*

$$\sigma_\psi(\sigma_1(A)(y,\eta))(\tilde{x},\tilde{\xi}) \neq 0, \text{ for } \tilde{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}, \tag{3.16}$$

$$\sigma_{\mathrm{e}}(\sigma_1(A)(y,\eta))(\tilde{x},\tilde{\xi}) \neq 0, \text{ for } \tilde{\xi} \in \mathbb{R}^{n+1}, \tag{3.17}$$

$$\sigma_{\psi,\mathrm{e}}(\sigma_1(A)(y,\eta))(\tilde{x},\tilde{\xi}) \neq 0, \text{ for } \tilde{\xi} \in \mathbb{R}^{n+1} \setminus \{0\}. \tag{3.18}$$

Proof. We represent the operators in (3.9) in coordinates $\tilde{x} = (\tilde{x}_1, \tilde{x}') \in \mathbb{R}^{n+1}$ for $\tilde{x}' = (\tilde{x}_2, \dots, \tilde{x}_{n+1})$. If U is a coordinates neighbourhood on X, identified via a fixed diffeomoephism with $B = \{x \in \mathbb{R}^n : |x| < b\}$ for some $b > 0$, we choose a chart $\chi : \mathbb{R}_+ \times U \to \Gamma$ where

$$\Gamma := \big\{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} : \tilde{x} = (r, rx), r = \tilde{x}_1 \in \mathbb{R}_+, x \in B\big\}.$$

In this description $\mathbb{R}^n_x$ is identified with the hyperplane $\{(1,\tilde{x}') : \tilde{x}' \in \mathbb{R}^n\}$ of $\mathbb{R}^{n+1}_{\tilde{x}}$. For the change of covariables $(\rho,\xi) \to \tilde{\xi}$, where (ρ,ξ) corresponds to (D_r, D_x) and $\tilde{\xi}$ to $D_{\tilde{x}}$ we compute the Jacobian belonging to the transformation $(d,\infty) \times B \to \Gamma_d$ for $\Gamma_d := \{\tilde{x} \in \Gamma : \tilde{x}_1 > d\}$ and some fixed $0 < d < 1$. We have

$$\partial_r = \sum_{j=1}^{n+1} q_{0j} \partial_{\tilde{x}_j}, \quad \partial_{x_k} = \sum_{j=1}^{n+1} q_{kj} \partial_{\tilde{x}_j}$$

for $q_{0j}(r,x) = \frac{\partial \tilde{x}_j}{\partial r}$, $q_{kj}(r,x) = \frac{\partial \tilde{x}_j}{\partial x_k}$. Then, for the covariables ρ, ξ_k, $\tilde{\xi}_j$ associated with ∂_r, ∂_{x_k} and $\partial_{\tilde{\xi}_j}$, respectively, we write

$$\begin{pmatrix} \rho \\ \xi \end{pmatrix} = \begin{pmatrix} 1 & x \\ \mathbf{0} & rI_n \end{pmatrix} \tilde{\xi}$$

with I_n being the $n \times n$-identity matrix and $x = (x_1, \dots, x_n)$, $\mathbf{0} = {}^{\mathrm{t}}(0 \cdots 0)$ denote the column vector of n zeros. We express (3.9) in variables $(\tilde{x}, y)$ and covariables $(\tilde{\xi}, \eta)$. First we have

$$\begin{aligned} \sigma_1(A)(y,\eta) : &= r^{-\mu} \sum_{j+|\alpha| \leq \mu} \Big\{ \sum_{|\beta| \leq \mu - (j+|\alpha|)} a_{j\alpha,\beta}(0,x,y) D_x^\beta \Big\} (-r\partial_r)^j (r\eta)^\alpha \\ &= r^{-\mu} \sum_{j+|\alpha|+|\beta| \leq \mu} a_{j\alpha,\beta}(0,x,y) D_x^\beta (-r\partial_r)^j (r\eta)^\alpha. \end{aligned} \tag{3.19}$$

For abbreviation from now on we set $a_{j\alpha,\beta}(x,y) := a_{j\alpha,\beta}(0,x,y)$. We now employ the identity $(-r\partial_r)^j = (-1)^j \sum_{l=0}^j S_{jl} r^l \partial_r^l$ where S_{jl} are just the Sterling numbers of second kind, see also [10, Lemma 2.2.4]. Using $S_{jj} = 1$ the expression (3.19) takes the form

$$\sigma_1(A)(y,\eta) := E + R \tag{3.20}$$

for

$$E = r^{-\mu} \sum_{j+|\alpha|+|\beta|=\mu} (-1)^j a_{j\alpha,\beta}(x,y) D_x^\beta r^j \partial_r^j (r\eta)^\alpha = \sum_{j+|\alpha|+|\beta|=\mu} (-1)^j a_{j\alpha,\beta}(x,y) (r^{-1} D_x)^\beta \partial_r^j \eta^\alpha$$

and

$$
\begin{aligned}
R &= r^{-\mu}\Big\{ \sum_{j+|\alpha|+|\beta|=\mu} (-1)^j a_{j\alpha,\beta}(x,y) D_x^\beta \{ \sum_{l=0}^{j-1} S_{jl} r^l \partial_r^l \} (r\eta)^\alpha \\
&+ \sum_{j+|\alpha|+|\beta|<\mu} (-1)^j a_{j\alpha,\beta}(x,y) D_x^\beta \{ \sum_{l=0}^{j} S_{jl} r^l \partial_r^l \} (r\eta)^\alpha \Big\} \\
&= \Big\{ \sum_{j+|\alpha|+|\beta|=\mu} (-1)^j a_{j\alpha,\beta}(x,y) (r^{-1}D_x)^\beta \{ \sum_{l=0}^{j-1} S_{jl} r^{l-j} \partial_r^l \} \eta^\alpha \\
&+ \sum_{j+|\alpha|+|\beta|<\mu} r^{-\mu+(j+|\alpha|+|\beta|)} (-1)^j a_{j\alpha,\beta}(x,y) (r^{-1}D_x)^\beta \{ r^{-j} \sum_{l=0}^{j} S_{jl} r^l \partial_r^l \} \eta^\alpha \Big\}.
\end{aligned}
$$

Assume that the operator A is σ_0-and $\tilde{\sigma}_0$-elliptic which contains the condition

$$
\sum_{j+|\alpha|+|\beta|=\mu} a_{j\alpha,\beta}(x,y) \xi^\beta (-i\tilde{\rho})^j \tilde{\eta}^\alpha \neq 0
$$

for all $(\xi, \tilde{\rho}, \tilde{\eta}) \neq 0$. This is equivalent to

$$
r^{-\mu} \sum_{j+|\alpha|+|\beta|=\mu} a_{j\alpha,\beta}(x,y) \xi^\beta (-ir\rho)^j (r\eta)^\alpha \neq 0
$$

for all $(\xi, \rho, \eta) \neq 0$, i.e., $T(x, y, r^{-1}\xi, \rho, \eta) \neq 0$ for all $(\xi, \rho, \eta) \neq 0$, where

$$
T(x, y, r^{-1}\xi, \rho, \eta) := \sum_{|\alpha|=\mu-(j+|\beta|)} \Big\{ \sum_{j+|\beta|\leq\mu} a_{j\alpha,\beta}(x,y) (r^{-1}\xi)^\beta (-i\rho)^j \Big\} \eta^\alpha, \tag{3.21}
$$

we now rephrase (3.21) which is the symbol of $E(y, \eta)$ in the variables $\tilde{x}$ and covariables $\tilde{\xi}$ and show the properties

$$
\sigma_\psi(E)(y,\eta))(\tilde{x}, \tilde{\xi}) \neq 0, \ \sigma_{\mathrm{e}}(E)(y,\eta))(\tilde{x}, \tilde{\xi}) \neq 0, \ \sigma_{\psi,\mathrm{e}}(E)(y,\eta))(\tilde{x}, \tilde{\xi}) \neq 0
$$

as in (3.17), (3.18). After that we will see that R has no influence to the exit ellipticity. In (3.21) we insert

$$
\rho = x'\tilde{\xi}, \text{ for } x' := (1 \ \ x); \quad \text{and} \quad r^{-1}\xi = P\tilde{\xi}, \text{ for } P := (\mathbf{0} \ \ I_n),
$$

where $x = f(\tilde{x}) = \left(\frac{\tilde{x}_2}{\tilde{x}_1}, \frac{\tilde{x}_3}{\tilde{x}_1}, \ldots, \frac{\tilde{x}_{n+1}}{\tilde{x}_1} \right)$. Then (3.21) takes the form

$$
\begin{aligned}
\tilde{T}(\tilde{x}, y, \tilde{\xi}, \eta) &= \sum_{|\alpha|=\mu-(j+|\beta|)} \Big\{ \sum_{j+|\beta|\leq\mu} a_{j\alpha,\beta}(x,y) (r^{-1}\xi)^\beta (-i\rho)^j \Big\} \eta^\alpha \\
&= \sum_{|\alpha|=\mu-(j+|\beta|)} \Big\{ \sum_{j+|\beta|\leq\mu} a_{j\alpha,\beta}(f(\tilde{x}),y) (P\tilde{\xi})^\beta (-ix'\tilde{\xi})^j \Big\} \eta^\alpha
\end{aligned} \tag{3.22}
$$

for $\beta = (\beta_1, \cdots, \beta_n)$. The vector $(P\tilde{\xi})^\beta$ can be written $(P\tilde{\xi})^\beta = {}^{\mathrm{t}}(\tilde{\xi}_2^{\beta_1}, \cdots, \tilde{\xi}_{n+1}^{\beta_n})$ and $(-ix'\tilde{\xi})^j = (-i(\tilde{\xi}_1 + x_2\tilde{\xi}_2 + \cdots + x_n\tilde{\xi}_{n+1}))^j$. Set $(P\tilde{\xi})^\beta(-ix'\tilde{\xi})^j = B_{j\beta}(x', \tilde{\xi})$ then $B_{j\beta}$ is a homogeneous polynomial in $\tilde{\xi}$ of order $j + |\beta|$, i.e.,

$$
B_{j\beta}(x', \delta\tilde{\xi}) = \delta^{j+|\beta|} B_{j\beta}(x', \tilde{\xi}),
$$

so the (3.22) is equal to $\sum_{|\alpha|=\mu-(j+|\beta|)} \Big\{ \sum_{j+|\beta|\le\mu} a_{j\alpha,\beta}(f(\tilde{x}),y)B_{j\beta}(g(\tilde{x}),\tilde{\xi})\Big\}\eta^\alpha$ for $x' = g(\tilde{x}) := (1\ \ f(\tilde{x}))$. Note that for $\delta \in \mathbb{R}_+$

$$\begin{aligned}\tilde{T}(\tilde{x},y,\delta\tilde{\xi},\delta\eta) &= \sum_{|\alpha|=\mu-(j+|\beta|)} \Big\{ \sum_{j+|\beta|\le\mu} a_{j\alpha,\beta}(f(\tilde{x}),y)B_{j\beta}(g(\tilde{x}),\delta\tilde{\xi})\Big\}(\delta\eta)^\alpha \\ &= \delta^\mu \tilde{T}(\tilde{x},y,\tilde{\xi},\eta).\end{aligned}$$

Thus we can write $\tilde{T}(\tilde{x},y,\tilde{\xi},\eta) = \sum_{|\tilde{\beta}|+|\alpha|=\mu} b_{\tilde{\beta}\alpha}(\tilde{x},y)\tilde{\xi}^{\tilde{\beta}}\eta^\alpha$ and it follows that

$$\tilde{T}(\tilde{x},y,\tilde{\xi},\eta) \neq 0, \text{ for } (\tilde{\xi},\eta) \neq 0.$$

In order to complete the proof it remains to note that in a similar reformulation of the remainder R in variables $(\tilde{x},y)$ and covariables $(\tilde{\xi},\eta)$ every term contains a power of $\tilde{x}_1 = r$ with a negative integer exponent. Therefore, the exit symbolic components σ_{e} and $\sigma_{\psi,\mathrm{e}}$ indicated in Theorem 3.2 vanish for $|\tilde{x}| \to \infty$. Hence they do not affect the exit ellipticity of the first summand E on the right of (3.20) stated before. □

Remark 3.3. *Let us consider instead of* (3.1) *a parameter-dependent operator*

$$A(\iota) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y,\iota)(-r\partial_r)^j(rD_y)^\alpha,$$

where $a_{j\alpha}(r,y,\iota) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X;\mathbb{R}^m_\iota))$, *and* $\mathrm{Diff}^\nu(X;\mathbb{R}^m_\iota)$ *is the set of all families of differential operators, locally in a coordinate neighborhood on X in variables $x \in \Sigma$ described by*

$$\sum_{|\beta|+|\tau|\le\nu} a_{\beta\tau}(x)D_x^\beta \iota^\tau$$

for coefficients $a_{\beta\tau}(x) \in C^\infty(\Sigma)$. Then, applying Theorem 3.2 *to $A(\iota)$ under the condition of σ_0, $\tilde{\sigma}_0$-ellipticity with parameter $\iota \in \mathbb{R}^m$, cf. Definition* 3.1, *the symbols* (3.17), (3.18) *are independent of ι.*

For purposes below we recall some notation on pseudo-differential operators on a manifold with conical exit to ∞. The simplest case of such a manifold is $\mathbb{R}^{n+1} \ni \tilde{x}$. We then have the space $S^{\mu;\nu}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ of symbols of order $\mu \in \mathbb{R}$, exit order $\nu \in \mathbb{R}$, defined as the set of all $a(\tilde{x},\tilde{\xi}) \in C^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$ satisfying the symbolic estimates

$$|D_{\tilde{x}}^\alpha D_{\tilde{\xi}}^\beta a(\tilde{x},\tilde{\xi})| \le c\langle\tilde{\xi}\rangle^{\mu-|\beta|}\langle\tilde{x}\rangle^{\nu-|\alpha|}$$

for all α, $\beta \in \mathbb{N}^{n+1}$ and $(\tilde{x},\tilde{\xi}) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, for constants $c = c(\alpha,\beta) > 0$. The associated pseudo-differential operators form spaces

$$L^{\mu;\nu}(\mathbb{R}^{n+1}) := \big\{\mathrm{Op}_{\tilde{x}}(a) : a(\tilde{x},\tilde{\xi}) \in S^{\mu;\nu}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})\big\}.$$

There is also an analogue of symbols and operators, classical in $\tilde{x}$ and $\tilde{\xi}$. For the (nuclear) Fréchet spaces $S^\mu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})$ and $S^\nu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})$ we set

$$S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}) := S^\mu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})\hat{\otimes}_\pi S^\nu_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})$$

where $\hat{\otimes}_\pi$ indicates the completed projective tensor product. We will write subscript "(cl)" when a consideration is valid both in the classical and the general case. The space $L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1})$ is defined in terms of $S^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$. Observe that

$$\mathrm{Op}_{\tilde{x}}: S^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}) \to L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$$

is an isomorphism. In this way the spaces $L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ are equipped with corresponding Fréchet topologies. Note that $L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}) = \langle\tilde{x}\rangle^{\nu} L^{\mu;0}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$. Moreover, there are natural inclusions $L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}) \hookrightarrow L^{\mu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$. The operators in $L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1})$ have principal symbols

$$\sigma_\psi(\cdot)(\tilde{x},\tilde{\xi}),\ \sigma_{\mathrm{e}}(\cdot)(\tilde{x},\tilde{\xi}),\ \text{and}\ \sigma_{\psi,\mathrm{e}}(\cdot)(\tilde{x},\tilde{\xi}). \tag{3.23}$$

Ellipticity is defined in an analogous manner as (3.12), (3.13), (3.14). If (3.11) $\in L^{\mu;-\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ is elliptic, there is a parametrix $\tilde{P}\in L^{-\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1})$ belonging to the triple of inverted symbols, and $\tilde{P}\tilde{A}-1$, $\tilde{A}\tilde{P}-1$ have kernels in $\mathcal{S}(\mathbb{R}^{n+1}_{\tilde{x}}\times\mathbb{R}^{n+1}_{\tilde{x}'})$.
It will be necessary also to refer to parameter-dependent operators, with an extra covariable $\zeta\in\mathbb{R}^d$. In this case we consider symbols $a(\tilde{x},\tilde{\xi},\zeta)\in S^{\mu;\nu}(\mathbb{R}^{n+1}_{\tilde{x}}\times\mathbb{R}^{n+1+d}_{\tilde{\xi},\zeta})$ satisfying the estimates

$$|D^{\alpha}_{\tilde{x}}D^{\beta}_{\tilde{\xi},\zeta}a(\tilde{x},\tilde{\xi},\zeta)|\le c\langle\tilde{\xi},\zeta\rangle^{\mu-|\beta|}\langle\tilde{x}\rangle^{\nu-|\alpha|}$$

for all $\alpha\in\mathbb{N}^{n+1}$, $\beta\in\mathbb{N}^{n+1+d}$ and $\tilde{x}\in\mathbb{R}^{n+1}$, $(\tilde{\xi},\zeta)\in\mathbb{R}^{n+1+d}$, for constants $c=c(\alpha,\beta)>0$. Similarly as before we have also classical symbols in $\tilde{x},\tilde{\xi},\zeta$. The associated pseudo-differential operators form spaces

$$L^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1};\mathbb{R}^d_\zeta) := \big\{\mathrm{Op}_{\tilde{x}}(a): a(\tilde{x},\tilde{\xi},\zeta)\in S^{\mu;\nu}_{(\mathrm{cl})}(\mathbb{R}^{n+1}_{\tilde{x}}\times\mathbb{R}^{n+1+d}_{\tilde{\xi},\zeta})\big\}.$$

In the classical case instead of (3.23) we have parameter-dependent symbols

$$\sigma_\psi(\cdot)(\tilde{x},\tilde{\xi},\zeta),\ \sigma_{\mathrm{e}}(\cdot)(\tilde{x},\tilde{\xi},\zeta),\ \text{and}\ \sigma_{\psi,\mathrm{e}}(\cdot)(\tilde{x},\tilde{\xi},\zeta). \tag{3.24}$$

Parameter-dependent ellipticity means non-vanishing of the components of (3.24) in

$$\mathbb{R}^{n+1}\times(\mathbb{R}^{n+1+d}\setminus\{0\}),\ (\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}^{n+1+d},\ \text{and}\ (\mathbb{R}^{n+1}\setminus\{0\})\times(\mathbb{R}^{n+1+d}\setminus\{0\}),$$

cf. (3.12), (3.13), and (3.14), respectively. Then a parameter-dependent elliptic $\tilde{A}(\zeta)\in L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1};\mathbb{R}^d_\zeta)$ has a parametrix $\tilde{P}(\zeta)\in L^{-\mu;-\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1};\mathbb{R}^d_\zeta)$ such that

$$\tilde{P}(\zeta)\tilde{A}(\zeta)-1,\ \tilde{A}(\zeta)\tilde{P}(\zeta)-1$$

have kernels in $\mathcal{S}(\mathbb{R}^d_\zeta,\mathcal{S}(\mathbb{R}^{n+1}_{\tilde{x}}\times\mathbb{R}^{n+1}_{\tilde{x}'}))$. This entails the invertibility of

$$\tilde{A}(\zeta): H^{s;g}(\mathbb{R}^{n+1})\to H^{s-\mu;g-\nu}(\mathbb{R}^{n+1}) \tag{3.25}$$

for every $s,g\in\mathbb{R}$ when $|\zeta|$ is sufficiently large.

3.2 Elements of the edge symbolic calculus

In this subsection we establish a version of edge calculus, based on operator-valued amplitude functions. In order to formulate them in new form we recall the notion of parameter-dependent Mellin symbols.

Definition 3.4. *The space $M^\mu_{\mathcal{O}}(X;\mathbb{R}^l_\lambda)$ of parameter-dependent holomorphic symbols of order $\mu\in\mathbb{R}$ with parameters $\lambda\in\mathbb{R}^l$, is defined as the set of all $h(w,\lambda)\in\mathcal{A}(\mathbb{C},L^\mu_{\mathrm{cl}}(X;\mathbb{R}^l_\lambda))$ such that $h|_{\Gamma_\beta\times\mathbb{R}^l}\in L^\mu_{\mathrm{cl}}(X;\Gamma_\beta\times\mathbb{R}^l)$ for every $\beta\in\mathbb{R}$, uniformly in compact β-intervals.*

The space $M^\mu_{\mathcal{O}}(X;\mathbb{R}^l_\lambda)$ is Fréchet in a natural way.
By a discrete Mellin asymptotic type R we understand a sequence

$$R=\{(r_j,n_j)\}_{j\in\mathbb{I}}\subset\mathbb{C}\times\mathbb{N} \tag{3.26}$$

for some index set $\mathbb{I}\subseteq\mathbb{Z}$ such that $\Pi_{\mathbb{C}}R:=\{r_j\}_{j\in\mathbb{I}}$ intersects the strip $\{c\le\operatorname{Re}w\le c'\}$ in a finite set for every $c\le c'$.
A function $\chi\in C^\infty(\mathbb{C})$ is called an R-excision function if $\chi(w)=0$ for $\operatorname{dist}(w,\Pi_{\mathbb{C}}R)<\varepsilon_0$, $\chi(w)=1$ for $\operatorname{dist}(w,\Pi_{\mathbb{C}}R)>\varepsilon_1$, for some $0<\varepsilon_0<\varepsilon_1$.

Definition 3.5. *Let $M^{-\infty}_R(X)$ denote the space of all $f\in\mathcal{A}(\mathbb{C}\setminus\Pi_{\mathbb{C}}R,L^{-\infty}(X))$ which are meromorphic with poles at all r_j of multiplicity n_j+1, and finite rank Laurent coefficients at $(w-r_j)^{-(k+1)}$, $0\le k\le n_j$, and $\chi f|_{\Gamma_\beta}\in L^{-\infty}(X;\Gamma_\beta)$ for any R-excision function χ and every $\beta\in\mathbb{R}$, uniformly in compact β-intervals.*

The space $M^{-\infty}_R(X)$ is a union of Fréchet spaces in a natural way.
Let us now recall some notation on weighted spaces on the infinite stretched cone $X^\wedge=\mathbb{R}_+\times X$ in the variables (r,x), for a smoothing closed manifolds X.
Recall that the formula (1.40), i.e. the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)$ for $s,\gamma\in\mathbb{R}$ is defined as the completion of $u(r,x)\in C^\infty_0(\mathbb{R}_+\times\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)}:=\Big\{\int_{\mathbb{R}^n}\int_{\Gamma_{\frac{n+1}{2}-\gamma}}\langle w,\xi\rangle^{2s}|(M_{r\to w}F_{x\to\xi}u)(w,\xi)|^2đwđ\xi\Big\}^{1/2}. \tag{3.27}$$

Moreover, we employ the spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$, see the relation (1.87) and the space $\mathcal{K}^{s,\gamma}(X^\wedge)$ defined by (1.95).
For purposes below we set

$$H^{s;g}_{\mathrm{cone}}(X^\wedge)=\langle r\rangle^{-g}H^s_{\mathrm{cone}}(X^\wedge),\ \ s,g\in\mathbb{R}. \tag{3.28}$$

Recall that for $X=S^n$ and any cut-off function ω we have a natural identification

$$(1-\omega)H^s_{\mathrm{cone}}(X^\wedge)=(1-\omega)H^s(\mathbb{R}^{n+1}).$$

The spaces (1.95) for $s,\gamma\in\mathbb{R}$, are Hilbert spaces with group action $\kappa=\{\kappa_\delta\}_{\delta\in\mathbb{R}_+}$, given by (2.25). Incidentally we use relation (2.26) in Remark 2.10.
It will be essential also to employ subspaces of distributions with asymptotic of type P, see Subsection 2.2, particular the following spaces :

$$\mathcal{K}^{s,\gamma}_\Theta(X^\wedge),\mathcal{E}_P(X^\wedge),\mathcal{K}^{s,\gamma}_P(X^\wedge)$$

For any strictly positive smooth function $r\to[r]$ on $\mathbb{R}_+$ such that $[r]=r$ for $r>C$ for some $C>0$ we set

$$\mathcal{K}^{s,\gamma;e}(X^\wedge):=[r]^{-e}\mathcal{K}^{s,\gamma}(X^\wedge),\ \mathcal{K}^{s,\gamma;e}_P(X^\wedge):=[r]^{-e}\mathcal{K}^{s,\gamma}_P(X^\wedge),$$

$$\mathcal{K}^{\infty,\gamma;\infty}(X^\wedge):=\bigcap_{s,e\in\mathbb{R}}\mathcal{K}^{s,\gamma;e}(X^\wedge),\ \mathcal{K}^{\infty,\gamma;\infty}_P(X^\wedge):=\bigcap_{s,e\in\mathbb{R}}\mathcal{K}^{s,\gamma;e}_P(X^\wedge),$$

for every $s, \gamma, e \in \mathbb{R}$. Note that specific asymptotics of type P have been observed in classical papers on elliptic boundary value problems in domains with conical singularities, cf. Kondratyev [32], or Egorov and Schulze [11].

Remark 3.6. *For any asymptotic type P associated with the weight data (γ, Θ) there is a $\delta > 0$ such that for $\Delta := (-\delta, 0]$ we have continuous embeddings*

$$\mathcal{K}_P^{s,\gamma;e}(X^\wedge) \hookrightarrow \mathcal{K}_\Delta^{s,\gamma;e}(X^\wedge).$$

In fact, it suffices to set $\delta = \operatorname{dist}(\Pi_{\mathbb{C}} P, \Gamma_{\frac{n+1}{2}-\gamma})$.
Now we recall the operator-valued symbol space

$$S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}) \tag{3.29}$$

for H and $\tilde{H}$ are Hilbert spaces with group action κ and $\tilde{\kappa}$, respectively, cf. Definition1.10. In concrete cases we set

$$H = \mathcal{K}^{s,\gamma}(X^\wedge), \; \tilde{H} = \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge). \tag{3.30}$$

cf. (1.95).

Remark 3.7. *[54, Subsection 3.2.1 Proposition 5] For every $s, \gamma, e \in \mathbb{R}$ the operator $\mathcal{M}_\varphi$ of multiplication by a function $\varphi \in C^\infty([0, R))$ induces a continuous operator $\mathcal{M}_\varphi : \mathcal{K}^{s,\gamma;e}(X^\wedge) \to \mathcal{K}^{s,\gamma;e}(X^\wedge)$ and can be interpreted as an element*

$$\mathcal{M}_\varphi \in S^0(\mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{s,\gamma;e}(X^\wedge)),$$

and $\varphi \to \mathcal{M}_\varphi$ defines a continuous operator

$$C^\infty([0, R)) \to S^0(\mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{s,\gamma;e}(X^\wedge)).$$

For references below we need the following remark which is an immediate consequence of the definition of classical operator-valued symbols.

Remark 3.8. *Let $a(y, \eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \tilde{H}))$ be a function such that*

$$a(y, \delta\eta) = \delta^\mu \tilde{\kappa}_\delta a(y, \eta) \kappa_\delta^{-1}$$

for all $\delta \geq 1$, $|\eta| \geq C$, for a constant $C > 0$. Then we have

$$a(y, \eta) \in S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{s,\gamma;e}(X^\wedge)).$$

Definition 3.9. (i) *Let $\mathcal{R}_G^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for $\mu \in \mathbb{R}$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta), \Theta = (\vartheta, 0]$, denote the space of all*

$$g(y, \eta) \in \bigcap_{s,e \in \mathbb{R}} S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu;\infty}(X^\wedge))$$

such that

$$g(y, \eta) \in \bigcap_{s,e \in \mathbb{R}} S_{\mathrm{cl}}^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;e}(X^\wedge), \mathcal{K}_P^{\infty,\gamma-\mu;\infty}(X^\wedge)),$$

$$g^*(y,\eta) \in \bigcap_{s,e\in\mathbb{R}} S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,-\gamma+\mu;e}(X^\wedge),\mathcal{K}^{\infty,-\gamma;\infty}_Q(X^\wedge)),$$

and for some g-dependent asymptotic types P and Q, the pointwise formal adjoint refers to the $\mathcal{K}^{0,0}(X^\wedge)$-scalar product.
(ii) *By $\mathcal{R}^\mu_{M+G}(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$ and $\Theta=(-(k+1),0],k\in\mathbb{N}$, we denote the space of all $m(y,\eta)+g(y,\eta)$ for $g(y,\eta)\in\mathcal{R}^\mu_G(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ and smoothing Mellin symbols $m(y,\eta)$ of the form*

$$m(y,\eta):=r^{-\mu}\omega_\eta\sum_{j=0}^{k} r^j\sum_{|\alpha|\le j}\mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha\omega'_\eta \tag{3.31}$$

for arbitrary $f_{j\alpha}\in C^\infty(\Omega,M^{-\infty}_{R_{j\alpha}}(X))$, and Mellin asymptotic types $R_{j\alpha}$, weights $\gamma_{j\alpha}\in\mathbb{R}$, satisfying

$$\gamma-j\le\gamma_{j\alpha}\le\gamma,\ \Pi_{\mathbb{C}}R_{j\alpha}\cap\Gamma_{\frac{n+1}{2}-\gamma_{j\alpha}}=\emptyset,$$

and cut-off functions ω,ω' on the r half-axis where $\omega_\eta(r)=\omega(r[\eta])$.
(iii) *By $\mathcal{R}^\mu(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta),\Theta=(-(k+1),0]$ we denote the space of all edge symbols, i.e., operator functions $a(y,\eta)$ of the form*

$$a(y,\eta)=r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega'+\varphi\mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\varphi'+(m+g)(y,\eta) \tag{3.32}$$

for cut-off functions ω,ω' on the r half-axis, for an

$$h(r,y,w,\eta)=\tilde{h}(r,y,w,r\eta),\ \tilde{h}(r,y,w,\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,M^\mu_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})), \tag{3.33}$$

$(m+g)(y,\eta)\in\mathcal{R}^\mu_{M+G}(\Omega\times\mathbb{R}^q,\boldsymbol{g}),\varphi,\varphi'\in C_0^\infty(\mathbb{R}_+)$, and $p_{\mathrm{int}}(r,y,\rho,\eta)\in C^\infty(\mathbb{R}_+\times\Omega,L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\rho,\eta}))$.

The elements $a(y,\eta)$ in $\mathcal{R}^\mu(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ are the (operator-valued) amplitude functions of the edge calculus, see, e.g., [55]. Originally they have been employed in the form (3.35) below, but here we refer to a "pure" Mellin representation close to the edge in the sense of [17].
There is also a well-known cone pseudo-differential calculus on the infinite stretched cone $X^\wedge$, consisting of spaces $L^\mu(X^\wedge,\boldsymbol{g})$ of operators of order $\mu\in\mathbb{R}$, referring to the weight data $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, cf. [54] or [55]. Then $a(y,\eta)\in\mathcal{R}^\mu(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ belongs to $L^\mu(X^\wedge,\boldsymbol{g})$ for every fixed $(y,\eta)\in\Omega\times\mathbb{R}^q$. Operators in $L^\mu(X^\wedge,\boldsymbol{g})$ have a symbolic structure, consisting of the interior symbol from the inclusion $L^\mu(X^\wedge,\boldsymbol{g})\subset L^\mu_{\mathrm{cl}}(X^\wedge)$ and the conormal symbols. The highest order conormal symbol in this notation is

$$\sigma_M(a)(y,w)=h(0,y,w,0)+f_{00}(y,w), \tag{3.34}$$

cf. the notation in (3.31).
In the following in (3.32) we drop the term with "int" because for a suitable choice of of ω,ω' it can be integrated in the first summand containing the Mellin operators. The operators families (3.32) play the role of edge amplitude functions. Those are of another form than those in expositions of the edge pseudo-differential calculus, cf. [55], where they are written as follows. Let $\boldsymbol{\omega''}\prec\boldsymbol{\omega}\prec\boldsymbol{\omega'}$ be cut-off functions on the half-axis and write

$\boldsymbol{\omega}_\eta(r) := \boldsymbol{\omega}(r[\eta])$ where $\eta \to [\eta]$ is any strictly positive smooth function such that $[\eta] = |\eta|$ for $|\eta| \geq C$ for some $C > 0$,

$$a(y,\eta) = r^{-\mu}\omega\Big\{\boldsymbol{\omega}_\eta \mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\boldsymbol{\omega}'_\eta + (1-\boldsymbol{\omega}_\eta)\mathrm{Op}_r(p)(y,\eta)(1-\boldsymbol{\omega}''_\eta)\Big\}\omega' \\ + (m+g)(y,\eta) \tag{3.35}$$

for a

$$p(r,y,\rho,\eta) = \tilde{p}(r,y,r\rho,r\eta), \tag{3.36}$$

where

$$\tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})), \tag{3.37}$$

such that p and h are related via a Mellin quantisation that says

$$\mathrm{Op}_M^\beta(h)(y,\eta) = \mathrm{Op}_r(p)(y,\eta) \tag{3.38}$$

modulo $C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q))$ for every $\beta \in \mathbb{R}$.
More precisely, we have the following result, cf. [17, Theorem 3.2].

Theorem 3.10. (i) *For every $p(r,y,\rho,\eta)$ of the form* (3.36) *and any $\varphi(r) \in C_0^\infty(\mathbb{R}_+), \varphi(r) \equiv 1$ close to $r=1$, there is an*

$$\tilde{h}(r,y,w,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})), \tag{3.39}$$

such that $h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta)$ satisfies the relation

$$\mathrm{Op}_r(p)(y,\eta) - \mathrm{Op}_M^\beta(h)(y,\eta) = \mathrm{Op}_r(q)(y,\eta)$$

for $q(r,r',y,\rho,\eta) = \tilde{q}(r,r',y,r\rho,r\eta)$, $\tilde{q}(r,r',y,\tilde{\rho},\tilde{\eta}) = (1-\varphi(r'/r))\tilde{p}(r,y,\tilde{\rho},\tilde{\eta})$, i.e.,

$$\mathrm{Op}_r(q)(y,\eta) \in C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q_\eta)).$$

(ii) *For every* (3.39) *and any $\psi(r) \in C_0^\infty(\mathbb{R}_+)$, $\psi \equiv 1$ close to $r=1$, there is a $p(r,y,\rho,\eta)$ of the form* (3.36) *such that*

$$\mathrm{Op}_M^\beta(h)(y,\eta) - \mathrm{Op}_r(p)(y,\eta) = \mathrm{Op}_M^\beta(1-\psi(r'/r))h)(y,\eta)$$

where the remainder belongs to $C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q_\eta))$.

As a consequence of what is done in [18] we have the following remark.

Remark 3.11. *We apply Theorem* 3.10 (ii) *to a Mellin symbol $\tilde{h}(w,\tilde{\eta})$ and obtain a Fourier symbol $\tilde{p}(\tilde{\rho},\tilde{\eta})$. when we feed in $\tilde{p}(\tilde{\rho},\tilde{\eta})$ in Theorem* 3.10 (i) *we get another Mellin symbol $\tilde{h}_1(w,\tilde{\eta})$, and then $\tilde{h}(w,\tilde{\eta}) - \tilde{h}_1(w,\tilde{\eta}) \in M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})$. A similar observation is true when we admit an extra C^∞-dependence on $(r,y) \in \mathbb{R}_+ \times \Omega$. Moreover, we can apply the procedure the other way around, i.e., starting from $\tilde{p}$, obtain an $\tilde{h}$ by* (i) *and then by* (ii) *a $\tilde{p}_1$ where $\tilde{p}(\tilde{\rho},\tilde{\eta}) - \tilde{p}_1(\tilde{\rho},\tilde{\eta}) \in L^{-\infty}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$.*

Relation (3.38) shows that (3.32) is an element of $C^\infty(\Omega, L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^q))$. As such it has a parameter-dependent homogeneous principal symbol close to $r=0$ where $\omega(r)=\omega'(r)=1$ and locally on X in variables $x\in\Sigma$, $\Sigma\subseteq\mathbb{R}^n$ open, of the form

$$\sigma_0(a)(r,x,y,\rho,\xi,\eta):=r^{-\mu}p_{(\mu)}(r,x,y,\rho,\xi,\eta)=r^{-\mu}\tilde{p}_{(\mu)}(r,x,y,r\rho,\xi,r\eta) \tag{3.40}$$

for a function $\tilde{p}_{(\mu)}(r,x,y,\tilde{\rho},\xi,\tilde{\eta})$ which is homogeneous in $(\tilde{\rho},\xi,\tilde{\eta})\neq 0$ and smooth in r up to zero. In this notation, close to $r=0$ we have

$$\tilde{\sigma}_0(a)(r,x,y,\rho,\xi,\eta):=\tilde{p}_{(\mu)}(r,x,y,\rho,\xi,\eta). \tag{3.41}$$

Moreover,

$$\sigma_1(a)(y,\eta)=r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta)+\sigma_1(m+g)(y,\eta),$$

for $h_0(r,y,w,\eta)=\tilde{h}(0,y,w,r\eta)$, where

$$\sigma_1(m)(y,\eta):=r^{-\mu}\omega_{|\eta|}\sum_{j=0}^{k}r^j\sum_{|\alpha|=j}\mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})(y)\eta^\alpha\omega'_{|\eta|}, \tag{3.42}$$

$\omega_{|\eta|}(r)=\omega(r|\eta|)$ and $\sigma_1(g)(y,\eta)=g_{(\mu)}(y,\eta)$ as the homogeneous principal part of $g(y,\eta)$ as a classical symbol of order μ.
Observe that for any cut-off functions ω,ω' and an excision function $\chi(\eta)$ for every $g(y,\eta)\in\mathcal{R}^\mu_G(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ we have

$$\omega\chi(\eta)g(y,\eta)\omega'\in\mathcal{R}^\mu_G(\Omega\times\mathbb{R}^q,\boldsymbol{g}),\ \ \sigma_1(g)(y,\eta)=\sigma_1(\omega\chi g\omega')(y,\eta). \tag{3.43}$$

In fact, the multiplication of a Green symbol g by an excision function χ gives us a Green symbol again since $(1-\chi)g$ is a smoothing Green symbol. The multiplication by ω or ω' preserves the property of being Green, cf. also [17, Remark 3.12]. The second relation of (3.43) is a consequence of (1.53).

Remark 3.12. *It is useful to express $\sigma_0(a)$ and $\tilde{\sigma}_0(a)$ in terms of the Mellin symbol h occurring in* (3.32) *which is connected with p via* (3.38)*. Also h can be locally expressed by symbols, in this case*

$$h(r,x,y,w,\xi,\eta)=\tilde{h}(r,x,y,w,\xi,r\eta)$$

for $\tilde{h}(r,x,y,w,\xi,r\eta)\in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\Gamma_{\frac{n+1}{2}-\gamma}\times\mathbb{R}^{n+q}_{\xi,\tilde{\eta}})$ with holomorphic extension as elements in $S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\Gamma_\beta\times\mathbb{R}^{n+q}_{\xi,\tilde{\eta}})$ for every $\beta\in\mathbb{R}$ where the homogeneous principal symbol

$$\tilde{h}_{(\mu)}(r,x,y,i\rho,\xi,\tilde{\eta})$$

computed for $\beta=0$ is independent of β. Then, cf. [55, Theorem 3.2.7], we have

$$\tilde{h}_{(\mu)}(r,x,y,i\rho,\xi,\tilde{\eta})=\tilde{p}_{(\mu)}(r,x,y,-r\rho,\xi,\tilde{\eta}). \tag{3.44}$$

Remark 3.13. $\sigma_0(a)(y,\eta)=0$ *implies* $\tilde{h}(r,y,w,\tilde{\eta})\in C^\infty(\overline{\mathbb{R}}_+\times\Omega, M^{\mu-1}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}}))$.

In fact, from (3.40) and (3.44) it follows that $\tilde{h}_{(\mu)}$ vanishes for all $z\in\Gamma_0$. However, this entails $\tilde{h}_{(\mu)}=0$ for $z\in\Gamma_\beta$ for every β which shows that $\tilde{h}$ itself is of order $\mu-1$.

3.3 Order filtrations

We now consider the space $\mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ of edge symbols for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, cf. Definition 3.9 (iii), from the point of view of a natural filtration

$$\mathcal{R}^{\mu} \supset \mathcal{R}^{\mu-1} \supset \cdots \supset \mathcal{R}^{-\infty}.$$

For $\mathcal{R}^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}) \subset S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$ this structure has a simple meaning. The elements $(m+g)(y,\eta) \in \mathcal{R}^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ have a sequence of (twisted) homogeneous components $\sigma_1^{\mu-j}(m+g)(y,\eta)$, $j \in \mathbb{N}$, where $\sigma_1 := \sigma_1^{\mu}$, and we can define $\mathcal{R}^{\mu-(j+1)}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for j-independent weight data $\boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0])$ as the subspace of all $(m+g)(y,\eta)$ such that $\sigma_1^{\mu-l}(m+g)(y,\eta) = 0$ for all $l = 0, \cdots, j$.

Definition 3.14. *Let* $\mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, $j \in \mathbb{N}$, $\boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0])$, *be the space of all operator families* (3.32) *of the form*

$$a(y,\eta) = r^{-\mu}\omega \mathrm{Op}_M^{\gamma-n/2}(r^j h)(y,\eta)\omega' + \varphi \mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\varphi' + (m+g)(y,\eta)$$

for cut-off functions ω, ω' *on the* r *half-axis, for arbitrary*

$$h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta),\ \tilde{h}(r,y,w,\tilde{\eta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{\mu-j}(X; \mathbb{R}^q_{\tilde{\eta}})),$$

$(m+g)(y,\eta) \in \mathcal{R}^{\mu-j}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}), \varphi, \varphi' \in C_0^{\infty}(\mathbb{R}_+)$, *and* $p_{\mathrm{int}}(r,y,\rho,\eta) \in C^{\infty}(\mathbb{R}_+ \times \Omega, L^{\mu-j}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\rho,\eta}))$.

Note that, as for $j = 0$, we can drop the term with p_{int} when we choose the cut-off functions in a suitable manner. $\mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ in general is not a classical symbol space. The corresponding filtration is by no means obvious. We do not really employ this in this paper, but in order to illustrate the structure we consider here the case $\dim X = 0$, $\mu = 0$, and $\boldsymbol{g} = (0, 0, (-(k+1), 0])$.

Proposition 3.15. $a(y,\eta) \in \mathcal{R}^0(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, *and* $\sigma_0(a) = 0$, $\sigma_1(a) = 0$, *implies* $a(y,\eta) \in \mathcal{R}^{-1}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$.

Proof. Without loss of generality we consider the case without dependence on y. The Mellin edge amplitude functions of order zero have the form (3.32), i.e.,

$$a(\eta) = \omega \mathrm{Op}_M(h)(\eta)\omega' + \varphi \mathrm{Op}_r(p_{\mathrm{int}})(\eta)\varphi' + (m+g)(\eta).$$

We employ the fact that for a suitable choice of ω, ω' we can drop the term with "int". In other words we ignore it from now on, i.e.,

$$a(\eta) = \omega \mathrm{Op}_M(h)(\eta)\omega' + (m+g)(\eta),$$

for

$$h(r,w,\eta) = \tilde{h}(r,w,r\eta),\ \ \tilde{h}(r,w,\tilde{\eta}) \in C_0^{\infty}(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^0(\mathbb{R}^q_{\tilde{\eta}})).$$

Then

$$a(\eta) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+). \tag{3.45}$$

The continuity for the Mellin term is a consequence of a Mellin analogue of the Calderón-Vaillancourt Theorem. For the other summands it is evident. Then, according to Remark 3.13, the relation $\sigma_0(a) = 0$ has the consequence that

$$\tilde{h}(r, w, \tilde{\eta}) \in C_0^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-1}(\mathbb{R}^q_{\tilde{\eta}})). \tag{3.46}$$

Thus,

$$\tilde{h}(0, w, \tilde{\eta}) \in M_{\mathcal{O}}^{-1}(\mathbb{R}^q_{\tilde{\eta}}).$$

Moreover, we have

$$\sigma_1(a)(\eta) = \mathrm{Op}_M(h_0)(\eta) + \sigma_1(m+g)(\eta), \quad h_0(r, w, \eta) = \tilde{h}(0, w, r\eta). \tag{3.47}$$

Let us write

$$\begin{aligned} a(\eta) = \omega \mathrm{Op}_M(h - h_0)(\eta)\omega' + \omega \mathrm{Op}_M(h_0)(\eta)\omega' \\ + m_0(\eta) + g_0(\eta) + m^{-1}(\eta) + g^{-1}(\eta) \end{aligned} \tag{3.48}$$

for

$$m_0(\eta) := \omega_\eta \sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{Op}_M^{\gamma_{j\alpha}}(f_{j\alpha})\eta^\alpha \omega'_\eta, \quad g_0(\eta) := \omega\chi(\eta)\sigma_1(g)(\eta)\omega',$$

and

$$m^{-1}(\eta) := (m - m_0)(\eta), \quad g^{-1}(\eta) := (g - g_0)(\eta) \in \mathcal{R}^{-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}). \tag{3.49}$$

Applying Taylor's formula in the first r-variable in $\tilde{h}(r, w, r\eta) = h(r, w, \eta)$ it follows that

$$\omega \mathrm{Op}_M(h - h_0)(\eta)\omega' = \omega r \mathrm{Op}_M(h^{-1})(\eta)\omega' \tag{3.50}$$

for some

$$h^{-1}(r, w, \eta) = \tilde{h}^{-1}(r, w, r\eta), \quad \tilde{h}^{-1}(r, w, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^{-1}(\mathbb{R}^q_{\tilde{\eta}})). \tag{3.51}$$

Thus (3.50) belongs to $\mathcal{R}^{-1}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$. It remains to verify that

$$\omega \mathrm{Op}_M(h_0)(\eta)\omega' + m_0(\eta) + g_0(\eta) \in \mathcal{R}^{-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}).$$

Because of (3.49), (3.50) we have

$$\begin{aligned} \sigma_1(a)(\eta) &= \sigma_1\Big(\omega \mathrm{Op}_M(h_0)\omega' + m_0 + g_0\Big)(\eta) \\ &= \mathrm{Op}_M(h_0)(\eta) + \sigma_1(m_0)(\eta) + \sigma_1(g_0)(\eta) = 0. \end{aligned}$$

We now employ the fact that $\sigma_1(a)(\eta)$ has the structure of an η-dependent family of operators in the cone calculus, i.e., $L^0(\overline{\mathbb{R}}_+, \boldsymbol{g}),\ \eta \neq 0$. Vanishing of these operators implies that the conormal symbols vanish, cf. [54, Subsection 1.3.1, Theorem 4]. In particular, we then have

$$\sigma_M^0(\mathrm{Op}_M(h_0)(\eta) + \sigma_1(m_0)(\eta))(w) = \tilde{h}(0, w, 0) + f_{00}(w) = 0.$$

This shows that we already have $\sigma_1(\omega \mathrm{Op}_M(h_0)\omega' + m_0)(\eta) = 0$ and

$$\omega(\mathrm{Op}_M(h_0)(\eta) - \mathrm{Op}_M(\tilde{h}(0, w, 0))(\eta))\omega' \in \mathcal{R}^{-1}(\Omega \times \mathbb{R}^q, \boldsymbol{g}),$$

hence $\omega \mathrm{Op}_M(h_0)(\eta)\omega' + m_0(\eta) \in \mathcal{R}^{-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, and consequently $\sigma_1(g_0)(\eta) = 0$ which gives us $g_0(\eta) \in \mathcal{R}^{-1}_G(\Omega \times \mathbb{R}^q, \boldsymbol{g})$. □

Lemma 3.16. *From* (3.48), (3.51) *and* $\tilde{h}^{-1}(0,w,r\eta) = h_0^{-1}(r,w,\eta)$ *we obtain*

$$\sigma_1^{-1}(a)(\eta) = r\mathrm{Op}_M(h_0^{-1})(\eta) + \sigma_1^{-1}(m^{-1}+g^{-1})(\eta),$$

where σ_1^{-1} *indicates the homogeneous principal edge symbol of order* -1 *of the respective operators, and it follows a family of continuous operators for* $\eta \neq 0$,

$$(1-\omega)\sigma_1^{-1}(a)(\eta) : L^2(\mathbb{R}_+) \to H^1(\mathbb{R}_+). \tag{3.52}$$

Proof. Let us write $h_0^{-1}(r,w,\eta) =: f(w,r)$ for fixed $\eta \neq 0$. The space $u \in H^1(\mathbb{R}_+)$ is characterised by the conditions $u \in L^2(\mathbb{R}_+)$ and $\partial_r u \in L^2(\mathbb{R}_+)$. In other words we have to prove

$$\|(1-\omega)r\mathrm{Op}_M(f)u\|_{L^2(\mathbb{R}_+)} \le c_0\|u\|_{L^2(\mathbb{R}_+)}, \tag{3.53}$$

$$\|\partial_r(1-\omega)r\mathrm{Op}_M(f)u\|_{L^2(\mathbb{R}_+)} \le c_1\|u\|_{L^2(\mathbb{R}_+)}, \tag{3.54}$$

for all $u \in L^2(\mathbb{R}_+)$, and some $c_0, c_1 > 0$. For (3.53) we have

$$\|(1-\omega)r\mathrm{Op}_M(f)u\|_{L^2(\mathbb{R}_+)}$$

$$= \|(1-\omega)\iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}+i\rho)} rf(\frac{1}{2}+i\rho,r)u(r')\frac{dr'}{r'}đ\rho\|_{L^2(\mathbb{R}_+)}.$$

Setting $f_0(\rho,r) := (1-\omega(r))rf(\frac{1}{2}+i\rho,r)$ we can apply the following Mellin analogue of a version of Calderón-Vailancourt's Theorem, cf. [25].
Let $F(\rho,r)$ be a function in $C^\infty(\mathbb{R}\times\mathbb{R}_+)$ such that

$$\pi(F) := \sup\Big\{|(r\partial_r)^k D_\rho^l F(\rho,r)| : (\rho,r)\in\mathbb{R}\times\mathbb{R}_+,\ 0\le k\le 1,\ 0\le l\le 1\Big\} < \infty. \tag{3.55}$$

Then $\mathrm{Op}_M(F) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is continuous, and we have

$$\|\mathrm{Op}_M(F)\|_{\mathcal{L}(L^2(\mathbb{R}_+))} \le c\pi(F)$$

for some $c > 0$. Applying this for $F(\rho,r) = f_0(\rho,r) \in S^0_{\mathrm{cl}}(\mathbb{R}_\rho\times\mathbb{R}_r)$, where $(\rho,r)\in\mathbb{R}\times\mathbb{R}_+$ is treated as a two-dimensional covariable, it suffices to note that the required estimate holds. For (3.54) we have

$$\partial_r(1-\omega)r\mathrm{Op}_M(f) = \varphi\mathrm{Op}_M(f) + (1-\omega)\mathrm{Op}_M(f) + (1-\omega)r\partial_r\mathrm{Op}_M(f)$$

for some $\varphi \in C_0^\infty(\mathbb{R}_+)$. Then the first two terms can be treated as (3.53). Therefore, we look at the third term. In this case

$$\begin{aligned}&\|(1-\omega)r\partial_r\mathrm{Op}_M(f)u\|^2_{L^2(\mathbb{R}_+)}\\ &= \|(1-\omega)r\partial_r\iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}+i\rho)} f(\frac{1}{2}+i\rho,r)u(r')\frac{dr'}{r'}đ\rho\|^2_{L^2(\mathbb{R}_+)}.\end{aligned} \tag{3.56}$$

Because of $\partial_r\Big(r^{-(\frac{1}{2}+i\rho)}\Big) = -(\frac{1}{2}+i\rho)r^{-(\frac{1}{2}+i\rho)}r^{-1}$ the right hand side of (3.56) can be estimated by

$$\|(1-\omega)\iint \left(\frac{r}{r'}\right)^{-(\frac{1}{2}+i\rho)}(\frac{1}{2}+i\rho)f(\frac{1}{2}+i\rho,r)u(r')\frac{dr'}{r'}đ\rho\|^2_{L^2(\mathbb{R}_+)}$$

and

$$\|(1-\omega)\iint\Big(\frac{r}{r'}\Big)^{-(\frac{1}{2}+i\rho)}r(\partial_r f)(\frac{1}{2}+i\rho,r)u(r')\frac{dr'}{r'}đ\rho\|^2_{L^2(\mathbb{R}_+)}.$$

Let us set

$$f_1(\rho,r):=(1-\omega(r))(\frac{1}{2}+i\rho)f(\frac{1}{2}+i\rho,r),\ f_2(\rho,r):=(1-\omega(r))(r\partial_r f)(\frac{1}{2}+i\rho,r).$$

For $F(\rho,r)=f_i(\rho,r), i=1,2$, we apply once again the Calderón-Vailancourt Theorem. In this case we have $f_1(\rho,r)\in S^0_{\mathrm{cl}}(\mathbb{R}_\rho\times\mathbb{R}_r),\ \ f_2(\rho,r)\in S^{-1}_{\mathrm{cl}}(\mathbb{R}_\rho\times\mathbb{R}_r)$ and hence

$$(r\partial_r)^k D^l_\rho f_1(\rho,r)\in S^{-l}_{\mathrm{cl}}(\mathbb{R}_\rho\times\mathbb{R}_r),\ \ (r\partial_r)^k D^l_\rho f_2(\rho,r)\in S^{-1-l}_{\mathrm{cl}}(\mathbb{R}_\rho\times\mathbb{R}_r),$$

for all $0\le k,\ l\le 1$. Every $p(\rho,r)\in S^{\nu}_{\mathrm{cl}}(\mathbb{R}\times\mathbb{R})$ for any $\nu\in\mathbb{R}$ satisfies the symbolic estimate $\sup|p(\rho,r)|\le c\langle\rho,r\rangle^{\nu}$. Then the relations (3.55) are satisfied for $F(\rho,r)=f_i(\rho,r), i=1,2$. □

Theorem 3.17. *The conditions*

$$\sigma_0(a)(r,\rho,\eta)=0,\ \ (r,\rho,\eta)\in\overline{\mathbb{R}}_+\times(\mathbb{R}^{1+q}\setminus\{0\}), \tag{3.57}$$

$$\sigma_1(a)(\eta)=0,\ \ \eta\neq 0, \tag{3.58}$$

imply that

$$\sigma_1^{-1}(a)(\eta):L^2(\mathbb{R}_+)\to\mathcal{K}^{1,0}_P(\mathbb{R}_+) \tag{3.59}$$

for some asymptotic type P, associated with the weight data $(0,0,(-1,0])$.

Proof. Let us write $\sigma_1^{-1}(a)(\eta)=(1-\omega)\sigma_1^{-1}(a)(\eta)+\omega\sigma_1^{-1}(a)(\eta)$. Then for (3.59) we employ (3.52) together with

$$\omega\sigma_1^{-1}(a)(\eta)=\omega r\mathrm{Op}_M(h_0^{-1})(\eta)+\omega\sigma_1^{-1}(m^{-1})(\eta)+\omega\sigma_1^{-1}(g^{-1})(\eta),$$

and we conclude the continuities

$$\omega r\mathrm{Op}_M(h_0^{-1})(\eta)+\omega\sigma_1^{-1}(m^{-1})(\eta):L^2(\mathbb{R}_+)\to\omega\mathcal{H}^{1,1}(\mathbb{R}_+),$$

$$\omega\sigma_1^{-1}(g^{-1})(\eta):L^2(\mathbb{R}_+)\to\omega\mathcal{K}^{\infty,0}_P(\mathbb{R}_+),$$

using $(1-\omega)H^1(\mathbb{R}_+)+\omega\mathcal{H}^{1,1}(\mathbb{R}_+)+\omega\mathcal{K}^{\infty,0}_P(\mathbb{R}_+)\subseteq\mathcal{K}^{1,0}_P(\mathbb{R}_+)$. □

3.4 Edge quantization

For the following consideration we need some preparations. Given a closed C^∞ manifold $X, n=\dim X$, with a system of charts $\kappa_l:U_l\to\mathbb{R}^n, l=1,\dots,N$, we consider the cylinder $\mathbb{R}\times X$ with the charts $\mathrm{id}_{\mathbb{R}}\times\kappa_l:\mathbb{R}\times U_l\to\mathbb{R}\times\mathbb{R}^n$. The cylinder can also be equipped with the structure of $X^{\asymp}$, a manifold with conical exits to infinity $r\to\pm\infty$, and we define a diffeomorphism

$$\chi:\mathbb{R}\times X\to X^{\asymp}$$

by the local transformations

$$\chi:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\times\mathbb{R}^n,\ \chi(r,x)=(r,[r]x).$$

Theorem 3.18. *[55, Proposition 3.2.19] For any*

$$\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}),\ p(r,\rho,\eta) := \tilde{p}(r\rho, r\eta),$$

and cut-off functions ω, ω', for every fixed $\eta \neq 0$ we have

$$\chi_*(1-\omega)\mathrm{Op}_r(p)(\eta)(1-\omega') \in L^{\mu;\mu}(X^{\asymp}), \tag{3.60}$$

with χ_* denoting the operator push forward under χ.

Remark 3.19. *A consequence of the details of the proof of Theorem* 3.18 *is the following. Let*

$$\tilde{p}(\tilde{\rho},\tilde{\eta},\tilde{\zeta}) \in L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^{1+q+d}_{\tilde{\rho},\tilde{\eta},\tilde{\zeta}}),\ \ d>0,\ \ p(r,\rho,\eta,\zeta) := \tilde{p}(r, r\rho, r\eta, r\zeta).$$

Moreover, assume that $\tilde{p}(\tilde{\rho},\tilde{\eta},\tilde{\zeta})$ is parameter-dependent elliptic of order μ, with parameters $(\tilde{\rho},\tilde{\eta},\tilde{\zeta}) \in \mathbb{R}^{1+q+d}_{\tilde{\rho},\tilde{\eta},\tilde{\zeta}}$. Then the operator (3.60)*, here depending on (η,ζ) rather than η, is parameter-dependent exit elliptic for every fixed $\eta \neq 0$, with parameter $\zeta \in \mathbb{R}^d$, cf. notation at the end of Section* 3.1*.*

The following considerations are crucial for the interpretation of edge amplitude functions (3.32). Earlier investigations on edge pseudo-differential operators mainly employed symbols of the form (3.35). Those are known to be elements of

$$S^{\mu}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^{\wedge}),\mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})),\ s\in\mathbb{R}.$$

However, (3.32) needs some care in terms of the mapping behaviour of Mellin operators for $r\to\infty$, i.e., in "standard" Sobolev spaces, involved in $\mathcal{K}$-spaces, rather than Mellin Sobolev spaces at ∞. Such a formulation from (3.35) to (3.32) has been obtained already in [17]. What we do here is to employ Mellin representations from the very beginning and take them as the primary objects in the edge calculus.

Lemma 3.20. *Let*

$$b(\eta) = \chi(\eta) r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(\eta) \tag{3.61}$$

for some $h(r,w,\eta) = \tilde{h}(w, r\eta)$, $\tilde{h}(w,\tilde{\eta}) \in M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})$. Then we have

$$b(\eta) \in C^{\infty}(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}),\mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))) \tag{3.62}$$

for every $s\in\mathbb{R}$.

Proof. The smoothness of $b(\eta)$ in $\eta\in\mathbb{R}^q$ is straightforward. Therefore, the main issue is to show that

$$b(\eta): \mathcal{K}^{s,\gamma}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}) \tag{3.63}$$

is continuous. Let us write

$$\begin{aligned} b(\eta) &= \omega b(\eta) + (1-\omega)b(\eta) \\ &= \omega b(\eta)\omega' + \omega b(\eta)(1-\omega') + (1-\omega)b(\eta)\omega'' + (1-\omega)b(\eta)(1-\omega'') \end{aligned}$$

for cut-off functions $\omega'' \prec \omega \prec \omega'$. Then, by virtue of [17, Proposition A.8] we have

$$g_1(\eta) := \omega b(\eta)(1-\omega'),\ g_2(\eta) := (1-\omega)b(\eta)\omega'' \in \mathcal{R}^{\mu}_{G}(\mathbb{R}^q;\boldsymbol{g})$$

for $\boldsymbol{g} = (\gamma, \gamma - \mu, (-\infty, 0])$. This entails the continuity (3.63) for $g_i(\eta)$ instead of $b(\eta)$, $i = 1, 2$. Moreover, we have as desired

$$\omega b(\eta)\omega' \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))).$$

Thus it remains to consider $(1-\omega)b(\eta)(1-\omega'')$. For convenience we assume $X = S^n$. In this case we have

$$(1-\omega)\mathcal{K}^{s,\gamma}((S^n)^\wedge) = (1-\omega)H^s(\mathbb{R}^{n+1})$$

for any cut-off function ω. For general X it suffices to consider distributions that are supported in a set $\left\{\tilde{x} \in \mathbb{R}^{n+1} : |\tilde{x}| > R, \tilde{x}/|\tilde{x}| \in V\right\}$ for some $R > 0$ and a coordinate neighbourhood V on S^n. The simple details combined with suitable charts on X and a partition of unity are left to the reader. In other words we verify the continuity

$$(1-\omega)b(\eta)(1-\omega'') : H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1}), \tag{3.64}$$

for every $s \in \mathbb{R}$. From Theorem 3.10 (ii) we have

$$\mathrm{Op}_M^{\gamma-n/2}(h)(\eta) = \mathrm{Op}_r(p)(\eta) + \mathrm{Op}_M^{\gamma-n/2}((1-\varphi(r'/r))h)(\eta) \tag{3.65}$$

for $p(r,\rho,\eta) = \tilde{p}(r\rho, r\eta)$ as in (3.36), (3.37). Applying Remark 3.11 we can modify relation (3.65) by applying Theorem 3.10 (i) to $\tilde{p}(\tilde{\rho},\tilde{\eta})$ and obtain another $\tilde{h}_1(w,\tilde{\eta}) \in M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde{\eta}})$ such that

$$\tilde{h}_1(w,\tilde{\eta}) = \tilde{h}(w,\tilde{\eta}) + \tilde{l}(w,\tilde{\eta}),\ \tilde{l}(w,\tilde{\eta}) \in M_{\mathcal{O}}^{-\infty}(X;\mathbb{R}^q_{\tilde{\eta}}). \tag{3.66}$$

For $h_1(r,w,\eta) = \tilde{h}_1(w,r\eta)$, where $\mathrm{Op}_r(p)(\eta) - \mathrm{Op}_M^{\gamma-n/2}(h_1)(\eta) = \mathrm{Op}_r(q)(\eta)$ and $q(r,r',\rho,\eta) = \tilde{q}(r,r',r\rho,r\eta)$, $\tilde{q}(r,r',\tilde{\rho},\tilde{\eta}) = (1-\varphi(r'/r))\tilde{p}(\tilde{\rho},\tilde{\eta})$. From (3.66) it follows that

$$\mathrm{Op}_M^{\gamma-n/2}(h)(\eta) = \mathrm{Op}_M^{\gamma-n/2}(h_1)(\eta) - \mathrm{Op}_M^{\gamma-n/2}(l)(\eta) = Q_1(\eta) - Q_2(\eta) - Q_3(\eta)$$

for $Q_1(\eta) := \mathrm{Op}_r(p)(\eta)$, $Q_2(\eta) := \mathrm{Op}_r(q)(\eta)$, $Q_3(\eta) := \mathrm{Op}_M^{\gamma-n/2}(l)(\eta)$, and $l(r,w,\eta) = \tilde{l}(w,r\eta)$. Thus

$$(1-\omega)b(\eta)(1-\omega'') = B_1(\eta) - B_2(\eta) - B_3(\eta)$$

for

$$B_j(\eta) := (1-\omega)\chi(\eta)r^{-\mu}Q_j(\eta)(1-\omega''),\ j = 1,2,3.$$

The first summand $B_1(\eta)$ on the right represents an operator in $L^{\mu;0}(\mathbb{R}^{n+1})$, cf. Theorem 3.18. This yields the desired continuity (3.64) of $B_1(\eta)$. The second summand is of the form

$$\begin{aligned} &B_2(\eta)u(r) \\ &= (1-\omega(r))\chi(\eta)r^{-\mu}\iint e^{i(r-r')\rho}(1-\varphi(r'/r))\tilde{p}(\tilde{\rho},\tilde{\eta})(1-\omega''(r'))u(r')dr'đ\rho. \end{aligned} \tag{3.67}$$

Since $\eta \neq 0$ is fixed we drop the excision factor $\chi(\eta)$ and simply write B_2 rather than $B_2(\eta)$. Thus, setting

$$f_N(r,r') = (1-\omega(r))(r'-r)^{-N}r^{N-\mu}(1-\varphi(r'/r))(1-\omega''(r'))$$

from (3.67) we obtain by integration by parts

$$B_2u(r) = \iint e^{i(r-r')\rho}f_N(r,r')(D_{\tilde{\rho}}^N\tilde{p})(r\rho,r\eta)u(r')dr'đ\rho$$

for every $N \in \mathbb{N}$. Since N is arbitrary the kernel of B_2 belong to $\mathcal{S}(\mathbb{R} \times \mathbb{R}, L^{-\infty}(X))$ and is supported in (r, r') by $[\varepsilon, \infty) \times [\varepsilon, \infty)$ for some $\varepsilon > 0$, because of the involved factors $1 - \omega$ and $1 - \omega''$. Thus $B_2 \in L^{-\infty;0}(\mathbb{R}^{n+1})$, and it follows the continuity $H^s(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1})$ which implies the mapping property (3.64).
For $B_3 := B_3(\eta)$ we again drop the factor $\chi(\eta)$. We have the continuity

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(l) : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{\infty,\gamma-\mu}(X^\wedge)$$

for every $s \in \mathbb{R}$. We now employ the fact that

$$(1 - \tilde{\omega}(r))l(w, r\eta) \in \mathcal{S}(\mathbb{R}, M_{\mathcal{O}}^{-\infty}(X)) \tag{3.68}$$

for any cut-off function $\tilde{\omega}(r), \eta \neq 0$ fixed. We have $\mathrm{Op}_M^\beta(l) = r^\beta \mathrm{Op}_M(T^{-\beta}l)r^{-\beta}$ for every $\beta \in \mathbb{R}$, but in the case of holomorphic l we have $\mathrm{Op}_M^\beta(l) = \mathrm{Op}_M^\vartheta(l)$ for any $\beta, \vartheta \in \mathbb{R}$, as an operator $C_0^\infty(X^\wedge) \to C^\infty(X^\wedge)$ cf. [55]. In particular,

$$\begin{aligned}(1-\omega)r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(l)(1-\omega'') &= (1-\omega)r^{-\mu}\mathrm{Op}_M(l)(1-\omega'')\\ &= (1-\omega)r^{-\mu}r^L\mathrm{Op}_M(T^{-L}l)r^{-L}(1-\omega'')\\ &= (1-\omega)r^{-\mu}r^L\mathrm{Op}_M^{\gamma-n/2}(T^{-L}l)r^{-L}(1-\omega'').\end{aligned}$$

We use that for given $s, \gamma \in \mathbb{R}$ there is an $L \geq 0$ such that

$$(1 - \omega(r))r^{-L} : H^s_{\mathrm{cone}}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge) \tag{3.69}$$

is continuous, cf. [17, Lemma A.5]. Moreover,

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(T^{-L}l) : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{\infty,\gamma-\mu}(X^\wedge) \tag{3.70}$$

is continuous. Then it remains to note that for every $L \in \mathbb{R}$,

$$(1 - \omega)r^L : \mathcal{H}^{\infty,\gamma-\mu}(X^\wedge) \to H^{s;g}_{\mathrm{cone}}(X^\wedge) \tag{3.71}$$

is continuous for a suitable $g = g(\gamma, L) \in \mathbb{R}$, cf. [17, Lemma A.5], and notation (3.28). Thus

$$(1 - \omega)r^{L+g} : \mathcal{H}^{\infty,\gamma-\mu}(X^\wedge) \to H^{s;0}_{\mathrm{cone}}(X^\wedge) \tag{3.72}$$

is continuous. Now the desired continuity of B_3 follows from (3.69)-(3.72). Note that similar arguments for the characterisation of B_3 are given in [17, Proposition A.11]. □

Definition 3.21. *An element $h(w, \lambda) \in M_{\mathcal{O}}^\mu(X; \mathbb{R}^l_\lambda)$ is called elliptic if for some $\beta \in \mathbb{R}$ the family*

$$h|_{\Gamma_\beta \times \mathbb{R}^l} \in L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^l)$$

is parameter-dependent elliptic (of order μ).

Recall that ellipticity as in this definition is independent of the choice of β, since the parameter-dependent homogeneous principal symbol of $h|_{\Gamma_\beta \times \mathbb{R}^l}$ of order μ is independent of β.

Remark 3.22. *Lemma* 3.20 *admits a parameter-dependent variant. Since the dimension q of η-variables is arbitrary anyway we may start with*

$$h(r, w, \eta, \zeta) = \tilde{h}(w, r\eta, r\zeta)$$

for $\tilde{h}(w, \tilde{\eta}, \tilde{\zeta}) \in M^{\mu}_{\mathcal{O}}(X; \mathbb{R}^{q+d}_{\tilde{\eta},\tilde{\zeta}})$. Now if $\tilde{h}(w, \tilde{\eta}, \tilde{\zeta})$ is parameter-dependent elliptic in the sense of Definition 3.21 *then $\tilde{p}(\tilde{\rho}, \tilde{\eta}, \tilde{\zeta})$ in Remark* 3.19 *is parameter-dependent elliptic in $L^{\mu}_{\mathrm{cl}}(X; \Gamma_{\beta} \times \mathbb{R}^{1+q+d}_{\tilde{\rho},\tilde{\eta},\tilde{\zeta}})$, and hence the conclusion of Remark* 3.19 *yields for every fixed $\eta \neq 0$ that*

$$(1-\omega)\mathrm{Op}_M^{\gamma-n/2}(h)(\eta, \zeta)(1-\omega') \in L^{\mu;\mu}(X^{\asymp})$$

is parameter-dependent elliptic with parameter $\zeta \in \mathbb{R}^d$ in the exit calculus for $r \to \infty$.

Theorem 3.23. *Let $a(y, \eta) \in \mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ be of the form* (3.32)*. Then we have*

$$a(y, \eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge}))$$

for every $s \in \mathbb{R}$.

Proof. As noted before we may drop the term p_{int} when we choose the cut-of function ω, ω' in an appropriate way and change, if necessary, the Green summand. For convenience we consider symbols $a(\eta)$, i.e., without dependence on y; the general case can be treated by a simple modification. Choose an excision function $\chi(\eta)$ and write $a(\eta) = \chi(\eta)a(\eta) + (1 - \chi(\eta))a(\eta)$. We first show that

$$a(\eta) \in C^{\infty}(\mathbb{R}^q, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \tag{3.73}$$

for every $s \in \mathbb{R}$. Let us focus on

$$a(\eta) := r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(\eta)\omega'. \tag{3.74}$$

Concerning the Mellin plus Green part, see [55, Proposition 3.3.20]. The multiplication by ω' transforms $\mathcal{K}^{s,\gamma}(X^{\wedge})$ to $\mathcal{H}^{s,\gamma}(X^{\wedge})$. We refer to the continuity of

$$r^{-\mu}\omega\mathrm{Op}_M^{\gamma-n/2}(h)(\eta) : \mathcal{H}^{s,\gamma}(X^{\wedge}) \to \mathcal{H}^{s-\mu,\gamma-\mu}(X^{\wedge}) \tag{3.75}$$

which is well-known, since $\omega(r)\tilde{h}(r, w, r\eta)$ is smooth up to zero and of bounded support with respect to r. A similar conclusion applies for the derivatives with respect to η. Since (3.75) is automatically a map to $\mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})$, the property (3.73) is proved, and hence it follows that

$$(1-\chi(\eta))a(\eta) \in S^{-\infty}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})).$$

Thus it suffices to show

$$\chi(\eta)a(\eta) \in S^{\mu}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^{\wedge}), \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})).$$

The Mellin action in (3.74) is combined with the multiplication by ω. Therefore, we may assume that

$$\tilde{h}(r, w, \tilde{\eta}) \in C_0^{\infty}([0, R)_0, M^{\mu}_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$$

for some $R > 0$; here

$$C_0^\infty([0,R)_0) := \{\varphi(r) \in C_0^\infty(\overline{\mathbb{R}}_+),\ \varphi(r) \equiv 0 \text{ for } r > R\}.$$

By virtue of $C_0^\infty([0,R)_0, M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde\eta})) = C_0^\infty([0,R)_0)\hat{\otimes}_\pi M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde\eta})$ we have a representation

$$\tilde h(r,w,\tilde\eta) = \sum_{j=0}^\infty \lambda_j \varphi_j(r)\tilde h_j(w,\tilde\eta)$$

convergent in the space $C_0^\infty([0,R)_0)\hat{\otimes}_\pi M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde\eta})$, for

$$\varphi_j(r) \subset C_0^\infty([0,R)_0),\ \tilde h_j(w,\tilde\eta) \in M_{\mathcal{O}}^\mu(X;\mathbb{R}^q_{\tilde\eta}),\ j \in \mathbb{N},$$

and

$$\lim_{j\to\infty} h_j(r) = 0,\ \lim_{j\to\infty} \varphi_j(r) = 0,\ (\lambda_j)_{j\in\mathbb{N}} \subset \mathbb{C},\ \sum_{j=0}^\infty |\lambda_j| < \infty.$$

This gives us a representation

$$\chi(\eta)a(\eta) = \sum_{j=0}^\infty \lambda_j \mathcal{M}_{\varphi_j} b_j(\eta)\mathcal{M}_{\omega'}, \tag{3.76}$$

for

$$b_j(\eta) = \chi(\eta) r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_j)(\eta),\ \ h_j(r,w,\eta) = \tilde h_j(w,r\eta).$$

In view of Remark 3.7 we may ignore $\mathcal{M}_{\omega'}$ in (3.76). In other words it suffices to show the convergence of $\sum_{j=0}^\infty \lambda_j \mathcal{M}_{\varphi_j} b_j(\eta)$ in $S^\mu(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$. First because of Remark 3.7 we have

$$\mathcal{M}_{\varphi_j} \in S^0(\mathbb{R}^q;\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

and

$$\lim_{j\to\infty} \mathcal{M}_{\varphi_j} = 0 \text{ in } S^0(\mathbb{R}^q;\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)).$$

Therefore, (3.76) is convergent as desired, if we show that

$$b_j(\eta) \in S^\mu(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{3.77}$$

and

$$\lim_{j\to\infty} b_j(\eta) = 0 \text{ in } S^\mu(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)). \tag{3.78}$$

(3.77) is a consequence of

$$b_j(\eta) \in C^\infty(\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{3.79}$$

and

$$b_j(\delta\eta) = \delta^\mu \kappa_\delta b_j(\eta)\kappa_\delta^{-1} \tag{3.80}$$

for all $\delta \geq 1$, $|\eta| \geq R$, for a constant $R > 0$, see Remark 3.8 and Lemma 3.20. The convergence of $\tilde h_j(w,\tilde\eta)$ to zero in $M_{\mathcal{O}}^\mu(X;\mathbb{R}^q)$ has the consequence that $\tilde h_j(w,r\eta)$ tends to zero in $C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^\mu(X;\Gamma_{\frac{n+1}{2}-\gamma}\times\mathbb{R}^q))$. Thus

$$b_j(\eta)|_{|\eta|\leq R} \in C^\infty(\{|\eta|\leq R\},\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)))$$

tends to zero in this space. Moreover, $b_j(\eta)|_{|\eta|\geq R}$ tends to zero in the space of functions in

$$C^\infty(\{|\eta| \geq R\}, \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)))$$

that are homogeneous in the sense before for $|\eta| \geq 1$, $|\eta| \geq R$. This entails (3.78). In other words, $\chi(\eta)a(\eta)$ is treated. $\square$

3.5 Mellin characterization of Kegel spaces at infinity

The Kegel spaces (1.95) can be characterised purely in terms of Mellin symbols

$$h(r, w, \eta) = \tilde{h}(w, r\eta)$$

for $\tilde{h}(w, \tilde{\eta}) \in M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})$, cf. Definition 3.4. Here we apply considerations of the preceding section. Those are of independent meaning as an idea to define Kegel spaces over a cone with singular X.
By $S^\mu_{\mathrm{cl},\mathcal{O}}(\Sigma \times \mathbb{R}^n)$ for $\Sigma \subseteq \mathbb{R}^n$ open, we denote the space of all

$$p(x, w, \xi) \in \mathcal{A}(\mathbb{C}_w, S^\mu_{\mathrm{cl}}(\Sigma \times \mathbb{R}^n))$$

such that

$$p|_{\Sigma \times \Gamma_\beta \times \mathbb{R}^n} \in S^\mu_{\mathrm{cl}}(\Sigma \times \Gamma_\beta \times \mathbb{R}^n)$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals.

Theorem 3.24. *For every $\gamma, \mu \in \mathbb{R}$, there is an element*

$$h(r, w, \eta) = \tilde{h}(w, r\eta), \text{ for an } \tilde{h}(w, \tilde{\eta}) \in M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})$$

such that for $\eta \neq 0$, $|\eta|$ sufficiently large

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.81}$$

is an isomorphism for every $s \in \mathbb{R}$.

Proof. We construct h in terms of symbols in local coordinates $x \in \mathbb{R}$ on X with covariables ξ, starting with

$$\tilde{f}_{\mathrm{loc}}(i\rho, \xi, \tilde{\eta}, \tilde{\zeta}, \vartheta) := (C + |\rho|^2 + |\xi|^2 + |\tilde{\eta}|^2 + |\tilde{\zeta}|^2 + |\vartheta|^2)^{\mu/2} \tag{3.82}$$

for an extra parameters $\tilde{\zeta} \in \mathbb{R}^d$, $d > 1$, $\vartheta \in \mathbb{R}$, and a constant $C > 0$, cf. [46, Subsection 3]. Fix a covering $\{U_1, \ldots, U_N\}$ on X, a subordinate partition of unity, $\{\varphi_1, \ldots, \varphi_N\}$, and charts $\chi_j : U_j \to \mathbb{R}^n$. Moreover, we choose functions $\psi_j \in C_0^\infty(U_j)$ that are equal to 1 on $\operatorname{supp} \varphi_j$, $j = 1, \ldots, N$. Then we form an element

$$\tilde{f}(i\rho, \tilde{\eta}, \tilde{\zeta}, \vartheta) := \sum_{j=1}^N \varphi_j\big\{(\chi_j^{-1})_*\mathrm{Op}_x(\tilde{f}_{\mathrm{loc}}(i\rho, \tilde{\eta}, \tilde{\zeta}, \vartheta))\big\}\psi_j$$

belonging to $L^\mu_{\mathrm{cl}}(X; \Gamma_0 \times \mathbb{R}^{q+d+1}_{\tilde{\eta},\tilde{\zeta},\vartheta})$. Applying kernel cut-off to $\tilde{f}$ gives us an element

$$\tilde{h}(w, \tilde{\eta}, \tilde{\zeta}, \vartheta) \in M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^{q+d+1}_{\tilde{\eta},\tilde{\zeta},\vartheta})$$

which is elliptic in the sense of Definition 3.21 with parameters $(\tilde{\eta}, \tilde{\zeta}, \vartheta)$. We then form

$$h(r, w, \eta, \zeta, \vartheta) := \tilde{h}(w, r\eta, r\zeta, \vartheta).$$

The operator

$$H(\eta, \zeta, \vartheta) := r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(\eta, \zeta, \vartheta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.83}$$

belongs to the cone algebra $L^\mu(X^\wedge, \boldsymbol{g})$ on $X^\wedge$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\Theta = (-\infty, 0]$, for every fixed $\eta \neq 0$, $\zeta \in \mathbb{R}^d$, $\vartheta \in \mathbb{R}$. The leading conormal symbol, cf. (3.34),

$$\sigma_M(\mathrm{Op}_M^{\gamma-n/2}(h))(w, \vartheta) = h(0, w, 0, 0, \vartheta) : H^s(X) \to H^{s-\mu}(X), \tag{3.84}$$

(responsible for the ellipticity close to $r = 0$) is parameter-dependent elliptic in $L^\mu_{\mathrm{cl}}(X, \Gamma_\beta \times \mathbb{R}_\vartheta)$ for every $\beta \in \mathbb{R}$. In addition it is an element of $M^\mu_{\mathcal{O}_w}(X; \mathbb{R}_\vartheta)$. It is well-known that for $(|\mathrm{Im}\, w|^2 + |\vartheta|^2)^{1/2} \geq D$ sufficiently large it is elliptic as a parameter-dependent operator in $L^\mu_{\mathrm{cl}}(X, \Gamma_\beta \times \mathbb{R})$, uniformly in compact β-intervals. For every $b \leq b'$ we can choose $D = D(b, b')$ so large that $h(0, w, 0, 0, \vartheta)$ defines a bijective operator family (3.84) for every $b \leq \beta \leq b'$. Since D is increasing together with $|\vartheta|$ it suffices to choose $|\vartheta|$ sufficiently large. Then we obtain the bijectivity of (3.84) for every $w \in \Gamma_{\frac{n+1}{2}-\gamma}$.
By construction, the operator (3.83) is also elliptic in the standard sense on the open manifold $X^\wedge$. In order to obtain a Fredholm operator (3.83) we also observe the exit ellipticity at the conical exit of $X^\wedge$ for $r \to \infty$. Here we refer to the tools of the preceding section. More precisely, we want to observe parameter-dependent exit ellipticity of our operator. In order to obtain an isomorphism when the parameter is sufficiently large we reinterpret our variables η as $(\eta, \zeta) \in \mathbb{R}^{q+d}$ for some $d \geq 1$. This is possible since the dimension of η is arbitrary anyway. The conclusions of Section 3.4 hold in analogous form for (η, ζ) rather than η, but now, if $\eta \neq 0$ we may admit also $\zeta = 0$.
Summing up the operator family (3.83) for fixed $\eta \neq 0$ is elliptic of order μ in the cone algebra on $X^\wedge$, according to the following symbolic components:
(i) the conormal symbol (3.84) which is responsible for a neighborhood of the tip of the cone, here with parameter $\vartheta \in \mathbb{R}$;
(ii) the interior symbol σ_0 over $X^\wedge = \mathbb{R}_+ \times X$ as an open manifold, here with parameter $(\zeta, \vartheta) \in \mathbb{R}^{d+1}$, and the associated reduced symbol $\tilde{\sigma}_0$;
(iii) the exit symbol with ellipticity for $r \to \infty$ of order $(\mu; 0)$, in this case with parameter $\zeta \in \mathbb{R}^d$, cf. the considerations at the end of Section 3.2.
Now, applying the tools of the cone calculus, because of (i) we find a parameter-dependent parametrix $P_0(\vartheta)$ of (3.83) in $\{(r, x) \in X^\wedge : 0 < r < R\}$ for any $r > 0$, where $P_0(\vartheta)$ belongs to the parameter-dependent cone calculus. At the same time, using (ii), namely, that

$$H(\eta, \zeta, \vartheta) \in L^\mu_{\mathrm{cl}}(X^\wedge; \mathbb{R}^{d+1}_{\zeta,\vartheta}) \tag{3.85}$$

is parameter-dependent elliptic, we find a parameter-dependent parametrix $P_1(\zeta, \vartheta)$ of (3.85) in for fixed $\eta \neq 0$. From (iii) we see that there is a parameter-dependent parametrix $P_\infty(\zeta)$ of (3.85) of order $(-\mu; 0)$ in the exit pseudo-differential calculus. Then, choosing cut-off functions $\omega'' \prec \omega \prec \omega'$ on the r half-axis, the operator family

$$P(\zeta, \vartheta) := \omega P_0(\vartheta)\omega' + (1-\omega)P_\infty(\zeta)(1-\omega'')$$

is a parametrix of $H(\eta, \zeta, \vartheta)$ for the chosen $\eta \neq 0$ within the parameter-dependent cone calculus. Note that for any $\varphi, \varphi' \in C_0^\infty(\mathbb{R}_+)$ the operators

$$\varphi P_0(\vartheta)\varphi',\ \varphi P_1(\zeta, \vartheta)\varphi',\ \varphi P_\infty(\zeta)\varphi'$$

coincide modulo $\mathcal{S}(\mathbb{R}^{d+1}, L^{-\infty}(X^\wedge))$. Therefore, the choice of $\omega, \omega', \omega''$ is not essential. What we obtain is that

$$P(\zeta,\vartheta)H(\eta,\zeta,\vartheta) = 1 - G_{\mathrm{L}}(\zeta,\vartheta),\;\; G_{\mathrm{L}}(\zeta,\vartheta) \in L_G(X^\wedge, \boldsymbol{g}_{\mathrm{L}}; \mathbb{R}^{d+1}),$$

$$H(\eta,\zeta,\vartheta)P(\zeta,\vartheta) = 1 - G_{\mathrm{R}}(\zeta,\vartheta),\;\; G_{\mathrm{R}}(\zeta,\vartheta) \in L_G(X^\wedge, \boldsymbol{g}_{\mathrm{R}}; \mathbb{R}^{d+1})$$

for $\boldsymbol{g}_{\mathrm{L}} := (\gamma, \gamma, \Theta)$, $\boldsymbol{g}_{\mathrm{R}} := (\gamma-\mu, \gamma-\mu, \Theta)$. For every $\varepsilon > 0$ there are $(\zeta_\varepsilon, \vartheta_\varepsilon) \in \mathbb{R}^{d+1}$ of sufficiently large absolute value such that

$$\|G_{\mathrm{L}}(\zeta_\varepsilon,\vartheta_\varepsilon)\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge))} < \varepsilon,\;\; \|G_{\mathrm{R}}(\zeta_\varepsilon,\vartheta_\varepsilon)\|_{\mathcal{L}(\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))} < \varepsilon.$$

This gives us the invertibility of (3.83) for given $s \in \mathbb{R}$. This property is then independent of s, and hence we can set $h(r,w,\eta) := h(r,w,\eta,\zeta_\varepsilon,\vartheta_\varepsilon)$. □

Remark 3.25. *Observe that we have* $H(\delta\eta, \delta\zeta, \vartheta) = \delta^\mu \kappa_\delta H(\eta,\zeta,\vartheta)\kappa_\delta^{-1}$ *for all* $\delta \in \mathbb{R}_+$. *Therefore, if*

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(\eta,\zeta_\varepsilon,\vartheta_\varepsilon) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is an isomorphism then also $r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(\delta\eta,\delta\zeta_\varepsilon,\vartheta_\varepsilon)$ *is an isomorphism for every* $\delta \in \mathbb{R}_+$.

As consequence of Theorem 3.24 we have the following result.

Theorem 3.26. *Relation* (3.81) *contains an intrinsic characterization of Kegel spaces for* $r \to \infty$, *namely,*

$$r^s\mathrm{Op}_M^{\gamma-s-n/2}(h)(\eta) : \mathcal{K}^{0,\gamma-s}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge), \tag{3.86}$$

$\eta \neq 0$, $|\eta|$ *sufficiently large, only using the elementary ingredients,*

$$\mathcal{K}^{0,\gamma-s}(X^\wedge) = \mathrm{k}^{\gamma-s}(r)\mathcal{K}^{0,0}(X^\wedge) = \mathrm{k}^{\gamma-s}(r)r^{-n/2}L^2(X^\wedge)$$

for any strictly positive $\mathrm{k}(r) \in C^\infty(\mathbb{R}_+)$ *such that* $\mathrm{k}(r) = r$ for $0 < r < c_0$, $\mathrm{k}(r) = 1$ for $r > c_1$, for $0 < c_0 < c_1$. *In other words the space* $\mathcal{K}^{s,\gamma}(X^\wedge)$ *is characterized purely in terms of a parameter-dependent Mellin operator of order* $-s$ *and weight* $\gamma - s$, *where the parameter is involved as the variable* r *connected with* η *as a factor.*

4 The filtration of the edge algebra

4.1 Edge symbols

Let M be a manifold with edge Y. The definition can be found in the Introduction also in several monographs or papers, cf. [55] or [63]. In particular, Y is a smooth manifold of dimension $q > 0$ such that $M \setminus Y$ is smooth as well, and M is locally near Y described by a Cartesian product

$$X^\Delta \times \Omega,\;\; X^\Delta := (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X), \tag{4.1}$$

for an open set $\Omega \subseteq \mathbb{R}^q$ and a smooth manifold X (closed in our case).
The main ingredient of edge symbols that we want to explain here are operator functions of the form

$$h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta) \tag{4.2}$$

for

$$\tilde{h}(r,y,w,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}})), \tag{4.3}$$

with $\mu \in \mathbb{R}$ being the order. The meaning of $M^\mu_{\mathcal{O}}(\cdot)$ can be seen in the Definition 3.4.
We systematically employ pseudo-differential operators with operator-valued symbols (4.2), or, more generally,

$$f(r,y,w,\eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^q)), \tag{4.4}$$

based on the weighted Mellin transform

$$M_\gamma u(w) = \int_0^\infty r^w u(r) \frac{dr}{r}\Big|_{\Gamma_{\frac{1}{2}-\gamma}}$$

with $\gamma \in \mathbb{R}$ being a given weight. We then write

$$\mathrm{Op}^\gamma_M(f)(y,\eta)u(r) = \iint \left(\frac{r}{r'}\right)^{-(1/2-\gamma+i\rho)} f(r,y,1/2-\gamma+i\rho,\eta)u(r')\frac{dr'}{r'}đ\rho,$$

$đ\rho = (2\pi)^{-1}d\rho$, for functions $u(r') \in C^\infty_0(\mathbb{R}_+, C^\infty(X))$. Later on the action is extended to more general distributions in $\mathbb{R}_+$.
Operator families (4.3) appear in the following well-known Mellin quantization results, cf. [55, Theorem 3.2.7], or [18, Theorem 2.3]. Let

$$p(r,y,\rho,\eta) := \tilde{p}(r,y,r\rho,r\eta), \quad \tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})). \tag{4.5}$$

Then there is an $h(r,y,w,\eta)$ as (4.2), (4.3) such that on functions in $C^\infty_0(X^\wedge)$

$$\mathrm{Op}_r(p)(y,\eta) = \mathrm{Op}^\gamma_M(h)(y,\eta) \mod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q_\eta)) \tag{4.6}$$

for every $\gamma \in \mathbb{R}$. Conversely, for any h we find a p with the indicated properties such that (4.6) holds, and the resulting operator functions $\tilde{p}$ and $\tilde{h}$ are unique modulo $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{-\infty}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$ and $C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{-\infty}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta}}))$, respectively.

Remark 4.1. *For purposes below we formulate a simple consequence of the latter Mellin quantization theorem. For*

$$p(r,y,\rho,\eta) := \tilde{p}(r,y,r\rho,r\eta), \quad \tilde{p}(r,y,\tilde{\rho},\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^{\mu-j}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$$

for any fixed $j \in \mathbb{N}$, we find an

$$h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta) \quad \text{for} \quad \tilde{h}(r,y,w,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^{\mu-j}_{\mathcal{O}}(X;\mathbb{R}_{\tilde{\eta}}))$$

such that

$$\mathrm{Op}_r(r^j p)(y,\eta) = \mathrm{Op}^\gamma_M(r^j h)(y,\eta) \mod C^\infty(\Omega, L^{-\infty}(X^\wedge;\mathbb{R}^q_\eta))$$

Conversely, for h we find a p with the indicated properties.

Recall that there is well-known kernel cut-off operator based on the Mellin transform

$$\mathbb{V}_M(\psi) : C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R}^q)) \to C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M^\mu_{\mathcal{O}_w}(X;\mathbb{R}^q))$$

for a cut-off function $\psi \in C_0^\infty(\mathbb{R}_+)$, $\psi \equiv 1$ in a neighborhood of $r = 1$, such that

$$\mathbb{V}_M(\psi) f|_{\Gamma_\beta} = f \mod C^\infty(\overline{\mathbb{R}}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^q)).$$

This shows that the space of symbols (4.3) is “nearly” as rich as (4.4). The choice of $\gamma \in \mathbb{R}$ is arbitrary. For normalizing weights we often replace γ by $\gamma - n/2$ for $n = \dim X$. In the edge algebra we interpret

$$r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)$$

for h as in (4.3) as an operator-valued symbol, i.e., an element of

$$S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}) \tag{4.7}$$

for suitable Hilbert spaces of weighted distributions on $X^\wedge := \mathbb{R}_+ \times X$, cf. the space $S^\mu(\Omega \times \mathbb{R}^q; H, \tilde{H}), S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \tilde{H}), S^{(\mu)}(\Omega \times \mathbb{R}^q \setminus \{0\}; H, \tilde{H})$ corresponding with Fréchet spaces, cf. the Definition 1.10.
In concrete cases we set

$$H = \mathcal{K}^{s,\gamma}(X^\wedge),\ \tilde{H} = \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge). \tag{4.8}$$

In addition we employ the Kegel spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ cf. (1.95) before.
With the spaces $\mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ of Definition 3.9 we recall the following results.

Remark 4.2. *Changing the cut-off functions ω, ω' in* (3.32) *leaves remainders of the form $\varphi\mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\varphi'$ for $\varphi, \varphi' \in C_0^\infty(\mathbb{R}_+)$, and $p_{\mathrm{int}}(r,y,\rho,\eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\rho,\eta}))$. Such terms could be added in the definition of $a(y,\eta)$; however, by a suitable choice of ω, ω' they can be integrated in the Mellin action. So without loss of generality we employ $a(y,\eta)$ in the form* (3.32)*.*

4.2 The edge algebra

We now turn to edge operators of order $\mu - j$, $j \in \mathbb{N}$, associated with weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$. The properties of the corresponding filtration are the main issue of this section. By definition our manifold M with edge Y contains a neighborhood $W \supset Y$ with the structure of a locally trivial X^Δ bundle over Y. That means we have a system of singular charts

$$\chi : V \to X^\Delta \times \Omega \tag{4.9}$$

of neighborhoods $V \subset M$ of points y on the edge, where

$$\chi|_{V \setminus Y} : V \setminus Y \to X^\wedge \times \Omega. \tag{4.10}$$

If $\tilde{V}$ is another neighborhood of points $\tilde{y}$ and

$$\chi : \tilde{V} \to X^\Delta \times \tilde{\Omega}$$

the corresponding singular charts, then for $V \cap \tilde{V}$ we have restrictions

$$\chi|_{V\cap\tilde{V}} : V \cap \tilde{V} \to X^\Delta \times D, \quad \tilde{\chi}|_{V\cap\tilde{V}} : V \cap \tilde{V} \to X^\Delta \times \tilde{D}$$

for open subsets $D \subseteq \Omega$, $\tilde{D} \subseteq \tilde{\Omega}$, such that the transition maps

$$X^\Delta \times D \to X^\Delta \times \tilde{D}$$

are bundle isomorphisms between the corresponding (trivial) X^Δ-bundles over D and $\tilde{D}$, respectively. Those are fiberwise, i.e., over every $y \in \Sigma$, quotient maps

$$\overline{\mathbb{R}}_+ \times X \to (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$$

for diffeomorphisms

$$C : \overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X, \tag{4.11}$$

where $\overline{\mathbb{R}}_+ \times X$ is regarded as a manifold with smooth boundary. The following local constructions refer to a fixed chart $G \to \mathbb{R}^q$ on Y, $q = \dim Y > 0$.
For a Hilbert space H with group action κ we have the abstract edge space

$$\mathcal{W}^s(\mathbb{R}^q, H), \;\; s \in \mathbb{R}, \tag{4.12}$$

cf. Definition 1.17 and employ these constructions to so-called (local) weighted edge spaces cf. (2.30)

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \tag{4.13}$$

of smoothness $s \in \mathbb{R}$ and subspaces cf. (2.47)

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)) \tag{4.14}$$

with (here constant discrete) asymptotics of type P, cf. notation in Section 2.2. Weighted edge spaces (4.13) and subspaces (4.14) with asymptotics have been studied in [54], see also [55]. The abstract version (4.12) has been introduced in [63]. More functional analytic properties have been studied in [23].
We now recall global spaces

$$H^{s,\gamma}(M) \quad \text{and} \quad H_P^{s,\gamma}(M) \tag{4.15}$$

cf (2.56) and (2.57) of Subsection 2.2. The (compact) manifold B is replaced by M with edge Y. The first space of (4.15) is regarded as the set of all $u \in H^s_{\text{loc}}(M \setminus Y)$ such that for any singular chart $\chi : V \to X^\Delta \times \mathbb{R}^q$ and the induced $\chi|_{V\setminus Y} : V \setminus Y \to X^\wedge \times \mathbb{R}^q$ we have

$$u|_{V\setminus Y} = f \circ \chi^{-1}|_{V\setminus Y}$$

for some $f \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$. Similarly, the second space of (4.15) is the set of all $u \in H^s_{\text{loc}}(M \setminus Y)$ such that

$$u|_{V\setminus Y} = f \circ \chi^{-1}|_{V\setminus Y}$$

for some $f \in \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge))$. A similar definition applies when M is a non-compact manifold with edge. In that case instead of (4.15) we write

$$H^{s,\gamma}_{\text{loc}}(M) \quad \text{and} \quad H^{s,\gamma}_{P,\text{loc}}(M), \tag{4.16}$$

respectively, and we have also the corresponding spaces with subscript "comp".
On a compact manifold M with edge Y we choose a system of singular charts $\chi_j : V_j \to X^\Delta \times \mathbb{R}^q$, $j = 1, \ldots, N$, of the kind (4.9), where $G_j := V_j \cap Y$ form an open covering $\{G_1, \ldots, G_N\}$ of Y. Let $\{\varphi_1, \ldots, \varphi_N\}$ be a subordinate partition of unity, and let $\{\varphi'_1, \ldots, \varphi'_N\}$ be a system of functions in $C_0^\infty(G_j), \varphi_j \prec \varphi'_j$ for all j. Moreover, fix cut-off functions $\omega, \omega', \omega''$ on M, i.e., continuous functions on M that are smooth on $M \setminus Y$ and $\equiv 1$ close to Y, and supported by a small neighborhood of Y, where $\omega'' \prec \omega \prec \omega'$.

Definition 4.3. *Let M be a compact manifold with edge Y.*
(i) *An operator $C : C_0^\infty(M \setminus Y) \to C^\infty(M \setminus Y)$ is smoothing in the edge algebra, i.e., $C \in L^{-\infty}(M, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, if (say for compact M) the operators C, C^* extend to continuous maps*

$$C : H^{s,\gamma}(M) \to H_P^{\infty,\gamma-\mu}(M), \;\; C^* : H^{s,-\gamma+\mu}(M) \to H_Q^{\infty,-\gamma}(M) \tag{4.17}$$

for every $s \in \mathbb{R}$, for asymptotic types P, Q, depending on C.
(ii) *The space $L^\mu(M, \boldsymbol{g})$ of edge operators on M for $\mu \in \mathbb{R}$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, is the set of all $A \in L^\mu_{\mathrm{cl}}(M \setminus Y)$ of the form*

$$A = \Big\{ \sum_{j=1}^{N} A_j + (1-\omega) A_{\mathrm{int}} (1 - \omega'') + C : A_{\mathrm{int}} \in L^\mu_{\mathrm{cl}}(M \setminus Y),\, C \in L^{-\infty}(M, \boldsymbol{g}), \\ A_j = \omega \varphi_j (\mathrm{int}\, \chi_j^{-1})_* \mathrm{Op}_y(a_j) \varphi_j' \omega',\, a_j(y,\eta) \in \mathcal{R}^\mu(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g}) \Big\}, \tag{4.18}$$

where $\mathrm{int}\, \chi_j := \chi_j|_{V_j \setminus Y}$.
(iii) *By $L^\mu_{M+G}(M, \boldsymbol{g})$ ($L^\mu_G(M, \boldsymbol{g})$) we denote the set of all $A \in L^\mu(M, \boldsymbol{g})$ such that $A_{\mathrm{int}} = 0$ and $a_j \in \mathcal{R}^\mu_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ ($\mathcal{R}^\mu_G(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$) for all j.*

It is well-known, cf. [55], that $A \in L^\mu(M, \boldsymbol{g})$ induces continuous operators

$$A : H^{s,\gamma}(M) \to H^{s-\mu,\gamma-\mu}(M), \;\; H_P^{s,\gamma}(M) \to H_Q^{s-\mu,\gamma-\mu}(M) \tag{4.19}$$

for all $s \in \mathbb{R}$ and asymptotic types P certain resulting Q, depending on P and A. Continuity results (4.19) are based on local continuity of $\mathrm{Op}(a)$ for $a \in \mathcal{R}^\mu_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ in weighted edge Sobolev spaces (4.13) or (4.14). The latter follows from Theorem **??** also using that $\mathcal{R}^\mu_{M+G}(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ is contained in $S^\mu(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}_P^{s,\gamma}(X^\wedge), \mathcal{K}_Q^{s-\mu,\gamma-\mu}(X^\wedge))$ for asymptotic types P with some resulting Q. Continuity results between abstract edge spaces are proved in [54] and [72], an operator-valued analogue of Hwang's proof [25] of the Calderón-Vaillancourt Theorem.
The assumption of compact M in Definition 4.3 has been made for convenience. A s-mall modification allows us also to admit the paracompact case, using a corresponding locally finite system of charts. Instead of the space in (4.19) we then have to take com/loc-analogues, cf. (4.16). Recall that

$$L^\mu_G(M, \boldsymbol{g}) \subset L^\mu_{M+G}(M, \boldsymbol{g}) \subset L^{-\infty}(M \setminus Y).$$

An operator $A \in L^\mu(M, \boldsymbol{g}) \subset L^\mu_{\mathrm{cl}}(M \setminus Y)$ has its standard homogeneous principal symbol of order μ

$$\sigma_0(A) \in C^\infty(T^*(M \setminus Y) \setminus 0). \tag{4.20}$$

Moreover, since A is edge-degenerate close to Y, in the splitting of variables $(r, x, y) \in \mathbb{R}_+ \times \Sigma \times \mathbb{R}^q$ and covariables (ρ, ξ, η) the function (4.20) takes the form

$$\sigma_0(A)(r, x, y, \rho, \xi, \eta) = r^{-\mu} \tilde{\sigma}_0(A)(r, x, y, r\rho, \xi, r\eta) \tag{4.21}$$

for a function $\tilde{\sigma}_0(A)(r, x, y, \tilde{\rho}, \xi, \tilde{\eta})$ that is homogeneous in $(\tilde{\rho}, \xi, \tilde{\eta}) \neq 0$ of order μ and smooth up to $r = 0$.
Observe that $\sigma_0(A)$ can be locally close to Y expressed in terms of the operator-valued

symbol $a(y,\eta) \in \mathcal{R}^\mu(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$, cf. Definition 4.3 (iii). From (4.6) and the subsequent observation in converse direction we see that the Mellin symbol h in (3.32), (3.33) belongs to $C^\infty(\mathbb{R}_+ \times \Omega, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^q_\eta))$. As such it has a parameter-dependent homogeneous principal symbol $p_{(\mu)}(r,x,y,\rho,\xi,\eta)$, $(\rho,\xi,\eta) \neq 0$. Thus $a(y,\eta)$ itself given by (3.32), has a parameter-dependent homogeneous principal symbol, called $\sigma_0(a)$, which is close to $r=0$ of the form

$$\sigma_0(a)(r,x,y,\rho,\xi,\eta) := r^{-\mu} p_{(\mu)}(r,x,y,\rho,\xi,\eta) = r^{-\mu} \tilde{p}_{(\mu)}(r,x,y,r\rho,\xi,r\eta) \tag{4.22}$$

for a $\tilde{p}_{(\mu)}(r,x,y,\tilde{\rho},\xi,\tilde{\eta}) \in S^{(\mu)}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times (\mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}} \setminus \{0\}))$, where $\Sigma \subset \mathbb{R}^n$ corresponds to a chart on X. Later on, when we talk about lower order symbols we also write

$$\sigma_0^\mu(a) := \sigma_0(a) \quad \text{and} \quad \sigma_0^\mu(A) := \sigma_0(A),$$

respectively. Moreover, the edge amplitude functions $a(y,\eta) \in \mathcal{R}^\mu(\mathbb{R}^q \times \mathbb{R}^q, \boldsymbol{g})$ involved in Definition 4.3 have a (twisted) homogeneous principal symbol, namely,

$$\sigma_1(a)(y,\eta) := r^{-\mu} \mathrm{Op}_M^{\gamma-n/2}(h_0)(y,\eta) + \sigma_1(m+g)(y,\eta), \tag{4.23}$$

$\eta \neq 0$, where

$$h_0(r,y,w,\eta) := \tilde{h}(0,y,w,r\eta), \tag{4.24}$$

cf. (3.33), and $\sigma_1(m+g)(y,\eta)$ is the (twisted) homogeneous principal symbol of $(m+g)(y,\eta)$ as a classical operator-valued symbol, i.e.,

$$(m+g)(y,\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)).$$

Using $\sigma_1(\cdot)(y,\eta)$ on edge amplitude functions we obtain $\sigma_1(A)(y,\eta)$ also for the operators $A \in L^\mu(M,\boldsymbol{g})$ themselves. For the definition we may refer to localized and properly supported representatives of operators, e.g., A_j as in Definition 4.3 (i) and to recover left symbols, similarly as for standard (scalar) pseudo-differential operators. For the resulting $a(y,\eta) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^q \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ we then have

$$\sigma_1(a)(y,\eta) = \lim_{\delta\to\infty} \delta^{-\mu} \kappa_\delta^{-1} a(y,\delta\eta)\kappa_\delta.$$

Summing up the local symbols which contain contributions of a partition of unity on Y we obtain the invariantly defined principal edge symbol $\sigma_1(A)(y,\eta)$, namely,

$$\sigma_1(A)(y,\eta) := \sum_{j=1}^N \varphi_j(y)\sigma_1(a_j)(y,\eta), \tag{4.25}$$

see (4.18).

4.3 The filtration of edge operator spaces

Order filtrations in the edge calculus are well-known and useful for dealing with lower order terms as soon as principal symbols vanish. We realize here the filtration by using an alternative representation of the edge calculus, cf. [17], based on amplitude functions as in Definition 3.9 (ii). At the same time we deepen the insight of the exit symbolic properties of operated-valued Mellin symbols on the infinite cone for $r \to \infty$. We also look at smoothing Mellin plus Green symbols; however, those are standard. In fact, when we define

$$\mathcal{R}^{\mu-j}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}) \text{ for } \boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0]), \tag{4.26}$$

for $j \in \mathbb{N} \setminus \{0\}$, we simply ask the homogeneous components of $(m+g)(y,\eta)$ of order l to be vanishing for all $0 \le l \le j-1$. More precisely, we have

$$\mathcal{R}^{\mu}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}) \subset S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)),$$

$s \in \mathbb{R}$, and any $(m+g)(y,\eta)$ has a sequence of homogeneous components

$$\sigma_1^{\mu-j}(m+g)(y,\eta) := (m+g)_{(\mu-j)}(y,\eta) \tag{4.27}$$

and

$$(m+g)_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(\Omega \times (\mathbb{R}^q \setminus \{0\}); \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\gamma-\mu}(X^\wedge)),\ j \in \mathbb{N},$$

cf. the generalities on classical operator-valued symbols in Section 4.1, where by notation $\sigma_1^{\mu}(m+g) := \sigma_1(m+g)$. Then the operator family $(m+g)(y,\eta)$ belongs to (4.26) if $\sigma_1^{\mu-l}(m+g)$ vanishes for all $0 \le l \le j-1$. As is well-known, we have

$$\mathcal{R}^{\mu-j}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}) = \mathcal{R}^{\mu-j}_{G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$$

when $j > k$ where k is involved in the weight interval contained in $\boldsymbol{g}$. The weight data $\boldsymbol{g}$ are independent of j. For general $a(y,\eta) \in \mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ the dominating term is the non-smoothing summand

$$r^{-\mu}\omega \mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta)\omega'. \tag{4.28}$$

For a simple model situation in [37] we illustrated the (unexpected) problem of understanding the homogeneous principal symbol of order μ of (4.28) for $r \to \infty$ in the frame of the exit pseudo-differential calculus on $X^\wedge$, for $\eta \neq 0$.
Let $\mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g}),\ j \in \mathbb{N}$, for $\boldsymbol{g}$ as in (4.26), be the space of all operator families of the form

$$r^{-\mu}\omega \mathrm{Op}_M^{\gamma-n/2}(r^j h)(y,\eta)\omega' + (m+g)(y,\eta)$$

for

$$h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta),\ \tilde{h}(r,y,w,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{\mu-j}(X; \mathbb{R}^q_{\tilde{\eta}})), \tag{4.29}$$

and $(m+g)(y,\eta) \in \mathcal{R}^{\mu-j}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$. Similarly as in Remark 4.2 we could add a term $\varphi \mathrm{Op}_r(p_{\mathrm{int}})(y,\eta)\varphi'$ for $p_{\mathrm{int}}(r,y,\rho,\eta) \in C^\infty(\mathbb{R}_+ \times \Omega, L^{\mu-j}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\rho,\eta}))$ which is contributed under changing ω, ω'.

Theorem 4.4. *Let $a(y,\eta) \in \mathcal{R}^{\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0])$, and assume $\sigma_i(a) = 0,\ i = 0, 1$. Then $a(y,\eta)$ is of the form*

$$a(y,\eta) = r^{-\mu+1}\omega \mathrm{Op}_M^{\gamma-n/2}(h_1)(y,\eta)\omega' + (m_1+g_1)(y,\eta)$$

for

$$h_1(r,y,w,\eta) = \tilde{h}_1(r,y,w,r\eta),\ \tilde{h}_1(r,y,w,\tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_{\mathcal{O}}^{\mu-1}(X; \mathbb{R}^q_{\tilde{\eta}}))$$

and $(m_1+g_1)(y,\eta) \in \mathcal{R}^{\mu-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$.

Proof. For convenience we consider the case of y-independent symbols. The modifications for the general case are evident. Let us first note that $a(\eta)$ is a family of operators taking values in the space $L^\mu(X^\triangle, \boldsymbol{g})$ for every fixed $\eta \in \mathbb{R}^q$, cf. Definition 5.20 (iii) below. Moreover, we have

$$\sigma_1(a)(\eta) \in L^\mu(X^\triangle, \boldsymbol{g})_{\text{exit}} \tag{4.30}$$

for every fixed $\eta \in \mathbb{R}^q \setminus \{0\}$, cf. Definition 5.20 (iv) below. In fact, $(m+g)(\eta)$ in (3.32) for fixed $\eta \in \mathbb{R}^q$ belongs to $L_{M+G}(X^\triangle, \boldsymbol{g})$ which is obvious. It remains to observe

$$r^{-\mu}\omega \mathrm{Op}_M^{\gamma-n/2}(h)(\eta)\omega' \in L^\mu(X^\triangle, \boldsymbol{g}), \tag{4.31}$$

cf. the expression (5.33). In fact, we have

$$h(r, w, \eta) = \tilde{h}(r, w, r\eta) \tag{4.32}$$

for $\tilde{h}(r, w, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}}))$. Thus, for fixed η the function (4.32) belongs to $C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_w}(X))$ and hence (4.31) just corresponds to the first summand on the right of (5.33). Setting $p = 0$, we see that (4.31) holds. In order to show (4.30) we first note that $\sigma_1(m+g)(\eta) \in L_{M+G}(X^\triangle, \boldsymbol{g})$. Moreover, the technique of the proof of [37, Lemma 53] shows that for any fixed $\eta \neq 0$ the operator $r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta)$ is of the form (5.35). From [11, Subsection 3.5], see also [37, Theorem 56], we have continuous operators

$$a(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \ \ s \in \mathbb{R}.$$

A consequence of $\sigma_0(a) = 0$ is that

$$\tilde{h}(r, w, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M^{\mu-1}_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})), \tag{4.33}$$

which implies

$$\tilde{h}(0, w, \tilde{\eta}) \in M^{\mu-1}_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}),$$

cf. [37, Remark 39]. We have (4.23) for $h_0(r, w, \eta) = \tilde{h}(0, w, r\tilde{\eta})$, and we write

$$\begin{aligned} a(\eta) = \omega r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h-h_0)(\eta)\omega' + \omega r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta)\omega' \\ + m_0(\eta) + g_0(\eta) + m^{\mu-1}(\eta) + g^{\mu-1}(\eta) \end{aligned} \tag{4.34}$$

for

$$m_0(\eta) := r^{-\mu}\omega_\eta \sum_{j=0}^{k} r^j \sum_{|\alpha|=j} \mathrm{Op}_M^{\gamma_{j\alpha}-n/2}(f_{j\alpha})\eta^\alpha \omega'_\eta, \quad g_0(\eta) := \omega\chi(\eta)\sigma_1(g)(\eta)\omega',$$

cf. Definition 3.9 (ii), and

$$m^{\mu-1}(\eta) := (m - m_0)(\eta), \quad g^{\mu-1}(\eta) := (g - g_0)(\eta) \in \mathcal{R}^{\mu-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}) \tag{4.35}$$

cf. notation (4.26) for $j = 1$. Taylor's formula in the first r-variable in $\tilde{h}(r, w, r\eta) = h(r, w, \eta)$ yields

$$\omega \mathrm{Op}_M^{\gamma-n/2}(h-h_0)(\eta)\omega' = \omega r \mathrm{Op}_M^{\gamma-n/2}(h^{-1})(\eta)\omega' \tag{4.36}$$

for some

$$h^{-1}(r, w, \eta) = \tilde{h}^{-1}(r, w, r\eta), \quad \tilde{h}^{-1}(r, w, \tilde{\eta}) \in C^\infty(\overline{\mathbb{R}}_+, M^{\mu-1}_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})). \tag{4.37}$$

Thus (4.36) belongs to $\mathcal{R}^{\mu-1}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$. It remains to verify that

$$\omega \mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta)\omega' + m_0(\eta) + g_0(\eta) \in \mathcal{R}^{\mu-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g}).$$

In fact, because of (4.35), (4.36) we have

$$\begin{aligned} \sigma_1(a)(\eta) &= \sigma_1\Big(\omega \mathrm{Op}_M(h_0)\omega' + m_0 + g_0\Big)(\eta) \\ &= \mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta) + \sigma_1(m_0)(\eta) + \sigma_1(g_0)(\eta) = 0. \end{aligned} \tag{4.38}$$

By virtue of $\sigma_1(a)(\eta) \in L^\mu(X^\Delta, \boldsymbol{g})_{\mathrm{exit}}$ for every fixed $\eta \neq 0$, cf. Definition 5.20 (iv), we can recover the sequence of conormal symbols in a unique way, cf. [54, Subsection 1.3.1, Theorem 4]. Relation (4.38) shows that all conormal symbols vanish. This is the case, in particular, for the leading component, and it follows that

$$\sigma_M^\mu\Big(\mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta) + \sigma_1(m_0)(\eta)\Big)(w) = \tilde{h}(0, w, 0) + f_{00}(w) = 0,$$

cf. notation in (4.24), (3.31), and formula (5.36) for $j = 0$. This shows

$$\sigma_1\Big(\omega \mathrm{Op}_M^{\gamma-n/2}(h_0)\omega' + m_0\Big)(\eta) = 0$$

and hence

$$\omega\Big(\mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta) - \mathrm{Op}_M^{\gamma-n/2}(\tilde{h}(0, w, 0))\Big)\omega' \in \mathcal{R}^{\mu-1}(\Omega \times \mathbb{R}^q, \boldsymbol{g}).$$

Thus $\omega \mathrm{Op}_M^{\gamma-n/2}(h_0)(\eta)\omega' + m_0(\eta) \in \mathcal{R}^{\mu-1}_{M+G}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$, and hence $\sigma_1(g_0)(\eta) = 0$ which entails $g_0(\eta) \in \mathcal{R}^{\mu-1}_G(\Omega \times \mathbb{R}^q, \boldsymbol{g})$. □

Every $a(y, \eta) \in \mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, (-(k+1), 0]), j \in \mathbb{N}, j \geq 1$, has again a pair of principal symbols, now of order $\mu - j$, namely,

$$(\sigma_0^{\mu-j}(a), \sigma_1^{\mu-j}(a)) =: \sigma^{\mu-j}(a).$$

For $a(y, \eta) \in \mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for any fixed $j \in \mathbb{N} \setminus \{0\}$, we define $\sigma_0^{\mu-j}(a)$ in a similar manner as in Section 4.2 for $j = 0$. In this case we employ Remark 4.1 which gives us the parameter-dependent homogeneous principal symbol $\tilde{p}_{(\mu-j)}(r, x, y, \tilde{\rho}, \xi, \tilde{\eta}) \in S^{(\mu-j)}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times (\mathbb{R}^{1+n+q}_{\tilde{\rho},\xi,\tilde{\eta}} \setminus \{0\}))$ with $\tilde{p}$ being related with $\tilde{h}$ in (4.29) via Mellin quantization. Then, similarly as (4.22) we set

$$\sigma_0^{\mu-j}(a)(r, x, y, \rho, \xi, \eta) := r^{-\mu+j}\tilde{p}_{(\mu-j)}(r, x, y, r\rho, \xi, r\eta). \tag{4.39}$$

Moreover, $a(y, \eta) \in \mathcal{R}^{\mu-j}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ has a principal edge symbol

$$\sigma_1^{\mu-j}(a)(y, \eta) := r^{-\mu+j}\mathrm{Op}_M^{\gamma-n/2}(h_0)(y, \eta) + \sigma_1^{\mu-j}(m + g)(y, \eta), \tag{4.40}$$

for $h_0(r, y, w, \eta) := \tilde{h}(0, y, w, r\eta)$ and $\sigma_1^{\mu-j}(m + g)(y, \eta)$ given by (4.27).

Theorem 4.5. *For $j \in \mathbb{N}\setminus\{0\}$ the space $\mathcal{R}^{\mu-(j+1)}(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0])$ is characterized as the set of all $a(y, \eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g})$ such that*

$$\sigma_0^{\mu-l}(a) = 0, \quad \sigma_1^{\mu-l}(a) = 0$$

for all $l = 0, \ldots, j$.

Proof. The result is iterative, and we can apply analogous arguments as for Theorem 4.4. □

Corollary 4.6. *Let M be a manifold with edge Y. Then there is a filtration of $L^\mu(M,\boldsymbol{g})$ for $\mu\in\mathbb{R}$, $\boldsymbol{g}=(\gamma,\gamma-\mu,(-(k+1),0])$, consisting of a sequence of subspaces $L^{\mu-l}(M,\boldsymbol{g})$, $l\in\mathbb{N}$, namely,*

$$L^\mu(M,\boldsymbol{g})\supset L^{\mu-1}(M,\boldsymbol{g})\supset\cdots\supset L^{\mu-l}(M,\boldsymbol{g})\supset\cdots\supset L^{-\infty}(M,\boldsymbol{g}),\tag{4.41}$$

where $L^{\mu-l}(M,\boldsymbol{g})\subset L^{\mu-l}_{\mathrm{cl}}(M\setminus Y)$ consists of operators A that are represented in an analogous manner as (4.18), here for $A_{\mathrm{int}}\in L^{\mu-l}_{\mathrm{cl}}(M\setminus Y)$, and $a_j(y,\eta)\in\mathcal{R}^{\mu-l}(\mathbb{R}^q\times\mathbb{R}^q,\boldsymbol{g})$ for all j. Similar filtrations hold for $L^\mu_{M+G}(M,\boldsymbol{g})$ and $L^\mu_G(M,\boldsymbol{g})$, respectively, where $L^{\mu-l}_{M+G}(M,\boldsymbol{g})$ ($L^{\mu-l}_G(M,\boldsymbol{g})$) consists of all $A\in L^{\mu-l}(M,\boldsymbol{g})$ such that $A_{\mathrm{int}}=0$ and $a_j\in\mathcal{R}^{\mu-l}_{M+G}(\mathbb{R}^q\times\mathbb{R}^q,\boldsymbol{g})$ ($\mathcal{R}^{\mu-l}_G(\mathbb{R}^q\times\mathbb{R}^q,\boldsymbol{g})$) for all j.

Note that

$$L^{\mu-l}_{M+G}(M,\boldsymbol{g})=L^{\mu-l}_G(M,\boldsymbol{g})\tag{4.42}$$

for $l>k$, where $k\in\mathbb{N}$ determines the finite weight interval in $\boldsymbol{g}$.

Proposition 4.7. *We have*

$$L^{\mu-l}(M,\boldsymbol{g})\bigcap L^{-\infty}(M\setminus Y)=L^{\mu-l}_{M+G}(M,\boldsymbol{g}).$$

Proof. Operators $A\in L^{\mu-l}(M,\boldsymbol{g})$ are locally close to the edge in the splitting of variables $(r,x,y)\in\mathbb{R}_+\times\Sigma\times\Omega$ for open $\Sigma\subseteq\mathbb{R}^n$ and $\Omega\subseteq\mathbb{R}^q$, corresponding to charts on X and Y, respectively, of the form

$$r^{-\mu+l}\mathrm{Op}_M^{\gamma-n/2}\mathrm{Op}_x(h)$$

for (local) Mellin symbols

$$h(r,x,y,w,\xi,\eta)=\tilde h(r,x,y,w,\xi,r\eta)$$

for

$$\tilde h(r,x,y,w,\xi,\tilde\eta)\in S^{\mu-l}_{\mathcal{O}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta}),$$

where $S^{\mu-l}_{\mathcal{O}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$ is the space of all

$$\tilde h(r,x,y,w,\xi,\tilde\eta)\in\mathcal{A}(\mathbb{C}_w,S^{\mu-l}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta}))$$

such that

$$\tilde h(r,x,y,\beta+i\rho,\xi,\tilde\eta)\in S^{\mu-l}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}_\rho\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$$

for all $\beta\in\mathbb{R}$, uniformly in compact β-intervals. If the respective operator belongs to $L^{-\infty}(M\setminus Y)$ then the operator A_{int} is smoothing (A_{int} can be identified with A regarded as an operator $C_0^\infty(M\setminus Y)\to C^\infty(M\setminus Y)$), and $h(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\eta)$ belongs to $S^{-\infty}(\mathbb{R}_+\times\Sigma\times\Omega\times\Gamma_{\frac{n+1}{2}-\gamma}\times\mathbb{R}^n_\xi\times\mathbb{R}^q_\eta)$. That means, the homogeneous components $h_{(\mu-l-j)}(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\eta)$ vanish for $r>0$ and all j. The symbol $\tilde h(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\tilde\eta)$ (no matter what the order is) can be reproduced as an asymptotic sum

$$\sum_{j=0}^\infty\chi(\rho,\xi,\tilde\eta)\tilde h_{(\mu-l-j)}(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\tilde\eta)$$

up to an element in $S^{-\infty}(\mathbb{R}_+\times\Sigma\times\Omega\times\Gamma_{\frac{n+1}{2}-\gamma}\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$. This gives us an $\tilde f(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\tilde\eta)\in S^{\mu-l}_{\mathrm{cl}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}_\rho\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$ which is smooth up to $r=0$. Applying a kernel cut-off operator to $\tilde f$ with respect to $w\in\Gamma_{\frac{n+1}{2}-\gamma}$ we recover $\tilde h\in S^{\mu-l}_{\mathcal{O}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$ modulo an element of

$$S^{-\infty}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}_\rho\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta}).$$

In the present case we have

$$\tilde h_{(\mu-l-j)}(r,x,y,\frac{n+1}{2}-\gamma+i\rho,\xi,\tilde\eta)=0$$

for all $j\in\mathbb{N}$. Thus $\tilde h$ itself belongs to $S^{-\infty}_{\mathcal{O}}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^n_\xi\times\mathbb{R}^q_{\tilde\eta})$. Now it suffices to note that

$$\omega r^{-\mu+j}\mathrm{Op}_y\mathrm{Op}_M^{\gamma-n/2}\mathrm{Op}_x(h)\omega'$$

is an element of $L^{\mu-l}_{M+G}(M,\boldsymbol{g})$, more precisely, the operator coming from the local Mellin symbols h for an open covering of X by coordinate neighborhoods and a sum, using a subordinate partition of unity. $\square$

Remark 4.8. *As a byproduct of the proof we obtain* $h(r,y,w,\eta)=\tilde h(r,y,w,r\eta)$ *for an* $\tilde h(r,y,w,\tilde\eta)\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,M^{\mu-l}_{\mathcal{O}}(X,\mathbb{R}^q_{\tilde\eta}))$ *such that*

$$\tilde h(r,y,w,\tilde\eta)|_{\mathbb{R}_+\times\Omega}\in C^\infty(\overline{\mathbb{R}}_+\times\Omega,M^{-\infty}_{\mathcal{O}}(X,\mathbb{R}^q_{\tilde\eta}))$$

automatically belongs to $C^\infty(\overline{\mathbb{R}}_+\times\Omega,M^{-\infty}_{\mathcal{O}}(X,\mathbb{R}^q_{\tilde\eta}))$.

In order to see that edge operator spaces of lower order coincide with the more common definition in terms of vanishing homogeneous principal terms we formulate the pair

$$\sigma^{\mu-l}(A):=(\sigma_0^{\mu-l}(A),\sigma_1^{\mu-l}(A))$$

of principal symbols of operators $A\in L^{\mu-l}(M,\boldsymbol{g})$. We define $\sigma_0^{\mu-l}(A)$ as the standard homogeneous principal symbol of A as an element of $L^{\mu-l}_{\mathrm{cl}}(M\setminus Y)$. Locally close to Y we can write

$$\sigma_0^{\mu-l}(A)(r,x,y,\rho,\xi,\eta)=r^{-\mu+l}\tilde\sigma_0^{\mu-l}(A)(r,x,y,r\rho,\xi,r\eta) \tag{4.43}$$

for a function $\tilde\sigma_0^{\mu-l}(A)(r,x,y,\tilde\rho,\xi,\tilde\eta)$ homogeneous in $(\tilde\rho,\xi,\tilde\eta)\neq 0$ of order $\mu-l$ and smooth up to $r=0$. As before for $l=0$ the symbols involved in (4.43) when restricted to a neighborhood close to Y, agree with the symbols involved in (4.39). Clearly, $\sigma_0^{\mu-l}$ vanishes on $L^{\mu-l}_{M+G}(M,\boldsymbol{g})$ since $L^{\mu-l}_{M+G}(M,\boldsymbol{g})\subset L^{-\infty}(M\setminus Y)$. Analogously as (4.25) for $A\in L^{\mu-l}(M,\boldsymbol{g})$ we define

$$\sigma_1^{\mu-l}(A)(y,\eta):=\sum_{j=1}^N\varphi_j(y)\sigma_1^{\mu-l}(a_j)(y,\eta),$$

using (4.40) for the local edge amplitude functions $a_j\in\mathcal{R}^{\mu-l}(\Omega\times\mathbb{R}^q,\boldsymbol{g})$, $j=1,\dots,N$. As a conclusion we recover the filtration property of the edge algebra.

Corollary 4.9. *Let* $A\in L^\mu(M,\boldsymbol{g})$ *and assume that*

$$\sigma_0^{\mu-l}(A)=0,\quad \sigma_1^{\mu-l}(A)=0 \text{ for all } l=0,\dots,j. \tag{4.44}$$

Then we have $A\in L^{\mu-(j+1)}(M,\boldsymbol{g})$.

Thus the property (4.44) for all j yields $A \in \bigcap_{j\in\mathbb{N}} L^{\mu-j}(M,\boldsymbol{g})$. Let us write for the moment

$$L^{-\infty}(\text{symbols}) := \{C \in L^{\mu}(M,\boldsymbol{g}) : \sigma^{\mu-j}(C) = 0 \quad \text{for all}\, j \in \mathbb{N}\} = \bigcap_{j\in\mathbb{N}} L^{\mu-j}(M,\boldsymbol{g}),$$

$$L^{-\infty}(\text{mapping}) := \{C \in L^{\mu}(M,\boldsymbol{g}) : C \text{ has the property (4.17)}\}.$$

Theorem 4.10. *We have*

$$L^{-\infty}(M,\boldsymbol{g}) = \bigcap_{j\in\mathbb{N}} L^{\mu-j}(M,\boldsymbol{g}) \tag{4.45}$$

for any $\mu \in \mathbb{R}$.

Proof. We have to show

$$L^{-\infty}(\text{mapping}) \subseteq L^{-\infty}(\text{symbols}) \tag{4.46}$$

and

$$L^{-\infty}(\text{symbols}) \subseteq L^{-\infty}(\text{mapping}). \tag{4.47}$$

Let us start with (4.46). We consider the case of compact M; the considerations in general only need some simple modifications. The space on the left hand side of (4.45) has been defined by the mapping properties (4.17). Such operators C belong to $L^{-\infty}(M \setminus Y)$ and hence $\sigma_0^{\mu-j}(C)$ vanishes for all $j \in \mathbb{N}$. We have to show that also $\sigma_1^{\mu-j}(C)$ vanishes for all $j \in \mathbb{N}$. To this end we pass to the operator $\varphi\omega C\varphi'\omega'$ for cut-off functions ω, ω' on M (i.e., $\equiv 1$ in a small neighborhood of Y, $\equiv 0$ outside another small neighborhood of Y) and factors $\varphi, \varphi' \in C_0^{\infty}(G)$ for a coordinate neighborhood G on Y. We then consider the operator in local coordinates under a chart $\chi|_{V\setminus Y} : V \setminus Y \to X^{\wedge} \times \mathbb{R}^q$ where V is a wedge neighborhood such that $V \setminus Y = G$, cf. formula (4.10) for $\Omega := \mathbb{R}^q$ where $\chi : V \to X^{\Delta} \times \mathbb{R}^q$ restricts to a chart $G = V \setminus Y \to \mathbb{R}^q$. Denoting $\varphi\omega C\varphi'\omega'$ in local coordinates again by C we have to show that the continuity of

$$C : \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^{\wedge})) \to \mathcal{W}^{\infty}(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma-\mu}(X^{\wedge}))$$

and

$$C^* : \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,-\gamma+\mu}(X^{\wedge})) \to \mathcal{W}^{\infty}(\mathbb{R}^q, \mathcal{K}_Q^{\infty,-\gamma}(X^{\wedge}))$$

for all s gives rise to $\sigma_1^{\mu-j}(C) = 0$ for all $j \in \mathbb{N}$. Since this holds for all s it suffices to consider the trivial group action on $\mathcal{W}^s(\mathbb{R}^q, \cdot)$-spaces, i.e., $\kappa = \mathrm{id}$, the action on $\mathcal{W}^{\infty}(\mathbb{R}^q, \cdot)$ is trivial anyway, cf. notation in Section 2. Moreover, it suffices to replace the spaces $\mathcal{K}_P^{\infty,\gamma-\mu}(X^{\wedge})$ and $\mathcal{K}_Q^{\infty,-\gamma}(X^{\wedge})$ by $E_1 := \mathcal{K}_{\Delta}^{\infty,\gamma-\mu}(X^{\wedge})$ and $E_2 := \mathcal{K}_{\Delta}^{\infty,\gamma}(X^{\wedge})$ for a weight interval $\Delta = (-\delta, 0]$ for some sufficiently small $\delta > 0$. The spaces E_1, E_2 are nuclear and Fréchet. It is then easy to recognise that C has a kernel in

$$c(y,y') \in C^{\infty}(\mathbb{R}^q \times \mathbb{R}^q, E_1 \hat{\otimes}_{\pi} H_1) \bigcap C^{\infty}(\mathbb{R}^q \times \mathbb{R}^q, H_2 \hat{\otimes}_{\pi} E_2) \tag{4.48}$$

for $H_1 = \mathcal{K}^{-s,-\gamma}(X^{\wedge})$, $\quad H_2 = \mathcal{K}^{s,-\gamma+\mu}(X^{\wedge})$. Then, similarly as in scalar smoothing operators, the integral operator

$$Cu(y) = \int c(y,y')u(y')dy'$$

can be written in the form

$$Cu(y) = \iint e^{i(y-y')\eta} c(y,y')\psi(\eta)u(y')e^{-i(y-y')\eta} dy' d\eta$$

for a $\psi \in C_0^\infty(\mathbb{R}^q)$ such that $\int \psi(y) đ\eta = 1$. Thus C has a double symbol

$$a(y,y',\eta) = c(y,y')\psi(\eta)e^{-i(y-y')\eta}$$

which is a Schwartz function in $\eta \in \mathbb{R}^q$ with values in $E_1 \hat{\otimes}_\pi H_1 \bigcap H_2 \hat{\otimes}_\pi E_2$ and with smooth dependence on $(y,y') \in \mathbb{R}^q \times \mathbb{R}^q$. By construction C is also properly supported with respect to (y,y')-variables. We can pass to a left symbol $a_{\mathrm{L}}(y,\eta) \sim \sum_{\alpha \in \mathbb{R}^q} \frac{1}{\alpha!} D_{y'}^\alpha \partial_y^\alpha a(y,y',\eta)|_{y=y'}$. Since $\sigma_1^{\mu-j}(C)$ only depends on the summands for $|\alpha| \leq j$ which are all Schwartz functions in η it follows that $\sigma_1^{\mu-j}(C)(y,\eta) = 0$ for all j.
The second part of the proof consists of verifying (4.47). In other words we prove the continuities (4.17) for any $C \in \bigcap_{j \in \mathbb{N}} L^{\mu-j}(M, \boldsymbol{g})$. The idea of proving (4.19) for $A \in L^{\mu-j}(M,\boldsymbol{g})$ for all j is analogous to that for $j = 0$. In the present case, for $C \in \bigcap_{j\in\mathbb{N}} L^{\mu-j}(M,\boldsymbol{g})$ we already know that $\sigma_0^{\mu-j}(C) = 0$ for all j, i.e., $C \in L^{-\infty}(M \setminus Y)$. Thus we have $C \in L^{\mu-j}(M,\boldsymbol{g}) \bigcap L^{-\infty}(M \setminus Y)$ and hence $C \in L_{M+G}^{\mu-j}(M,\boldsymbol{g})$. For sufficiently large j we even have $C \in L_G^{\mu-j}(M,\boldsymbol{g})$, cf. relation (4.42). This gives us the continuity of

$$C : H^{s,\gamma}(M) \to H_P^{s-\mu+j,\gamma-\mu}(M)$$

for every sufficiently large j, for some asymptotic type P depending on C. Since this holds for all j, it follows that C has the desired mapping property in (4.17). For C^* we can argue in an analogous manner, using that formal adjoints of operators in the edge calculus with weight data $\boldsymbol{g}$ belong the calculus with weight data $\boldsymbol{g}^* := (-\gamma+\mu, -\gamma, (-(k+1), 0])$. This completes the proof of Theorem 4.10. □

5 Auxiliary material

5.1 General notation

For $\Omega \subseteq \mathbb{R}^n$ open, the dual $(C_0^\infty(\Omega))' =: \mathcal{D}'(\Omega)$ is the space of all distribution on Ω, i.e., linear continuous operators

$$u : C_0^\infty(\Omega) \to \mathbb{C}.$$

For any $u \in \mathcal{D}'(\Omega)$ we also write $u(\varphi) = \langle u, \varphi\rangle$, $\varphi \in C_0^\infty(\Omega)$. The dual $(C^\infty(\Omega))' =: \mathcal{E}'(\Omega)$ is the subspace of all $u \in \mathcal{D}'(\Omega)$ for which the support supp$u$ is compact. Here generally, for a Hilbert space H we have the space of H-valued distributions $\mathcal{D}'(\Omega, H) := \mathcal{L}(C_0^\infty(\Omega), H)$. If E, F are locally convex vector space, $\mathcal{L}(E,F)$ will denote the space of all linear continuous operators $A : E \to F$. If $E = F$ we will also write $\mathcal{L}(E)$ instead of $\mathcal{L}(E,F)$.

5.2 Local calculus of pseudo-differential operators

Let formulate some simple properties of the scalar symbol spaces for the pseudo-differential calculus. If a property is valid both for general and classical symbols we write as subscript (cl). We denote by $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n)$ the subspace of symbols in $S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ that are independent of x. Those are closed subspaces of $S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$, and we have

$$S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n) = C^{\infty}(\Omega, S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n)).$$

Moreover, observe that there are continuous embeddings

$$S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n) \hookrightarrow S^{\tilde{\mu}}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$$

for any $\mu \le \tilde{\mu}$. Moreover, $a(x,\xi)\in S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$, $b(x,\xi)\in S^{\nu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ implies $(ab)(x,\xi)\in S^{\mu+\nu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$. Observe that $a(x,\xi)\in S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$, implies $D^{\alpha}_x D^{\beta}_{\xi} a(x,\xi)\in S^{\mu-|\beta|}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ for every α, $\beta\in\mathbb{N}^n$ as we see form (1.7). Note that

$$S^{\mu}(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n) \tag{5.1}$$

for every $\mu\in\mathbb{R}$. We set

$$S^{-\infty}(\Omega\times\mathbb{R}^n) := \bigcap_{\mu\in\mathbb{R}} S^{\mu}(\Omega\times\mathbb{R}^n).$$

Observe that

$$S^{-\infty}(\mathbb{R}^n)=\mathcal{S}(\mathbb{R}^n) \tag{5.2}$$

and $S^{-\infty}(\Omega\times\mathbb{R}^n)=C^{\infty}(\Omega,\mathcal{S}(\mathbb{R}^n))$.

Theorem 5.1. *Let $a_j(x,\xi)\in S^{\mu_j}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$, $j\in\mathbb{N}$, be an arbitrary sequence where $\mu_j\to-\infty$ as $j\to\infty$ and in the classical case $\mu_j=\mu-j$ for some $\mu\in\mathbb{R}$. Then there is an $a(x,\xi)\in S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ for $\mu:=\max\{\mu_j\}$, such that for every $\nu\le\mu$ there is an $N=N(\nu)$ with*

$$a(x,\xi)-\sum_{j=0}^{N} a_j(x,\xi)\in S^{\nu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n).$$

The symbol $a(x,\xi)$ is unique modulo $S^{-\infty}(\Omega\times\mathbb{R}^n)$ and called asymptotic sum of the symbols $a_j(x,\xi)$, written $a(x,\xi)\sim\sum_{j=0}^{\infty}a_j(x,\xi)$.

Remark 5.2. *For any excision function $\chi(\xi)$ (i.e., $\chi(\xi)\in C^{\infty}(\mathbb{R}^n)$, $\chi(\xi)=0$ for $|\xi|<\varepsilon_0$, $\chi(\xi)=1$ for $|\xi|>\varepsilon_1$, for some $0<\varepsilon_0<\varepsilon_1$) there are constants c_j such that*

$$a(x,\xi)=\sum_{j=0}^{\infty}\chi(\frac{\xi}{c_j})a_j(x,\xi) \tag{5.3}$$

converges in $S^{\mu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ and $a(x,\xi)\sim\sum_{j=0}^{\infty}a_j(x,\xi)$. Moreover, in the notation of Theorem 5.1 the sum $\sum_{j=N+1}^{\infty}\chi(\frac{\xi}{c_j})a_j(x,\xi)$ converges in the space $S^{\nu}_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$ for every $\nu\le\mu$ and $N=N(\nu)$.

Remark 5.3. *For every $c(x,\xi)\in S^{-\infty}(\Omega\times\mathbb{R}^n)$ the operator $\mathrm{Op}(c)$ has a kernel in $C^{\infty}(\Omega\times\Omega)$.*

In fact, for every $u \in C_0^\infty(\Omega)$ we have

$$\mathrm{Op}(c)u(x) = \iint e^{i(x-x')\xi} c(x,\xi)u(x')dx'đ\xi = \int \{ \int e^{i(x-x')\xi} c(x,\xi)đ\xi \} u(x')dx'.$$

It is now easy to see that the kernel $\int e^{i(x-x')\xi} c(x,\xi)đ\xi$ belongs to $C^\infty(\Omega \times \Omega)$. Let $L^{-\infty}(\Omega)$ be the space of all operators with kernel in $C^\infty(\Omega \times \Omega)$.

Note that asymptotic summations also occur in other contexts of analysis, e.g., in the Borel Theorem. It states that for every sequence $c_\alpha \in \mathbb{C},\ \alpha \in \mathbb{N}^n$, there exists a $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$D_x^\alpha \varphi(x)|_{x=0} = c_\alpha, \text{ for all } \alpha \in \mathbb{N}^n.$$

In other words the coefficients in the Taylor expansion of φ at $x = 0$

$$\varphi(x) = \sum \frac{1}{\alpha!} x^\alpha c_\alpha,\ c_\alpha = D_x^\alpha \varphi(x)|_{x=0}$$

can be prescribed in an arbitrary manner. Such φ can be found as a convergence sum in $C_0^\infty(\mathbb{R}^n)$, according to the following Borel Theorem.

Theorem 5.4. *Let $(c_\alpha)_{|\alpha| \le j,\, j \in \mathbb{N}}$ be an arbitrary sequence of complete summers. Then for any $\omega \in C_0^\infty(\mathbb{R}^n)$ with $\omega \equiv 1$ close to $x = 0$, there are constants $0 < d_\alpha$ tending to ∞ sufficiently fast as $|\alpha| \to \infty$, such that*

$$\varphi(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \omega(x d_\alpha) x^\alpha c_\alpha \tag{5.4}$$

converges in $C_0^\infty(\mathbb{R}^n)$. In addition, if $\widetilde{\varphi}$ is another element in $C^\infty(\mathbb{R}^n)$ of that kind then $\varphi - \widetilde{\varphi}$ vanishes at $x = 0$ of infinite order, i.e.,

$$D_x^\alpha(\varphi - \widetilde{\varphi})(x)|_{x=0} = 0, \text{ for all } \alpha \in \mathbb{N}^n.$$

Definition 5.5. *For any open set $\Omega \subseteq \mathbb{R}^n$, we set $L^\mu_{(\mathrm{cl})}(\Omega) := \{\mathrm{Op}_x(a) + C : a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n), C \in L^{-\infty}(\Omega)\}$. For $A \in L^\mu_{\mathrm{cl}}(\Omega), A = \mathrm{Op}_x(a) + C$, we set $\sigma_\psi(A)(x,\xi) := a_{(\mu)}(x,\xi)$.*

It can be easily verified that an $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ induces a continuous operator

$$A : C_0^\infty(\Omega) \to C^\infty(\Omega). \tag{5.5}$$

Incidentally the symbol $a(x,\xi)$ in Definition 5.5 will also be called a left symbol of A, written $a_{\mathrm{L}}(x,\xi)$. The meaning of the notation becomes clear by the following proposition.

Proposition 5.6. *The space $L^\mu_{(\mathrm{cl})}(\Omega)$ can be equivalently defined as*

$$L^\mu_{(\mathrm{cl})}(\Omega) = \{\mathrm{Op}(a_{\mathrm{D}}) : a_{\mathrm{D}}(x,x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\}$$

where $a_{\mathrm{D}}(x,x',\xi)$ is called a double symbol, or

$$L^\mu_{(\mathrm{cl})}(\Omega) = \{\mathrm{Op}(a_{\mathrm{R}}) + C : a_{\mathrm{R}}(x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n), C \in L^{-\infty}(\Omega)\}$$

where $a_{\mathrm{R}}(x',\xi)$ is called a right symbol of the respective operator.

For every $A = \mathrm{Op}(a_\mathrm{D})$, here is a left and a right symbol $a_\mathrm{L}(x,\xi)$ and $a_\mathrm{R}(x',\xi)$, respectively, such that

$$\mathrm{Op}(a_\mathrm{D}) = \mathrm{Op}(a_\mathrm{L}) \bmod L^{-\infty}(\Omega),\ \mathrm{Op}(a_\mathrm{D}) = \mathrm{Op}(a_\mathrm{R}) \bmod L^{-\infty}(\Omega),$$

and we have asymptotic expansions

$$a_\mathrm{L}(x,\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha D_{x'}^\alpha a_\mathrm{D}(x,x',\xi)|_{x'=x}. \tag{5.6}$$

$$a_\mathrm{R}(x',\xi) \sim \sum_{\alpha\in\mathbb{N}^n} (-1)^{|\alpha|}\frac{1}{\alpha!}\partial_\xi^\alpha D_{x}^\alpha a_\mathrm{D}(x,x',\xi)|_{x=x'}. \tag{5.7}$$

By virtue of the continuity (5.5) the operator A has a distributional kernel $K_A \in \mathcal{D}'(\Omega\times\Omega)$, the Schwartz kernel, which is uniquely determined by

$$\langle K_A, \varphi\psi\rangle := \int_{\Omega\times\Omega} (A\varphi)(x)\psi(y)dxdy$$

for arbitrary $\varphi, \psi \in C_0^\infty(\Omega)$.

Definition 5.7.

(i) *A set $M \subseteq \Omega\times\Omega$ is called proper, if for any compact subsets $N_1,\ N_2 \subset \Omega$ the intersections*

$$\pi_1^{-1}N_1 \cap M \text{ and } M \cap \pi_2^{-1}N_2$$

are compact in $\Omega\times\Omega$; here $\pi_1 : (x,y) = x,\ \pi_2 : (x,y) = y$ are canonical projections to the second component.

(ii) *The operator $A \in L^\mu(\Omega)$ is called properly support if* $\mathrm{supp}K_A$ *is a proper subset of $\Omega\times\Omega$.*

Note that every differential operator $A = \sum_{|\alpha|\le\mu} a_\alpha(x)D_x^\alpha$ is properly supported.

Remark 5.8.

(i) *A properly supported operator $A \in L^\mu(\Omega)$ induces continuous operator*

$$A : C_0^\infty(\Omega) \to C_0^\infty(\Omega),\ C^\infty(\Omega) \to C^\infty(\Omega).$$

(ii) *Every $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ can be written as $A = A_1 + C$ where $A_1 \in L^\mu_{(\mathrm{cl})}(\Omega)$ is properly supported and $C \in L^{-\infty}(\Omega)$.*

(iii) *Let $A \in L^\mu(\Omega)$ be properly supported. Then for $u \in C_0^\infty(\Omega)$ we have*

$$\begin{aligned} Au &= A\int e^{ix'\xi}\hat{u}(\xi)đ\xi = \int Ae^{ix'\xi}\hat{u}(\xi)đ\xi \\ &= \int e^{ix\xi}(e^{-ix\xi}A(e^{ix'\xi}))\hat{u}(\xi)đ\xi \\ &= \int e^{ix\xi}a(x,\xi)\hat{u}(\xi)đ\xi = \mathrm{Op}(a)u(x) \end{aligned}$$

for $a(x,\xi) = e^{-ix\xi}A(e^{ix'\xi})$.

It can be proved that $a(x,\xi) \in S^\mu(\Omega \times \mathbb{R}^n)$ (and in $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ for $A \in L^\mu_{\mathrm{cl}}(\Omega)$). In other words, for a properly supported operator A we can recover a symbol $a(x,\xi)$ in a unique way.

Given A in Remark 5.8 in the form $A = \mathrm{Op}(a_{\mathrm{D}})$ for a double symbol $a_{\mathrm{D}}(x,x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)\}$ we can find A_1 in the form $A_1 = \mathrm{Op}(a_1)$ for

$$a_1(x,x',\xi) = \omega(x,x')a_{\mathrm{D}}(x,x',\xi) \tag{5.8}$$

where $\omega(x,x') \in C^\infty(\Omega\times\Omega)$ is a function with proper support in $\Omega\times\Omega$ where $\mathrm{diag}(\Omega\times\Omega) \subset \mathrm{int\,supp}\,\omega$, and $\omega(x,x') = 1$ in a neighbourhood of $\mathrm{diag}(\Omega \times \Omega)$.

Given symbols $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, $b(x,\xi) \in S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, we define the Leibniz product $(a\#b)(x,\xi) \in S^{\mu+\nu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ by the asymptotic sum

$$(a\#b)(x,\xi) = \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}(\partial_\xi^\alpha a(x,\xi))D_x^\alpha b(x,\xi). \tag{5.9}$$

Here

$$\partial_\xi^\alpha := \Big(\frac{\partial}{\partial\xi_1}\Big)^{\alpha_1}\dots\Big(\frac{\partial}{\partial\xi_n}\Big)^{\alpha_n}.$$

The latter asymptotic sum is formed by using Theorem 5.1 which can be applied because of $(\partial_\xi^\alpha a(x,\xi))D_x^\alpha b(x,\xi) \in S^{\nu+\mu-|\alpha|}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, i.e., the order of summands in (5.9) tends to $-\infty$ as $|\alpha| \to \infty$. Note that

$$(a\#b)_{(\mu+\nu)}(x,\xi) = a_{(\mu)}(x,\xi)b_{(\nu)}(x,\xi).$$

Theorem 5.9. *Let $A \in L^\mu_{(\mathrm{cl})}(\Omega), B \in L^\nu_{(\mathrm{cl})}(\Omega)$, and assume that A or B is properly supported. Then we have $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(\Omega)$. Moreover, for $A = \mathrm{Op}(a), a \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n), B = \mathrm{Op}(b), b \in S^\nu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, we have*

$$AB = \mathrm{Op}(a\#b) \bmod L^{-\infty}(\Omega).$$

In the classical case we have

$$\sigma_\psi(AB) = \sigma_\psi(A)\sigma_\psi(B).$$

Remark 5.10. *The Leibniz multiplication is associative, i.e., we always have $(a\#b)\#c = a\#(b\#c)$.*

In the definition of symbol spaces $S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^m)$ we may admit $n = \dim\Omega$ to be independent of m. This gives rise to parameter-dependent symbol spaces $S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n+l}_{\xi,\lambda})$ where $(\xi,\lambda) \in \mathbb{R}^{n+l}$ is treated as a covariable.

Remark 5.11. *Let $a(x,\xi,\lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n+l}_{\xi,\lambda})$. Then for every fixed $\lambda_0 \in \mathbb{R}^l$ we have $a(x,\xi,\lambda_0) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n_\xi)$.*

Let us now generalize the notation (1.11) to the parameter-dependent case for a parameter $\lambda \in \mathbb{R}^l,\ l \in \mathbb{N}$. We set

$$L^\mu_{(\mathrm{cl})}(\Omega;\mathbb{R}^l) := \{A(\lambda) = \mathrm{Op}_x(a)(\lambda) + C(\lambda) : a(x,\xi,\lambda) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^{n+l}_{\xi,\lambda}),\ C \in L^{-\infty}(\Omega;\mathbb{R}^l)\},$$

where

$$L^{-\infty}(\Omega;\mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(\Omega)).$$

Here we use that $L^{-\infty}(\Omega)$ is a Fréchet space, under identifying operators $C \in L^{-\infty}(\Omega)$ with their kernels $c(x,x') \in C^\infty(\Omega \times \Omega)$.

5.3 Pseudo-differential operators on a smooth manifold

We need pseudo-differential operators also on a C^∞ manifold M, $n = \dim M$. On M we fix a Riemannian metric. This allows us also on M to identify smoothing operators C with their kernels via

$$Cu(x) = \int_M c(x,x')u(x')dx',\ c(x,x') \in C^\infty(M \times M),$$

where dx' is the measure associated with the Riemannian metric. The set of these operators will be denoted by $L^{-\infty}(M)$, and then we define

$$L^{-\infty}(M;\mathbb{R}^l) = \mathcal{S}(\mathbb{R}^l, L^{-\infty}(M)).$$

The space $L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l)$ is defined as the set of all families of operators

$$A(\lambda) : C_0^\infty(M) \to C^\infty(M)$$

such that for every $\varphi \in C_0^\infty(M)$, $\psi \in C^\infty(M)$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, the operator $\psi A(\lambda)\varphi$ is parameter-dependent smoothing, i.e., an element of $L^{-\infty}(M;\mathbb{R}^l)$.

For coordinate neighborhood $U \subseteq M$, and a chart $\chi : U \to \Omega,\ \Omega \in \mathbb{R}^n$ open, we define the space

$$L^\mu_{(\mathrm{cl})}(U;\mathbb{R}^l) = \{\chi^* \circ A(\lambda) \circ (\chi^*)^{(-1)} : A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega;\mathbb{R}^l)\},$$

with $\chi^* : C_0^\infty(\Omega) \to C_0^\infty(U)$, $C^\infty(\Omega) \to C^\infty(U)$ being the function pull back.

In our application the manifold M is either compact or a countable union of compact sets. Let $\{U_j\}_{j\in\mathbb{N}}$ be a countable covering of M by coordinate neighborhoods (finite for compact M) and $\{\varphi_j\}_{j\in\mathbb{N}}$ a subordinate partition of unity, $\varphi_j \in C_0^\infty(U_j)$. Moreover, let $\{\psi_j\}_{j\in\mathbb{N}}$ be a system of functions $\psi_j \in C_0^\infty(U_j)$ such that $\varphi_j \prec \psi_j$ for all j ($f \prec g$ means that $g \equiv 1$ on support of f). Then for every $A(\lambda) \in L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l_\lambda)$ we have

$$A(\lambda) = \sum_{j\in\mathbb{N}} A(\lambda)\varphi_j = \sum_{j\in\mathbb{N}} \psi_j A(\lambda)\varphi_j + C(\lambda)$$

for

$$C(\lambda) := \sum_{j\in\mathbb{N}} (1-\psi_j)A(\lambda)\varphi_j.$$

Here $C(\lambda) \in L^{-\infty}(M;\mathbb{R}^l)$ because of $(1-\psi_j)A(\lambda)\varphi_j \in L^{-\infty}(M;\mathbb{R}^l)$ for every j. The space $L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l)$ can be equivalently defined as follows:

$$L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^l) = \Big\{A(\lambda) = \sum_{j\in\mathbb{N}} \psi_j A_j(\lambda)\varphi_j + C(\lambda) : A_j(\lambda) \in L^\mu_{(\mathrm{cl})}(U_j;\mathbb{R}^l),\ C(\lambda) \in L^{-\infty}(M;\mathbb{R}^l)\Big\}.$$

For $A_j(\lambda) = A(\lambda)|_{U_j}$, it follows from the consideration before that, in fact, in the representation we may admit arbitrary $A_j(\lambda) \in L^\mu_{(\mathrm{cl})}(U_j;\mathbb{R}^l)$.

5.4 Ellipticity and parametrix in the standard calculus

Definition 5.12.

(i) *An* $a(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n)$ *is called elliptic of order* μ *if there exists a* $p(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n)$ *with the* $p(x,\xi)a(x,\xi) = 1 \bmod S^{-1}(\Omega \times \mathbb{R}^n)$*. Clearly, we then also have* $a(x,\xi)p(x,\xi) = 1 \bmod S^{-1}(\Omega \times \mathbb{R}^n)$*.*

(ii) *An operator* $A \in L^{\mu}(\Omega)$ *is called elliptic of order* μ *if for any representation* $A = \mathrm{Op}(a) + C,\ a(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n),\ C \in L^{-\infty}(\Omega)$*, the symbol* $a(x,\xi)$ *is called elliptic in the sense of* (i)*.*

Remark 5.13. *A symbol* $a(x,\xi) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ *is elliptic if and only if* $a_{(\mu)}(x,\xi) \neq 0$ *for all* $x \in \Omega,\ \xi \neq 0$*. In this case for any excision function* $\chi(\xi)$ *we can set* $p(x,\xi) = \chi(\xi)a^{-1}_{(\mu)}(x,\xi)$*.*

Proposition 5.14. *If* $a(x,\xi) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ *is elliptic then there exists a* $p(x,\xi) \in S^{-\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ *such that*

$$p\#a = 1 \quad \bmod S^{-\infty}(\Omega \times \mathbb{R}^n), \tag{5.10}$$

$$a\#p = 1 \quad \bmod S^{-\infty}(\Omega \times \mathbb{R}^n). \tag{5.11}$$

Proof. Choose $p_1 \in S^{-\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ such that $c := 1 - p_1 a$ belongs to $S^{-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ then $p_1\#a = 1 - c_1$ for $c_1 \in S^{-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$, hence, $p_1 a = p_1\#a$ modulo $S^{-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ cf.the expression (5.9) . According to Theorem 5.1 form the asymptotic sum

$$(1 - c_1)^{\#-1} := \sum_{j=0}^{\infty} c_1^{\#j} \in S^{0}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$$

where

$$c_1^{\#j} := c_1\#c_1\#....\#c_1, \quad (j \text{ times})$$

which belong to $S^{-j}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$. Then $(1 - c_1)^{\#-1}\#(p_1\#a) = (1 - c_1)^{\#-1}(1 - c_1) = 1 \bmod S^{-\infty}(\Omega \times \mathbb{R}^n)$. Thus we may set $p := (1 - c_1)^{\#-1}\#p_1$ and we obtain relation (5.10). In a similar manner we can construct $\tilde{p} \in S^{-\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ such that $a\#\tilde{p} = 1 \bmod S^{-\infty}(\Omega \times \mathbb{R}^n)$. A simple algebraic argument tells us $p = \tilde{p} \bmod S^{-\infty}(\Omega \times \mathbb{R}^n)$. Thus we also obtain relation (5.11). □

Theorem 5.15. *Let* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$ *be elliptic. Then there is a a properly supported parametrix* $P \in L^{-\mu}_{(\mathrm{cl})}(\Omega)$*, i.e.,*

$$PA = 1 - C_{\mathrm{L}}\,,\ AP = 1 - C_{\mathrm{R}} \tag{5.12}$$

for certain $C_{\mathrm{L}}\,, C_{\mathrm{R}} \in L^{-\infty}(\Omega)$*.*

Proof. Let $P_0 := \mathrm{Op}(p)$ for the symbol $p(x,\xi)$ of Proposition 5.14. Then, applying Remark 5.8 (ii) we find a properly supported operator $P \in L^{-\mu}(\Omega)$ such that $P = P_0 \bmod L^{-\infty}(\Omega)$. By virtue of Remark 5.8 (iii) we can write P in the form $P = \mathrm{Op}(p_1)$ for a unique $p_1(x,\xi) \in S^{-\mu}(\Omega \times \mathbb{R}^n)$. We have

$$p_1(x,\xi) = p(x,\xi) \bmod S^{-\infty}(\Omega \times \mathbb{R}^n).$$

In fact, to obtain $p_1(x,\xi)$, it suffices to apply the asymptotic expansion (5.6) to $\omega(x,\xi)P(x,\xi)$, see Proposition 5.6 .

$$PA = \mathrm{Op}(p_1)(\mathrm{Op}(a) + C) = \mathrm{Op}(p_1)\mathrm{Op}(a) + C_1$$

where $C_1 := \mathrm{Op}(p_1)C \in L^{-\infty}(\Omega)$. Moreover, using Theorem 5.9 we have

$$\mathrm{Op}(p_1)\mathrm{Op}(a) = \mathrm{Op}(p_1\#a) + C_2$$

for some $C_2 \in L^{-\infty}(\Omega)$. Because of $p_1\#a = (p_1 - p)\#a + p\#a$ and $p_1 - p \in S^{-\infty}(\Omega \times \mathbb{R}^n)$ it follows that

$$\mathrm{Op}(p_1\#a) = \mathrm{Op}(p\#a) + C_3$$

for $C_3 := \mathrm{Op}(p_1 - p) \in L^{-\infty}(\Omega)$. From (5.10) we have $p\#a = 1 + c_4$ for a $c_4 \in S^{-\infty}(\Omega \times \mathbb{R}^n)$ and hence

$$\mathrm{Op}(p\#a) = \mathrm{Op}(1) + \mathrm{Op}(c_4)$$

for $C_4 \in L^{-\infty}(\Omega)$. Thus

$$PA = 1 + C_1 + C_2 + C_3 + C_4$$

and we obtain the first relation of (5.12) for $C_\mathrm{L} = -(C_1 + C_2 + C_3 + C_4) \in L^{-\infty}(\Omega)$. In an analogous manner we find a right parametrix $Q \in L^{-\mu}(\Omega)$ such that $AQ = 1 - D_1$ for a $D_1 \in L^{-\infty}(\Omega)$. However, an elementary algebraic construction shows that $Q = P \bmod L^{-\infty}(\Omega)$ and hence $AQ = AP + D_2$ for a $D_2 \in L^{-\infty}(\Omega)$. Then

$$AP = AQ - D_2 = 1 - (D_1 + D_2) = 1 - C_\mathrm{R}$$

for $C_\mathrm{R} = D_1 + D_2$. □

5.5 Elementary aspects of kernel cut-off

Let us first consider symbols with constant coefficient $a(\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$. The case of symbols $a(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ will be admitted below as a simple generalization. First that $S^\mu(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ for every $\mu \in \mathbb{R}$. Motivated by the expression (1.10), namely

$$\mathrm{Op}(a)u(x) := \int k_a(x - x')u(x')dx' \tag{5.13}$$

for $k_a(\theta) := \int e^{i\theta\xi}a(\xi)đ\xi$. We consider the inverse Fourier transform applied to $a(\xi) \in S^\mu(\mathbb{R}^n)$ as a Schwartz distribution. According to (1.29) we obtain

$$k_a(\theta) = (F^{-1}_{\xi\to\theta}a)(\theta) \in S'(\mathbb{R}^n_\theta).$$

From (5.13) we see that $k_a(x - x')$ is the distributional kernel of the operator Op(a). We often refer to the following standard properties of the Fourier transform:

$$(-\theta)^\alpha k_a(\theta) = k(D^\alpha_\xi a)(\theta),\ D^\alpha_\theta k_a(\theta) = k(\xi^\alpha a)(\theta), \quad \alpha \in \mathbb{N}^n. \tag{5.14}$$

Lemma 5.16. *We have*

$$\chi(\theta)k_a(\theta) \in \mathcal{S}(\mathbb{R}^n_\theta) \tag{5.15}$$

for every excision function $\chi(\theta)$ in $\mathbb{R}^n$.

Proof. We employ the identities

$$|\theta|^{2N}e^{i\theta\xi} = (-1)^N\Delta_\xi^N e^{i\theta\xi},\ \Delta_\theta^M e^{i\theta\xi} = (-1)^M|\xi|^{2M}e^{i\theta\xi}$$

for every $N, M \in \mathbb{N}$. This yields after integration by parts

$$\begin{aligned}\chi(\theta)|\theta|^{2N}\Delta_\theta^M k(a)(\theta) &= \chi(\theta)|\theta|^{2N}(-1)^M\int e^{i\theta\xi}|\xi|^{2M}a(\xi)đ\xi \\ &= (-1)^{(N+M)}\chi(\theta)\int e^{i\theta\xi}\Delta_\xi^N(|\xi|^{2M}a(\xi))đ\xi\end{aligned} \tag{5.16}$$

as well as

$$\chi(\theta)k_a(\theta) = \chi(\theta)(-1)^N|\theta|^{-2N}\int e^{i\theta\xi}\Delta_\xi^N a(\xi))đ\xi \tag{5.17}$$

which follows from the second equation of (5.16) for $M = 0$. Since (5.17) holds for an arbitrary excision function χ and every N from $\Delta_\xi^N a(\xi)) \in S^{\mu-2N}(\mathbb{R}^n)$ we conclude $k_a(\theta) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, in particular, $\operatorname{sing\ supp} k_a \subseteq \{0\}$. This is equivalent to the pseudo-locality of the operator $\mathrm{Op}(a)$, i.e., the singular support of its distributional kernel is contained in $\operatorname{diag}(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore, in order to verify (5.15) it suffices so show that

$$\sup_{|\theta|\geq C}||\theta|^{2N}\Delta_\theta^M k_a(\theta)|$$

is bounded for some $C > 0$ and suitable sufficiently large M, N. But this follows from the absolute convergence of the integrals on the right hand side of (5.16) for arbitrary M by choosing N so large that $\mu + 2(M - N) < -n$. □

Let

$$\psi(\theta) := 1 - \chi(\theta)$$

which is a cut-off function, i.e., $\psi \in C_0^\infty(\mathbb{R}^n)$, $\psi(\theta) \equiv 1$ in a neighbourhood of $\theta = 0$. We call

$$V_F(\psi)a(\xi) := F_{\theta\to\xi}(\psi k_a)(\xi) \tag{5.18}$$

a kernel cut-off operator and $V_F(\chi) := F_{\theta\to\xi}(\chi k_a)(\xi)$ a kernel excision operator.

Proposition 5.17. *The operator* (5.18) *induces a continuous map*

$$V_F(\psi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\mathbb{R}^n), \tag{5.19}$$

where

$$V_F(\psi)a(\xi) = a(\xi) \mod S^{-\infty}(\mathbb{R}^n). \tag{5.20}$$

Proof. We have

$$k_a(\theta) = \psi(\theta)k(a)(\theta) + \chi(\theta)k_a(\theta). \tag{5.21}$$

Then (5.20) follows from (5.15). Because of

$$a(\xi) = V_F(\psi)a(\xi) + V_F(\chi)a(\xi)$$

and the continuity of $V_F(\chi) : S^\mu_{(\mathrm{cl})}(\mathbb{R}^n) \to S^{-\infty}(\mathbb{R}^n)$ which is a consequence of Lemma 5.16 we obtain the continuity of (5.19). □

Lemma 5.18. *For every $\psi(\theta) \in C_0^\infty(\mathbb{R}^n_\theta)$ such that $\psi(\theta) = 1$ in a neighborhood of $\theta = 0$. Then we have*

$$(1 - \psi(\theta))k_a(\theta) \in \mathcal{S}(\mathbb{R}^n_\theta) \tag{5.22}$$

which entails

$$\operatorname{sing\,supp}\, k_a \subseteq \{0\}$$

From $F_{\theta \to \xi} k_a(\theta) = a(\xi)$ we obtain

$$a(\xi) = (F_{\theta\to\xi}(1-\psi)k_a)(\theta) + (F_{\theta\to\xi}\psi k_a)(\theta) = a_0(\xi) + a_1(\xi) \tag{5.23}$$

for $a_0(\xi) \in \mathcal{S}(\mathbb{R}^n) = S^{-\infty}(\mathbb{R}^n)$, cf. (5.23), and hence $a_1(\xi) = a(\xi) - a_0(\xi) \in S^\mu(\mathbb{R}^n)$.

In other words, the symbol $a_1(\xi)$ as the Fourier transform of some compactly supported distribution in $\mathbb{R}^n_\theta$ belongs to $\mathcal{A}(\mathbb{C}^n)$, i.e., it is an entire function.

We write

$$a_1(\xi) = (V_\psi a)(\xi) = (F_{\theta\to\xi}\psi k_a)(\xi).$$

Clearly we can form $(V_\varphi a)(\xi)$ also for arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n_\theta)$. Let us write down the expression more explicitly.

We have

$$\begin{aligned}(V_\varphi a)(\xi) &= \int e^{-i\theta\xi}\varphi(\theta)\Big\{\int e^{i\theta\zeta}a(\zeta)đ\zeta\Big\}d\theta \\ &= \iint e^{i(\zeta-\xi)\theta}\varphi(\theta)a(\zeta)đ\zeta d\theta = \iint e^{i\eta\theta}\varphi(\theta)a(\eta+\xi)đ\eta d\theta.\end{aligned}$$

We interpret V_φ as a kernel cut-off operators, acting on a symbol and depending on φ, cf.

$$V : (\varphi, a) \to V_\varphi a \tag{5.24}$$

Let us have a look at other variants of kernel cut-off. We may admit $\varphi \in C_{\mathrm{b}}^\infty(\mathbb{R}^n)$ where

$$C_{\mathrm{b}}^\infty(\mathbb{R}^n) = \big\{\varphi(\theta) \in C^\infty(\mathbb{R}^n) : \sup\nolimits_{\theta\in\mathbb{R}^n}|D_\theta^\alpha\varphi(\theta)| < \infty \text{ for every } \alpha \in \mathbb{N}^n\big\},$$

cf. Theorem 5.19 below.

Theorem 5.19. *The kernel cut-off operator* (5.24) *defines a bilinear continuous operator*

$$V : C_{\mathrm{b}}^\infty(\mathbb{R}) \times S_{\mathrm{cl}}^\mu(\mathbb{R}^n) \to S_{\mathrm{cl}}^\mu(\mathbb{R}^n).$$

The symbol $V_\varphi a(\xi)$ admits asymptotic expansion

$$V_\varphi a(\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_\theta^\alpha\varphi(0)\partial_\xi^\alpha a(\xi).$$

Proof. (i) From the definition

$$V_\varphi a(\xi) = \iint e^{i\eta\theta}\varphi(\theta)a(\eta+\xi)đ\eta d\theta$$

we known that it is clear for the bilinear according to the oscillatory integral.
(ii) In order to prove $V_\varphi a(\xi) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n)$ we need the following identities:

$$e^{-i\theta\eta} = \langle\theta\rangle^{-2N}(1-\triangle_\eta)^N e^{-i\theta\eta}$$

and

$$e^{-i\theta\eta} = \langle\eta\rangle^{-2M}(1-\triangle_\theta)^M e^{-i\theta\eta}$$

for every $M,\ N \in \mathbb{N}, \theta \in \mathbb{R}, \eta \in \mathbb{R}^n$ we set $M = 1$, then

$$\begin{aligned} V_\varphi a(\xi) &= \iint e^{-i\eta\theta}\varphi(\theta)a(\xi-\eta)đ\eta d\theta \\ &= \int e^{-i\eta\theta}\langle\eta\rangle^{-2N}(1-\partial_\theta^2)^N\langle\theta\rangle^{-2}(1-\partial_\eta^2)\varphi(\theta)a(\xi-\eta)đ\eta d\theta \\ &= \int e^{-i\eta\theta}\langle\theta\rangle^{-2}\{(1-\partial_\theta^2)^N\varphi(\theta)\}a_N(\eta,\xi)đ\eta d\theta \end{aligned} \tag{5.25}$$

where

$$a_N(\eta,\xi) = (1-\partial_\eta^2)\{\langle\eta\rangle^{-2N}a(\xi-\eta)\} \tag{5.26}$$

for $N \in \mathbb{N}$ sufficiently large. The function (5.26) is a linear combination of the terms $(\partial_\eta^j\langle\eta\rangle^{-2N})(\partial_\xi^k)a(\xi-\eta)$ for $0 \le j,\ k \le 2$. From Peetre's inequality $\langle\xi-\eta\rangle^\mu \le c^{|\mu|}\langle\xi\rangle^\mu\langle\eta\rangle^{|\mu|}$ we have

$$\begin{aligned} |(\partial_\eta^j\langle\eta\rangle^{-2N})(\partial_\xi^k)a(\xi-\eta)| &\le |(\partial_\eta^j\langle\eta\rangle^{-2N})||(\partial_\xi^k)a(\xi-\eta)| \\ &\le c_1\langle\eta\rangle^{-2N}|(\partial_\xi^k)a(\xi-\eta)| \\ &\le c_1\langle\eta\rangle^{-2N}c_2\langle\xi-\eta\rangle^{\mu-k} \\ &\le c_1\langle\eta\rangle^{-2N}c_2\langle\xi\rangle^\mu\langle\eta\rangle^{|\mu|} \\ &\le c_3\langle\eta\rangle^{|\mu|-2N}\langle\xi\rangle^\mu \end{aligned} \tag{5.27}$$

for some $c_3 > 0$. This implies analogous estimates for the function (5.26) For N so large that $|\mu| - 2N \le 0$ we obtain

$$\begin{aligned} |V_\varphi a(\xi)| &\le \int |e^{-i\eta\theta}\langle\theta\rangle^{-2}\{(1-\partial_\theta^2)^N\varphi(\theta)\}a_N(\eta,\xi)|đ\eta d\theta \\ &\le \int |e^{-i\eta\theta}|đ\eta| \int \langle\theta\rangle^{-2}\{(1-\partial_\theta^2)^N\varphi(\theta)\}d\theta||a_N(\eta,\xi)| \\ &\le C\langle\xi\rangle^\mu \end{aligned} \tag{5.28}$$

for some $C > 0$ since the convergence integral

$$\int \langle\theta\rangle^{-2}\{(1-\partial_\theta^2)^N\varphi(\theta)\}d\theta,\ \int e^{-i\eta\theta}đ\eta.$$

Therefore it suffices to verify that $V_\varphi a(\xi) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$.
(iii) We apply the close graph theorem to prove the continuity of V.
(iv) Let

$$\varphi(\theta) = \sum_{|\alpha|\le N}\frac{1}{\alpha!}\partial_\theta^\alpha\varphi(0)\theta^\alpha + R_{N+1}(\theta) = \sum_{|\alpha|\le N}c_\alpha\theta^\alpha + R_{N+1}(\theta)$$

we choose $R_{N+1}(\theta)$ in the form $\theta^{N+1}\varphi_{N+1}(\theta)$ for

$$\varphi_{N+1}(\theta) = \frac{1}{\alpha!}\int_0^1 (1-t)^N (\partial_\theta^{N+1}\varphi)(t\theta)dt.$$

Then the function belong to $C_b^\infty(\mathbb{R})$. Integration by parts gives us

$$\begin{aligned}
&\int e^{-i\theta\xi} \sum_{|\alpha|\le N} c_\alpha \theta^\alpha \{ \int e^{i\theta\zeta} a(\zeta) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} \sum_{|\alpha|\le N} c_\alpha \{ \int \theta^\alpha e^{i\theta\zeta} a(\zeta) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} \sum_{|\alpha|\le N} c_\alpha \{ \int (D_\zeta^\alpha e^{i\theta\zeta}) a(\zeta) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} \sum_{|\alpha|\le N} c_\alpha \{ \int (-1)^{|\alpha|} e^{i\theta\zeta} (D_\zeta^\alpha a(\zeta)) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} \sum_{|\alpha|\le N} c_\alpha \{ \int (-1)^{|\alpha|} e^{i\theta\zeta} ((-i)^{|\alpha|} \partial_\zeta^\alpha a(\zeta)) đ\zeta \} d\theta \\
&= \sum_{|\alpha|\le N} c_\alpha i^{|\alpha|} \int e^{-i\theta\xi} \{ \int e^{i\theta\zeta} (\partial_\zeta^\alpha a(\zeta)) đ\zeta \} d\theta \\
&= \sum_{|\alpha|\le N} c_\alpha i^{|\alpha|} \partial_\xi^\alpha a(\xi) \\
&= \sum_{|\alpha|\le N} \frac{i^{|\alpha|}}{\alpha!} \partial_\theta^\alpha \varphi(0) \partial_\xi^\alpha a(\xi) \\
&= \sum_{|\alpha|\le N} \frac{(-1)^{|\alpha|}}{\alpha!} D_\theta^\alpha \varphi(0) \partial_\xi^\alpha a(\xi)
\end{aligned} \tag{5.29}$$

and

$$\begin{aligned}
&\int e^{-i\theta\xi} \varphi_{N+1}(\theta) \{ \int e^{i\theta\zeta} \theta^{N+1} a(\zeta) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} \varphi_{N+1}(\theta) \{ \int (D_\zeta^{N+1} e^{i\theta\zeta}) a(\zeta) đ\zeta \} d\theta \\
&= \int e^{-i\theta\xi} (-1)^{N+1} \varphi_{N+1}(\theta) \{ \int e^{i\theta\zeta} (D_\zeta^{N+1} a(\zeta)) đ\zeta \} d\theta \\
&= \iint e^{i\theta(\zeta-\xi)} (-1)^{N+1} \varphi_{N+1}(\theta) D_\zeta^{N+1} a(\zeta) đ\zeta d\theta \\
&= V_{(-1)^{N+1}\varphi_{N+1}(\theta)} (D_\xi^{N+1} a)(\xi) \in S_{\mathrm{cl}}^{\mu-(N+1)}(\mathbb{R}^n)
\end{aligned} \tag{5.30}$$

summing up the (5.29) and (5.30) for large N we get

$$V_\varphi a(\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{(-1)^{|\alpha|}}{\alpha!} D_\theta^\alpha \varphi(0) \partial_\xi^\alpha a(\xi).$$

$\square$

It will also make sense to discuss kernel cut-off on symbols $a(\xi,\lambda) \in S^{\mu}(\mathbb{R}^{n+l}_{\xi,\lambda})$, operating H with respect to the variable $\xi \in \mathbb{R}^n$, where λ remains untouched and is treated as a parameter. In other words, applying the kernel cut-off to a parameter-dependent symbol $a(x,\xi,\lambda) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n_{\xi})$, regarded as a λ-dependent family of symbols in $S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n_{\xi})$, cf. Remark 5.11, we also can form

$$(V_{\varphi}a)(x,\xi,\lambda)$$

where V_{φ} only acts on the ξ- variables.

Observe that

$$D^{\alpha}_{\xi,\lambda}V_{\varphi}a(x,\xi,\lambda) = V_{\varphi}D^{\alpha}_{\xi,\lambda}a(x,\xi,\lambda).$$

5.6 Asymptotics and operators on cones

The analysis of edge operators refers to spaces with discrete asymptotic and pseudo-differential operators on infinite cones. In this section we outline some necessary notation. Edge symbols take values in the cone algebra over the infinite cone X^{Δ}, cf. (4.1). So we briefly recall what we understand by the cone algebra. In cone pseudo-differential operators on an infinite cone X^{Δ} we start observing the behavior for $r \to \infty$, the conical exit to infinity. In this case the variables $(r,x) \in \mathbb{R}_+ \times X$ are considered for x in a coordinate neighborhood U on X that we identify with $x \in \mathbb{R}^n$. Then, a standard process via an open covering of X and a subordinate partition of unity gives us classical operators globally on $\mathbb{R}_+ \times X$ for $r \to \infty$, indicated by $L^{\mu;\nu}_{\mathrm{cl}}(\cdot)_{\mathrm{exit}}$ for a pair of orders $(\mu;\nu) \in \mathbb{R} \times \mathbb{R}$. The local definition is as follows. Consider the space

$$S^{\mu;\nu}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}}) \subset C^{\infty}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}})$$

defined by symbolic estimates

$$|D^{\alpha}_{\tilde{x}}D^{\beta}_{\tilde{\xi}}a(\tilde{x},\tilde{\xi})| \leq c_{\alpha\beta}\langle\tilde{\xi}\rangle^{\mu-|\beta|}\langle\tilde{x}\rangle^{\nu-|\alpha|}$$

for all $\alpha,\, \beta \in \mathbb{N}^{n+1}$ and $(\tilde{x},\tilde{\xi}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, for constants $c_{\alpha\beta} > 0$. The space $L^{\mu;\nu}(\mathbb{R}^{n+1})_{\mathrm{exit}}$ is defined as the set of all operators $\mathrm{Op}_{\tilde{x}}(a)$ for arbitrary $a(\tilde{x},\tilde{\xi}) \in S^{\mu;\nu}(\mathbb{R}^{n+1}_{\tilde{x}} \times \mathbb{R}^{n+1}_{\tilde{\xi}})$. The subspace of classical operators is defined in terms of symbols in $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{\xi}})\hat{\otimes}_{\pi}S^{\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1}_{\tilde{x}})$. The corresponding space with classical symbols is denoted by

$$L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}^{n+1})_{\mathrm{exit}}. \tag{5.31}$$

More details can be found in [55, Subsection 1.4]. This notation has an extension to $\mathbb{R}_+ \times X$ for a smooth manifold X, which gives us the spaces

$$L^{\mu;\nu}(\mathbb{R}_+ \times X)_{\mathrm{exit}} \text{ or } L^{\mu;\nu}_{\mathrm{cl}}(\mathbb{R}_+ \times X)_{\mathrm{exit}}.$$

Definition 5.20. *Let X be a closed smooth manifold.*
(i) *The space $L_G(X^{\Delta},\boldsymbol{g})$ of Green operators on X^{Δ} for $\mu \in \mathbb{R}$, $\boldsymbol{g} = (\gamma,\gamma-\mu,\Theta)$, is the set of all operators*

$$G \in \bigcap_{s,e\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,\gamma;e}(X^{\wedge}),\mathcal{K}^{\infty,\gamma-\mu;\infty}_{P}(X^{\wedge}))$$

such that

$$G^* \in \bigcap_{s,e\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,-\gamma+\mu;e}(X^\wedge), \mathcal{K}_Q^{\infty,-\gamma;\infty}(X^\wedge))$$

for some G-dependent asymptotic types P and Q.
(ii) *By $L_{M+G}(X^\Delta, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, (-(k+1), 0])$ and $k \in \mathbb{N}$, we define the space of all $M+G$ for $G \in L_G(X^\Delta, \boldsymbol{g})$ and smoothing Mellin operators*

$$M := r^{-\mu}\omega \sum_{j=0}^{k} r^j \mathrm{Op}_M^{\gamma_j - n/2}(f_j)\omega' \tag{5.32}$$

for cut-off functions ω, ω', and smoothing Mellin symbols $f_j(w) \in M_{R_j}^{-\infty}(X)$ with Mellin asymptotic types R_j, weights $\gamma_j \in \mathbb{R}$, satisfying the conditions

$$\gamma - j \leq \gamma_j \leq \gamma,\ \Pi_{\mathbb{C}} R_j \cap \Gamma_{\frac{n+1}{2}-\gamma_j} = \varnothing.$$

For $\boldsymbol{g} = (\gamma, \gamma-\mu, (-\infty, 0])$ we define $L_{M+G}(X^\Delta, \boldsymbol{g})$ as the intersection of the respective L_{M+G} spaces for $(\gamma, \gamma-\mu, (-(k+1), 0])$ over $k \in \mathbb{N}$.
(iii) *The space $L^\mu(X^\Delta, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\mu \in \mathbb{R}$, $\Theta = (-(k+1), 0], k \in N \cup \{\infty\}$, is defined as the set of all operators*

$$A = r^{-\mu}\big\{\omega \mathrm{Op}_M^{\gamma-n/2}(h)\omega' + (1-\omega)\mathrm{Op}_r(p)(1-\omega'')\big\} + M + C \tag{5.33}$$

for cut-off functions $\omega'' \prec \omega \prec \omega'$, arbitrary $h(r,w) \in C^\infty(\overline{\mathbb{R}}_+, M_{\mathcal{O}}^\mu(X))$ and $p(r,\rho)$ given by

$$p(r,\rho) := \tilde{p}(r, r\rho),\ \ \tilde{p}(r,\tilde{\rho}) \in C^\infty(\overline{\mathbb{R}}_+, L_{\mathrm{cl}}^\mu(X; \mathbb{R}_{\tilde{\rho}})),$$

M as in (ii) *and $C \in L^{-\infty}(X^\Delta, \boldsymbol{g})$, i.e.,*

$$C : H_{\mathrm{comp}}^{s,\gamma}(X^\Delta) \to H_{\mathrm{loc},P}^{\infty,\gamma-\mu}(X^\Delta),\ \ C^* : H_{\mathrm{comp}}^{s,-\gamma+\mu}(X^\Delta) \to H_{\mathrm{loc},Q}^{\infty,-\gamma}(X^\Delta)$$

for every $s \in \mathbb{R}$ and asymptotic types P and Q, depending on C.
(iv) *The space $L^\mu(X^\Delta, \boldsymbol{g})_{\mathrm{exit}}$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\mu \in \mathbb{R}, \Theta$ as in* (iii) *is defined as the set of all operators*

$$A = A_\psi + M + G \tag{5.34}$$

for

$$A_\psi = r^{-\mu}\big\{\omega \mathrm{Op}_M^{\gamma-n/2}(h)\omega' + \varphi \mathrm{Op}_r(p)\varphi' + \chi P_{\mathrm{exit}}\chi'\big\} \tag{5.35}$$

where h, p are as in (iii), *$\omega \prec \omega'$ are arbitrary cut-off functions, $\varphi, \varphi' \in C_0^\infty(\mathbb{R}_+)$, and $\chi \prec \chi'$ excision functions (i.e., $1-\chi \succ 1-\chi'$ are cut-off functions in the former sense), where $\omega + \varphi + \chi = 1$, $M + G \in L_{M+G}(X^\Delta, \boldsymbol{g})$ and*

$$P_{\mathrm{exit}} \in L_{\mathrm{cl}}^{\mu;0}(\mathbb{R}_+ \times X)_{\mathrm{exit}}.$$

Let us finally recall the notion of conormal symbols of operators in $L^\mu(X^\Delta, \boldsymbol{g})$. By that we understand the operator functions

$$\sigma_M^{\mu-j}(A)(w) = \frac{1}{j!}(\partial_r^j h)(0,w) + f_j(w), \tag{5.36}$$

$j = 0, \ldots, k$, with $k \in \mathbb{N}$ being involved in $\boldsymbol{g}$. For $j = 0$ we also write $\sigma_M(A) := \sigma_M^\mu(A)$, called the principal conormal symbol.

References

[1] M.F. Atiyah, *K-Theory*, Harvard University, Cambridge Mass.,1965.

[2] M.F. Atiyah, *Resolution of singularities and division of distributions*, Comm. Pure Appl. Math. **23** (1970), 15-150.

[3] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11-51.

[4] D. Calvo and B.-W. Schulze, *Edge symbolic structure of second generation*, Math. Nachr. **282** (2009), 348-367.

[5] D.-C.Chang, N. Habal, and B.-W. Schulze, *The edge algebra structure of the Zaremba problem*, NCTS Preprints in Mathematics **2013-6-002**, Taiwan, 2013. J. Pseudo-Differ. Oper. Appl. **5** (2014), 69-55, DOI: 10.1007/s11868-013-0088-7

[6] D.-C.Chang, T. Qian, and B.-W. Schulze, *Corner Boundary Value Problems*, Complex Analysis and Operator Theory, **9**, 5 (2014) 1157-1210. DOI:10.1007/s11785-014-0424-9.

[7] D.-C. Chang, X. Lyu, B.-W.Schulze, Recent developments on pseudo-differential operators (II), Tamkang J. Math. Vol. 46, No. 3 (2015), 281-347.

[8] S. Coriasco and B.-W. Schulze, *Edge problems on configurations with model cones of different dimensions*, Osaka J. Math. **43** (2006), 1-40.

[9] N. Dines, *Elliptic operators on corner manifolds*, Ph.D. thesis, University of Potsdam, 2006.

[10] Ch. Dorschfeldt, *Algebras of pseudo-differential operators near edge and corner singularities*, Math. Res. **102**, Akademie Verlag, Berlin, 1998.

[11] Ju. V. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications*, Oper. Theory: Adv. Appl. **93**, Birkhäuser Verlag, Basel, 1997.

[12] G.I. Eskin, *Boundary value problems for elliptic pseudodifferential equations*, Transl. of Nauka, Moskva, 1973, Math. Monographs, Amer. Math. Soc. **52**, Providence, Rhode Island 1980.

[13] H.-J. Flad, G. Harutyunyan, R. Schneider, and B.-W. Schulze, *Explicit Green operators for quantum mechanical Hamiltonians.I. The hydrogen atom*, arXiv:1003.3150v1 [math.AP], 2010. manuscripta math. **135** (2011), 497-519.

[14] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze *Explicit Green operators for quantum mechanical Hamiltonians. II. Edge singularities of the helium atom*, (in preparation).

[15] H.-J. Flad, G. Harutyunyan, and B.-W. Schulze *Asymptotic parametrices of elliptic edge operators*, arXiv: 1010.1453, 2010.J. Pseudo-Differ. Oper. Appl. **7**, 321 – 363, 2016. DOI: 10.1007/s11886-016-0159-7.

[16] H.-J. Flad, G. Harutyunyan, *Ellipticity of quantum mechanical Hamiltonians in the edge algebra*, Proc. AIMS Conference on Dynamical Systems, Differential Equations and Applications, Dresden, 2010.

[17] J.B. Gil, B.-W. Schulze, and J. Seiler, *Cone pseudo-differential operators in the edge symbolic calculus*, Osaka J. Math. **37** (2000), 221-260.

[18] J.B. Gil, B.-W. Schulze, and J. Seiler, *Holomorphic operator-valued symbols for edge-degenerate pseudo-differential operators*, Math. Res. **100**, "Differential Equations, Asymptotic Analysis, and Mathematical Physics", Akademie Verlag Berlin, 1997, pp. 113-137.

[19] R.C. Gunning and H Rossi, *Analytic Functions of Several Complex Variables*, Amer. Math. Soc (2009).

[20] N. Habal and B.-W. Schulze, *Holomorphic corner symbols*, J. Pseudo-Differ. Oper. Appl. **2**, 4 (2011), 419-465.

[21] G. Harutyunyan and B.-W. Schulze, *Elliptic mixed, transmission and singular crack problems*, European Mathematical Soc., Zürich, 2008.

[22] G. Harutyunyan and B.-W. Schulze, *The relative index for corner singularities*, Integr. Equ. Oper. Theory **54**, 3 (2006), 385-426.

[23] T. Hirschmann, *Functional analysis in cone and edge Sobolev spaces*, Ann. Global Anal. Geom. **8**, 2 (1990), 167-192.

[24] L. Hörmander, *The analysis of linear partial differential operators*, vol. 1 and 2, Springer-Verlag, New York, 1983.

[25] I.L. Hwang, *The L^2-boundedness of pseudodifferential operators*, Trans. Amer. Math. Soc. **302** (1987), 55-76.

[26] K.G. Jänich, *Vektorraumbündel und der Raum der Fredholm-Operatoren*, Math. Ann. **161** (1965), 129-142.

[27] P. Jeanquartier, *Transformation de Mellin et développements asymptotiques*, Enseign. Math. (2) **25** (1979), 285-308.

[28] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities*, Kluwer Academic Publ., Dordrecht, 2003.

[29] T. Krainer, *On the inverse of parabolic systems of partial differential equations of general form in an infinite space-time cylinder*, Oper. Theory Adv. Appl. **138**, Adv. in Partial Differential Equations "Parabolicity, Volterra Calculus, and Conical Singularities" (Albeverio, S. and Demuth, M. and Schrohe, E. and Schulze, B.-W., eds.), Birkhäuser Verlag, Basel, 2002, pp. 93-278.

[30] T. Krainer, *The calculus of Volterra Mellin pseudo-differential operators with operator-valued symbols*, Oper. Theory Adv. Appl. **138**, Adv. in Partial Differential Equations "Parabolicity, Volterra Calculus, and Conical Singularities" (S. Albeverio, M. Demuth, E. Schrohe, and B.-W. Schulze, eds.), Birkhäuser Verlag, Basel, 2002, pp. 47-91.

[31] J. Kohn, and L. Nirenberg, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 269-305.

[32] V.A. Kondratyev, *Boundary value problems for elliptic equations in domains with conical points*, Trudy Mosk. Mat. Obshch. **16**, (1967), 209-292.

[33] A. Kufner, L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific, New Jersey-London-Singapore-Hong Kong 2003.

[34] H. Kumano-go, *Pseudo-differential operators*, The MIT Press, Cambridge, Massachusetts and London, England, 1981.

[35] X. Liu and B.-W. Schulze, *Boundary value problems in edge representation*, Math. Nachr. **280**, 5-6 (2007), 1-41.

[36] X. Lyu, *Asymptotics in weighted corner spaces*, Asian-Eur. J. Math, **7**, 3 (2014) 1450050-1-1450050-36.

[37] X. Lyu and B.-W. Schulze, *Mellin operators in the edge calculus*, Complex Analysis and Operator Theory, **10**, 5 (2016), 965-1000.

[38] X. Lyu, B.-W. Schulze, and T. Qian *Order filtrations of the edge algebra*, J. Pseudo-Differ. Oper. Appl. **6**, 3 (2015), 279-305.

[39] L. Maniccia and B.-W. Schulze, *An algebra of meromorphic corner symbols*, Bull. des Sciences Math. **127**, 1 (2003), 55-99.

[40] F.W.J. Olver, *Asymptotics and special functions*, Academic Press, New York and London, 1974.

[41] V.S. Rabinovich, *Pseudo-differential operators in non-bounded domains with conical structure at infinity*, Mat. Sb. **80**, 4 (1969), 77-97.

[42] S. Rempel and B.-W. Schulze, *Parametrices and boundary symbolic calculus for elliptic boundary problems without transmission property*, Math. Nachr. **105**, (1982), 45-149.

[43] S. Rempel and B.-W. Schulze, *Asymptotics for elliptic mixed boundary problems* (*pseudo-differential and Mellin operators in spaces with conormal singularity*), Math. Res. **50**, Akademie-Verlag, Berlin, 1989.

[44] S. Rempel and B.-W. Schulze, *Complete Mellin and Green symbolic calculus in spaces with conormal asymptotics*, Ann. Glob. Anal. Geom. **4**, 2 (1986), 137-224.

[45] S. Rempel and B.-W. Schulze, *Pseudo-differential and Mellin operators in spaces with conormal singularity* (*boundary symbols*), Report R-Math-**01/84**, Karl-Weierstrass Institut, Berlin, 1984.

[46] W. Rungrotheera, *Parameter-dependent corner operators*, Asian-Eur. J. Math, **6** (2013), 1-29.

[47] W. Rungrottheera and B.-W. Schulze, *Holomorphic operator families on a manifold with edge*, J. Pseudo-Differ. Oper. Appl. **4**, 3 (2013), 297-315.

[48] W. Rungrottheera and B.-W. Schulze, *Weighted spaces on corner manifolds*, Complex Variables and Elliptic Equations, **59**, 12 (2014), 1706-1738. DOI:10.1080/17476933.2013.876416.

[49] H.H. Schäfer, *Topological vector spaces*, Graduate Texts in Mathematics, Springer-Verlag, New York, Heidelberg, Berlin, 1986.

[50] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities II*, Adv. in Partial Differential Equations “Boundary Value Problems, Schrödinger Operators, Deformation Quantization”, Akademie Verlag, Berlin, 1995, pp. 70-205.

[51] E. Schrohe and B.-W. Schulze, *Boundary value problems in Boutet de Monvel's calculus for manifolds with conical singularities I*, Adv. in Partial Differential Equations “Pseudo-Differential Calculus and Mathematical Physics”, Akademie Verlag, Berlin, 1994, pp. 97-209.

[52] E. Schrohe and B.-W. Schulze, *Edge-degenerate boundary value problems on cones*, Proc. “Evolution Equations and their Applications in Physical and Life Sciences”, Bad Herrenalb, Karlsruhe, 2000.

[53] B.-W. Schulze, *Pseudo-differential operators on manifolds with edges*, Teubner-Texte zur Mathematik, Symp. “Partial Differential Equations”, Holzhau 1988, **112**, Leipzig, 1989, pp. 259-287.

[54] B.-W. Schulze, *Pseudo-differential operators on manifolds with singularities*, North-Holland, Amsterdam, 1991.

[55] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J. Wiley, Chichester, 1998.

[56] B.-W. Schulze, *Operators with symbol hierarchies and iterated asymptotics*, Publications of RIMS, Kyoto University **38**, 4 (2002), 735-802.

[57] B.-W. Schulze, *Ellipticity and continuous conormal asymptotics on manifolds with conical singularities*, Math. Nachr. **136** (1988), 7-57.

[58] B.-W. Schulze, *Pseudo-differential boundary value problems, conical singularities, and asymptotics*, Akademie Verlag, Berlin, 1994.

[59] B.-W. Schulze and J. Seiler, *Edge operators with conditions of Toeplitz type*, J. of the Inst. Math. Jussieu, **5**, 1 (2006), 101-123.

[60] B.-W. Schulze and J. Seiler, *The edge algebra structure of boundary value problems*, Ann. Glob. Anal. Geom. **22** (2002), 197-265.

[61] B.-W. Schulze, *Elliptic complexes on manifold with conical singularities*, Teubner-Texte zur Mathematik **106**, “Seminar Analysis of the Karl-Weierstrass-Institute 1986/87”, BSB Teubner, Leipzig, 1988, pp. 170-223.

[62] B.-W. Schulze, *The iterative structure of corner operators*, arXiv: 0901.1967v1 [math.AP], 2009.

[63] B.-W. Schulze, *The iterative structure of the corner calculus*, Oper. Theory: Adv. Appl. **213**, Pseudo-Differential Operators: Analysis, Application and Computations (L. Rodino et al. eds.), Birkhäuser Verlag, Basel, 2011, pp. 79-103.

[64] B.-W. Schulze, *Edge-degenerate operators at conical exits to infinity*, Operator Theory: Advances and Applications **216**, Springer Verlag, Basel AG, 2011, pp. 41-64.

[65] B.-W. Schulze and Y. Wei, *Edge-boundary problems with singular trace conditions*, Ann. Glob. Anal. Geom. **35** (2009), 413-429.

[66] B.-W. Schulze and Y. Wei, *The Mellin-edge quantisation for corner operators*, arXiv:1201.6525v1 [math.AP]. Complex Analysis and Operator Theory, **8**, 4 (2014), 803-841; published online: 27 February 2013, 10.1007/s11785-013-0289-3.

[67] B.-W. Schulze, *Compositions in the edge calculus*, Elliptic and Parabolic Equations, Springer Proceedings in Mathematics and Statistics **119**, Springer Int. Publishing Switzerland 2015, J. Escher et al., pp. 247 - 271. DOI:10.1007/978$-3-319-12547-3_1 1$

[68] R.T. Seeley, *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625-626.

[69] R. Seeley, *Complex powers of an elliptic operator*, Proc. Sympos. Pure Math. **10** (1967), 288-307.

[70] J. Seiler, *Pseudodifferential calculus on manifolds with non-compact edges*, Ph.D. thesis, University of Potsdam, 1997.

[71] J. Seiler, *The cone algebra and a kernel characterization of Green operators*, Oper. Theory Adv. Appl. **125** Adv. in Partial Differential Equations “Approaches to Singular Analysis” (J. Gil, D. Grieser, and M. Lesch, eds.), Birkhäuser, Basel, 2001, pp. 1-29.

[72] J. Seiler, *Continuity of edge and corner pseudo-differential operators*, Math. Nachr. **205** (1999), 163-182.

[73] F. Treves, *Topological vector spaces*, Academic Press, New York, London, 1967.

[74] M.I. Vishik and G.I. Eskin, *Convolution equations in a bounded region*, Uspekhi Mat. Nauk **20**, 3 (1965), 89-152.

[75] M.I. Vishik and G.I. Eskin, *Convolution equations in bounded domains in spaces with weighted norms*, Mat. Sb. **69**, 1 (1966), 65-110.

[76] I. Witt, *On the factorization of meromorphic Mellin symbols*, Advances in Partial Differential Equations (Approaches to Singular Analysis) (S. Albeverio, M. Demuth, E. Schrohe, and B.-W. Schulze, eds.), Oper. Theory: Adv. Appl., Birkhäuser Verlag, Basel, 2002, pp. 279-306.