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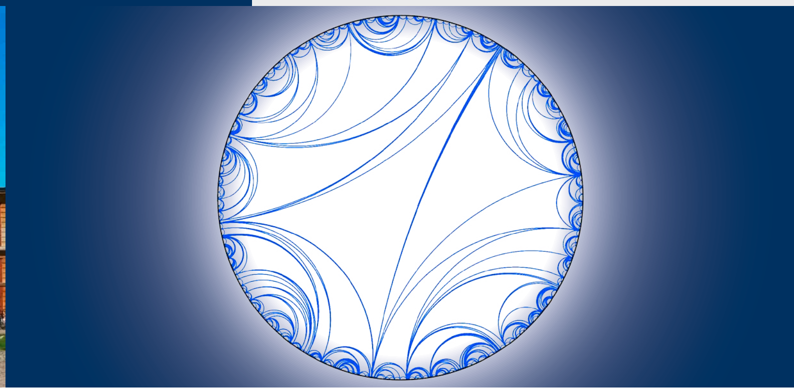

Alexander Shlapunov | Nikolai Tarkhanov

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GOLUSIN-KRYLOV FORMULAS IN COMPLEX ANALYSIS

ALEXANDER SHLAPUNOV AND NIKOLAI TARKHANOV

This paper is dedicated to the 130 th anniversary of Vladimir I. Smirnov.

Abstract. This is a brief survey of a constructive technique of analytic continuation related to an explicit integral formula of Golusin and Krylov (1933). It goes far beyond complex analysis and applies to the Cauchy problem for elliptic partial differential equations as well. As started in the classical papers, the technique is elaborated in generalised Hardy spaces also called Hardy-Smirnov spaces.

Contents

Introduction

The phenomenon of analytic continuation is of great interest in one complex variable and all of mathematics and its applications. It is well understood for data lying in the domain of analyticity. Otherwise there arises the problem in what sense the limit values should be taken on. In function theory in the unit disk $\mathbb{D} \subset \mathbb{C}$ one commonly considers the Hardy spaces $H^p(\mathbb{D})$ of holomorphic functions while the boundary data are prescribed on a subset of positive measure on the unit circle. The functions of Hardy classes converge to their limit values on the circle weakly in the sense that

$$\lim_{\varepsilon \to 0+} \int_{|\zeta|=1-\varepsilon} f(\zeta)g(\zeta)d\zeta = \int_{|\zeta|=1} f(\zeta)g(\zeta)d\zeta \tag{0.1}$$

for all smooth functions g with compact support in the plane. Equality (0.1) allows one to recover any function $f \in H^1(\mathbb{D})$ from its limit values on $\partial\mathbb{D}$.

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When considering analytic continuation from a boundary subset of a simply connected domain $\mathcal{D}$ in the complex plane, one is looking for a substitute for Hardy spaces in $\mathcal{D}$. To this end one can use the Riemann mapping theorem and parametrise the points of $\mathcal{D}$ by those of the unit disk under a conformal mapping $z = \mathfrak{c}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}$. On replacing the homothety $(1-\varepsilon)\mathbb{D}$ of $\mathbb{D}$ in (0.1) by its image by $\mathfrak{c}$ we arrive at diverse classes of holomorphic functions in $\mathcal{D}$ which can be specified through their weak limit values on the boundary of $\mathcal{D}$. A classical pattern is the scale $E^p(\mathcal{D})$ of Hardy-Smirnov spaces, where $1 \leq p \leq \infty$. A holomorphic function f on $\mathcal{D}$ belongs to $E^p(\mathcal{D})$ if and only if $\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta)) \in H^p(\mathbb{D})$. This just amounts to saying that the weak limit values of f on $\partial\mathcal{D}$ are of Lebesgue space $L^p(\partial\mathcal{D})$.

If the boundary of $\mathcal{D}$ is a rectifiable curve then the function $z = \mathfrak{c}(\zeta)$ is continuous on the closed disk $\overline{\mathbb{D}}$ and absolutely continuous on the unit circle. The scale of Hardy-Smirnov spaces has proved to suit perfectly for the study of classical boundary value problems for the Laplace equation in a plane domain bounded by a nonsmooth curve which has a finite number of singular points (see [GT13] and the references given there).

In contrast to boundary value problems for holomorphic functions similar to the Riemann-Hilbert problem, analytic continuation goes beyond the theory of Fredholm operators in Banach spaces. If the data are given on a nonempty open arc $\mathcal{S}$ in $\partial\mathcal{D}$, the problem of analytic continuation fails to be stable unless $\mathcal{S} = \partial\mathcal{D}$. This makes it a challenge for mathematicians. The focus is therefore on bounded domains $\mathcal{D}$ with smooth boundary, in which case $\mathfrak{c}'(\zeta)$ is bounded away from zero on the circle. In this case the Hardy-Smirnov spaces in $\mathcal{D}$ reduce to push-forwards of Hardy spaces in $\mathbb{D}$.

Hardy spaces have been of limited interest in the general theory of partial differential equations because of their rigidity. To some extent they are counterparts to Sobolev spaces $H^{1/p,p}(\mathcal{D})$, as if the trace theorem still holds for them. On the other hand, using Hardy spaces makes it technically easy to reduce a boundary value problem to an overdetermined system of integral equations in Lebesgue spaces L^p on the boundary.

It was G. M. Golusin, a PhD student of V. I. Smirnov, and V. I. Krylov [GK33], who found an abstract idea of suppressing function to construct an explicit formula for analytic continuation in Hardy spaces. They named their formula after T. Carleman who had used a simple trick to restore a holomorphic function in an angle on its bisectrix through the values of the function on an arc connecting both sides of the angle, see [Car26]. In [VGK83] the contribution of [GK33] was restored fairly in the designation of the formula.

Since then the formula of [GK33] attracted considerable attention of researchers in complex analysis. The monograph [Aiz93] sums up the results in this direction to a great extent, including those for functions of several variables. Moreover, the formula of [GK33] was precisely specified within the framework of ill-posed problems of analysis and geometry. The book [Tar95] gives an introduction into Hardy spaces of solutions to elliptic equations and contains a discussion of basic facts on explicit formulas for solutions to instable problems of mathematical physics up to date. Still, since the monographs were published there appeared a number of new developments of [GK33].

Among the new contributions into the Golusin-Krylov formulas in function theory we mention a numerical experiment with the Riemann hypothesis on zeros of the zeta function, see [Tar12]. In several complex variables the new developments concern explicit formulas for the Dolbeault cohomology classes through their values on a part of the boundary, see [Tar10]. Another formulation of the Cauchy problem for the Dolbeault complex and the related Goluzin-Krylov type formulas can be found in [FS13]. In the Cauchy problem for elliptic equations these are perhaps substantial applications of explicit formulas in inverse problems of cardiology, see [Sea06, Ch. 7]. They actually give some evidence to a familiar argument of M. A. Evgrafov and M. M. Postnikov from the early 1970 s that each result of pure mathematics finds its first applications not earlier than in 50 years. In boundary value problems for elliptic equations the progress consists in the development of boundary equation method whose explicit formulas are actually counterparts of Golusin-Krylov formulas, see [AT16]. Finally, one should mention Carleman type formulas for solutions of nonstandard Cauchy problems for parabolic equations, see [Ike09], [MMT16].

In this survery we focus on these new developments of results of the Saint Petersburg school of mathematics, which go back at least as far as V. I. Smirnov.

1. The problem of analytic continuation

Let $\mathcal{D}$ be a bounded simply connected domain in $\mathbb{C}$ whose boundary is a rectifiable Jordan curve. A holomorphic function f in $\mathcal{D}$ possesses angular limit values almost everywhere on $\partial\mathcal{D}$ and can be restored through these limit values by means of the Cauchy integral

$$f(z) = \int_{\partial\mathcal{D}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f(\zeta) d\zeta \tag{1.1}$$

for $z \in \mathcal{D}$ if and only if the integral

$$\int_{|\mathfrak{c}^{-1}(z)|=r} |f(z)|\,|dz|$$

is bounded uniformly in $0 < r < 1$. In this way we recover once again the Hardy-Smirnov space $E^1(\mathcal{D})$.

Theorem 1.1. *A holomorphic function f in $\mathcal{D}$ belongs to the Hardy-Smirnov space $E^1(\mathcal{D})$ if and only if it possesses angular limit values almost everywhere on $\partial\mathcal{D}$ and f can be restored through these limit values by Cauchy integral (1.1).*

Proof. This is a familiar result of V. I. Smirnov published in 1928. A proof can be found in [Gol66]. □

Suppose $f \in E^p(\mathcal{D})$, where $p \geq 1$. Then the angular limit values of f on $\partial\mathcal{D}$ constitute a function in $L^p(\partial\mathcal{D})$. By Theorem 1.1 f is the Cauchy integral (1.1) of these limit values. Hence it follows, that functions of $E^p(\mathcal{D})$ are Cauchy integrals of L^p -functions on the boundary curve. For non-extreme values $1 < p < \infty$ the inverse assertion is also true, and so the Hardy-Smirnov space $E^p(\mathcal{D})$ can be identified with a closed subspace of $L^p(\partial\mathcal{D})$, see [Dav82].

Corollary 1.2. *Let $g \in E^q(\mathcal{D})$, where $1 \leq q \leq \infty$. For each function $f \in E^p(\mathcal{D})$ mit $1/p + 1/q = 1$ it follows that*

$$\int_{\partial\mathcal{D}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \frac{g(\zeta)}{g(z)} f(\zeta) d\zeta = \begin{cases} f(z), & \textit{if} \quad z \in \mathcal{D}, \\ 0, & \textit{if} \quad z \in \mathbb{C} \setminus \overline{\mathcal{D}}. \end{cases}$$

This is a generalisation of the Cauchy integral formula. One certainly assumes that $g(z) \neq 0$, otherwise the left hand side of the formula does not make any proper sense.

Proof. By the Hölder inequality we conclude immediately that $fg \in E^1(\mathcal{D})$. Theorem 1.1 now implies that fg can be represented in $\mathcal{D}$ by the Cauchy integral formula. On applying (1.1) for fg and dividing both sides by $g(z) \neq 0$ one derives the formula for all $z \in \mathcal{D}$. For $z \notin \overline{\mathcal{D}}$ the desired equality follows from the Stokes formula. □

Note that whenever $z \in \mathcal{D}$ and $g(z) \neq 0$ we get

$$\frac{1}{2\pi\imath}\frac{1}{\zeta - z}\frac{g(\zeta)}{g(z)} = \frac{1}{2\pi\imath}\frac{1}{\zeta - z} + r_z(\zeta) \tag{1.2}$$

for all $\zeta \in \mathcal{D}$, where

$$r_z(\zeta) := \frac{1}{2\pi\imath}\frac{g(\zeta) - g(z)}{\zeta - z}\frac{1}{g(z)}$$

has a removable singularity at the point z. Hence, r_z is also a function of class $E^p(\mathcal{D})$.

Suppose f is a function on $\mathcal{D}$ of Hardy-Smirnov class $E^1(\mathcal{D})$. We look for an analytic formula which enables us to recover f in all of $\mathcal{D}$ through its angular limit values on a measurable subset $\mathcal{S} \subset \partial\mathcal{D}$ of positive measure. If the values $f(z)$ are known almost everywhere on the boundary, one can use for example the Cauchy integral.

The first formula of this kind was seemingly constructed by T. Carleman in his book [Car26]. As $\mathcal{D}$ he took an angular domain bounded through two sides of an angle and a rectifiable arc between the sides (denoted by $\mathcal{S}$). The formula allowed one to reconstruct a holomorphic function f in $\mathcal{D}$ on the hypotenuse through the values of f on $\mathcal{S}$. An idea of [Car26] was to introduce an additional function in the Cauchy integral formula, which suppressed the contribution of the integral over the complementary set $\partial\mathcal{D} \setminus \mathcal{S}$ through a limit passage. The abstract idea was perhaps first clarified later in [GK33], where it was designated as the idea of a suppressing function.

Suppose h is a bounded holomorphic function in $\mathcal{D}$ with the property that
1) $|h(z)| = 1$ almost everywhere on $\partial\mathcal{D} \setminus \mathcal{S}$;
2) $|h(z)| > 1$ for all $z \in \mathcal{D}$.

In Section 2 we discuss concrete cases where one can construct such a function explicitly.

We now consider the function

$$g(z) = (h(z))^\sigma = \exp(\sigma \log h(z)) \tag{1.3}$$

of $z \in \mathcal{D}$, where σ is a positive number. This function is holomorphic and bounded in $\mathcal{D}$, hence

$$\begin{aligned} f(z) &= \int_{\partial\mathcal{D}} \frac{1}{2\pi\imath}\frac{1}{\zeta - z}\Big(\frac{h(\zeta)}{h(z)}\Big)^\sigma f(\zeta)d\zeta \\ &= \int_{\partial\mathcal{D}} \frac{1}{2\pi\imath}\frac{1}{\zeta - z}\Big(\frac{h(\zeta)}{h(z)}\Big)^\sigma f(\zeta)d\zeta + \int_{\partial\mathcal{D}\setminus\mathcal{S}} \frac{1}{2\pi\imath}\frac{1}{\zeta - z}\Big(\frac{h(\zeta)}{h(z)}\Big)^\sigma f(\zeta)d\zeta \end{aligned}$$

for all $z \in \mathcal{D}$, which is due to the generalised Cauchy formula of Corollary 1.2. Since $|h(z)| = 1$ almost everywhere on $\partial\mathcal{D} \setminus \mathcal{S}$ and $|h(z)| > 1$ for all $z \in \mathcal{D}$, the integral

$$\int_{\partial\mathcal{D}\setminus\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} f(\zeta) d\zeta$$

tends to zero uniformly in z on compact subsets of $\mathcal{D}$, when $\sigma \to \infty$. We thus obtain

Theorem 1.3. *Suppose h is a suppressing function in $\mathcal{D}$. For each $f \in E^1(\mathcal{D})$ it follows that*

$$f(z) = \lim_{\sigma\to\infty} \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} f(\zeta) d\zeta \qquad (1.4)$$

uniformly in z on compact subsets of $\mathcal{D}$.

Furthermore one obtains

$$\begin{aligned} &\lim_{\sigma\to\infty} \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} f(\zeta) d\zeta \\ &= \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f(\zeta) d\zeta + \int_0^{\infty} d\sigma \frac{d}{d\sigma} \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} f(\zeta) d\zeta \\ &= \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f(\zeta) d\zeta + \int_0^{\infty} d\sigma \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} \log\frac{h(\zeta)}{h(z)} f(\zeta) d\zeta, \end{aligned}$$

the differentiation under the integral sigh is obviously possible here. Therefore, one can write instead of (1.4)

$$f(z) = \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f(\zeta) d\zeta + \int_0^{\infty} d\sigma \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{h(\zeta)}{h(z)}\Big)^{\sigma} \log\frac{h(\zeta)}{h(z)} f(\zeta) d\zeta \quad (1.5)$$

for all $z \in \mathcal{D}$.

The presence of limit passage in (1.4) or unproper integral in (1.5) displays the instable character of the problem of analytic continuation from a proper subset $\mathcal{S}$ of the boundary curve, see [Aiz93].

2. The Goluzin-Krylov formula

The Riemann mapping theorem yields readily a construction of suppressing function h in the case, when $\mathcal{S}$ is an arc of the boundary curve $\partial\mathcal{D}$, see [GK33]. More precisely, let $\mathcal{D}$ be a simply connected domain in $\mathbb{C}$ with rectifiable boundary curve $\partial\mathcal{D}$. Let moreover $\mathcal{S}$ be a nonempty subset of $\partial\mathcal{D}$. We choose a larger simply connected domain $\mathcal{D}' \subsetneq \mathbb{C}$ which contains $\mathcal{D}$ and whose boundary curve is formed by the closure of $\partial\mathcal{D} \setminus \mathcal{S}$ and by a rectifiable curve in the complement of $\overline{\mathcal{D}}$. Fix $z_0 \in \mathcal{D}' \setminus \overline{\mathcal{D}}$. By the Riemann theorem there is precisely one bijective conformal mapping $w : \mathcal{D}' \to \mathbb{D}$, such that

$$\begin{aligned} w(z_0) &= 0, \\ w'(z_0) &> 0, \end{aligned}$$

see Fig. 1.

Theorem 2.1. *Under the above notation, let f be an E^1-function in $\mathcal{D}$. Then it follows that*

$$f(z) = \lim_{\sigma\to\infty} \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{w(z)}{w(\zeta)}\Big)^{\sigma} f(\zeta) d\zeta \qquad (2.1)$$

uniformly in z on compact subsets of $\mathcal{D}$.

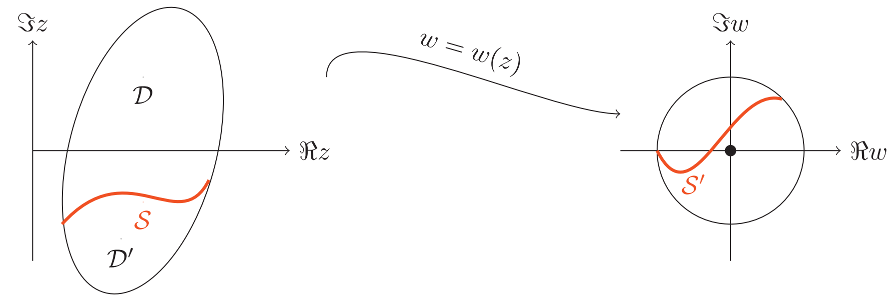

Fig. 1. Conformal image of $\mathcal{D}$.

Proof. It suffices to show that $h(z) = 1/w(z)$ is a suppressing function on $\mathcal{D}$. By the construction of $w = w(z)$ we get $|h(z)| = 1/|w(z)| \geq 1$ for all z in the closure of $\mathcal{D}'$. Since $w = w(z)$ maps the boundary of $\mathcal{D}'$ onto the unit circle $|w| = 1$, it follows that $|h(z)| = 1$ for all $z \in \partial\mathcal{D} \setminus \mathcal{S}$. Furthermore, $|h(z)| > 1$ holds for all $z \in \mathcal{D}$, for $w = w(z)$ maps the domain $\mathcal{D}'$ onto $\mathbb{D}$. Finally the function $h(z)$ is uniformly bounded in $\mathcal{D}$, for the image $w(\overline{\mathcal{D}})$ is bounded away from the origin. □

By (1.5), formula (2.1) can be equivalently formulated as

$$f(z) = \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f(\zeta) d\zeta + \int_0^\infty d\sigma \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} \Big(\frac{w(z)}{w(\zeta)} \Big)^\sigma \log \frac{w(z)}{w(\zeta)} f(\zeta) d\zeta \quad (2.2)$$

for $z \in \mathcal{D}$.

In particular, one can choose $w(z) = z$ in the case where $\mathcal{D}$ is a moon-shaped domain in $\mathbb{D}$ with $0 \notin \overline{\mathcal{D}}$ and $\mathcal{S} := \partial\mathcal{D} \cap \mathbb{D}$, see [Aiz93].

3. Criteria for analytic continuation

To our best knowledge, Theorem 2.1 gives the simplest explicit formula for analytic continuation in complex analysis. Based upon this formula, we show a criterion of analytic continuability into $\mathcal{D}$ for a function f_0 given on a nonempty open piece $\mathcal{S}$ of the boundary $\partial\mathcal{D}$. While polynomials of z are dense in the Banach space $C(\overline{\mathcal{S}})$ unless $\overline{\mathcal{S}} = \partial\mathcal{D}$, those functions on $\overline{\mathcal{S}}$ which extend analytically to $\mathcal{D}$ form a subspace of infinite codimension in $C(\overline{\mathcal{S}})$. In particular, a continuous functions $f_0 \not\equiv 0$ of compact support in $\mathcal{S}$ fail to have analytic continuation to the domain $\mathcal{D}$, which is a consequence of a familiar uniqueness theorem. The following result is due to [Aiz95].

Theorem 3.1. *Let $f_0 \in C(\overline{\mathcal{S}})$ satisfy $f_0 \not\equiv 0$. In order that there be a holomorphic function $f \in C(\mathcal{D} \cup \mathcal{S})$ in $\mathcal{D}$, such that $f(z) = f_0(z)$ for all $z \in \mathcal{S}$, it is necessary and sufficient that*

$$\limsup_{n\to\infty} \sqrt[n]{\Big| \int_{\mathcal{S}} \frac{f_0(z)}{(w(z))^{n+1}} dw(z) \Big|} = 1. \quad (3.1)$$

It suffices to prove this theorem in the case where $\mathcal{S}$ is a regular curve in $\mathbb{D}$, whose endpoints lie on the unit circle and which does not run through 0 (i.e. $0 \notin \mathcal{S}$). The curve $\mathcal{S}$ divides the disk $\mathbb{D}$ into two domains and we write $\mathcal{D}$ for the subdomain of $\mathbb{D}$ that does not contain the origin 0. In this way we obtain a bounded domain with piecewise smooth boundary which is referred to as lune. The boundary of $\mathcal{D}$ consists of two parts, one of the two is the curve $\mathcal{S}$ and the other an arc of the circle $\partial\mathbb{D}$, cf. Fig. 1. As is mentioned, in this case formula (2.1) holds for the identity map $w(z) := z$.

Proof. Necessity. Given a nonzero function $f_0 \in C(\overline{\mathcal{S}})$, we define the Cauchy-type integral of f_0 by

$$Cf_0\,(z) = \int_{\mathcal{S}} \frac{1}{2\pi\imath} \frac{1}{\zeta - z} f_0(\zeta) d\zeta$$

for $z \notin \overline{\mathcal{S}}$. This is a holomorphic function away from the closure of $\mathcal{S}$, and we denote by $C^{\pm} f_0$ the restrictions of Cf_0 to $\mathcal{D}$ and $\mathbb{C}\setminus\overline{\mathcal{D}}$, respectively. The Sokhotsky-Plemelj formula says that

$$\lim_{\varepsilon \to 0+} \Big(C^{+} f_0\,(\zeta + \varepsilon\nu(\zeta)) - C^{-} f_0\,(\zeta - \varepsilon\nu(\zeta)) \Big) = f_0(\zeta) \tag{3.2}$$

holds uniformly in ζ on compact subsets of $\mathcal{S}$, where $\nu(\zeta)$ is the inward unit normal vector to $\mathcal{S}$ at a point $\zeta \in \mathcal{S}$. In particular, if either of the functions $C^{\pm} f_0$ extends continuously to $\mathcal{S}$ then so does the other function. The limit in (3.2) is obviously zero, if $\zeta \in \partial\mathcal{D} \setminus \overline{\mathcal{S}}$.

Assume that there is a holomorphic function f in $\mathcal{D}$ which is continuous up to $\mathcal{S}$ and satisfies $f = f_0$ on $\mathcal{S}$. A simple manipulation with the Cauchy integral formula for f shows that the difference $C^{+} f_0 - f$ extends to a continuous (even C^{∞}) function on $\mathcal{D} \cup \mathcal{S}$. Since f is continuous on $\mathcal{D} \cup \mathcal{S}$, the integral $C^{+} f_0$ extends to a continuous function on $\mathcal{D} \cup \mathcal{S}$. By the above, $C^{-} f_0$ extends continuously to $\mathbb{D} \setminus \mathcal{D}$, too.

Consider the function

$$F(z) = \begin{cases} C^{+} f_0\,(z) - f(z), & \text{if} \quad z \in \mathcal{D} \cup \mathcal{S}, \\ C^{-} f_0\,(z), & \text{if} \quad z \in \mathbb{D} \setminus \overline{\mathcal{D}}, \end{cases}$$

in the disk $\mathbb{D}$. This function is holomorphic in $\mathbb{D} \setminus \mathcal{S}$ and continuous on all of $\mathbb{D}$, for $C^{+} f_0 - f = C^{-} f_0$ on $\mathcal{S}$, which is due to Sokhotsky-Plemelj formula (3.2). From the Morera theorem we easily deduce that F is actually holomorphic in the unit disk $\mathbb{D}$. Hence, the Taylor series of this function around the origin converges in all of $\mathbb{D}$. The series looks like

$$F(z) = \sum_{n=0}^{\infty} c_n z^n \tag{3.3}$$

for $|z| < 1$, where

$$c_n = \frac{1}{2\pi\imath} \int_{\mathcal{S}} \frac{f_0(\zeta)}{\zeta^{n+1}}\, d\zeta,$$

for $F = C^{-} f_0$ nearby the origin. From the Cauchy-Hadamard formula for the convergence radius of power series we readily conclude that $\limsup \sqrt[n]{|c_n|} \leq 1$. If this limit is less than 1, then the series (3.3) converges in a a disk about the origin of radius greater than 1. Hence, $C^{-} f_0$ extends to a holomorphic function in a neighbourhood of the closure of $\mathcal{S}$, and so does Cf_0. On applying the Sokhotsky-Plemelj formula once again we see that $f_0 \equiv 0$ on $\mathcal{S}$, a contradiction. This establishes (3.1), as desired.

Sufficiency. To prove the converse theorem, let f_0 be a continuous function on the closure of $\mathcal{S}$ satisfying (3.1). By assumption, the integral $C^{-} f_0$ is holomorphic in a disk of sufficiently small radius $\varepsilon > 0$ around the origin (take $\varepsilon < \operatorname{dist}(0, \mathcal{S})$).

Hence, $C^- f_0$ expands in this small disk as a power series whose coefficients are given by

$$c_n = \frac{1}{2\pi\imath} \int_{\mathcal{S}} \frac{f_0(\zeta)}{\zeta^{n+1}}\, d\zeta,$$

cf. (3.3). Condition (3.1) forces the power series (3.3) to actually converge in the unit disk $\mathbb{D}$ to a holomorphic function F. By the uniqueness theorem, the integral $C^- f_0$ extends holomorphically to all of $\mathbb{D}$, and this analytic continuation is F. Hence it follows that the integral $C^+ f_0$ extends to a continuous function on $\mathcal{D} \cup \mathcal{S}$. We now set

$$f(z) := C^+ f_0\,(z) - F(z)$$

for $z \in \mathcal{D} \cup \mathcal{S}$, thus obtaining a holomorphic function in $\mathcal{D}$ which is continuous up to $\mathcal{S}$ and satisfies $f(z) = f_0(z)$ for all $z \in \mathcal{S}$, as desired. □

The Riemann hypothesis is equivalent to the fact that the reciprocal function $1/\zeta(s)$ extends from the interval $(1/2, 1)$ to an analytic function in the quarter-strip $1/2 < \Re s < 1$, $\Im s > 0$. Theorem 3.1 allows one to rewrite the condition of analytic continuability in an elegant form amenable to numerical experiments, see [AAL99], [Tar12].

4. Analytic continuation in higher dimensions

For n-dimensional vectors $v_1, \ldots, v_N$ with entries in a ring and nonnegative integers $n_1, \ldots, n_N$ with $n_1 + \ldots + n_N = n$, we denote by $D_{n_1,\ldots,n_N}(v_1, \ldots, v_N)$ the determinant of order n whose first n_1 columns are v_1, the next n_2 columns are v_2 etc., the last n_N columns are v_N. We compute the determinant by columns, i.e., we define

$$\det(v_{ij}) = \sum_I (-1)^{\varepsilon_I} v_{i_1 1} \ldots v_{i_n n}$$

where ε_I denotes the signature of the permutation $I = (i_1, \ldots, i_n)$ of the integers $(1, \ldots, n)$.

Let $v = v(z, \zeta, t)$ be a smooth function on $O \times [0, 1]$ with values in $\mathbb{C}^n$, U being an open set not intersecting the diagonal $\{z = \zeta\}$ in $\mathbb{C}^n_z \times \mathbb{C}^n_\zeta$. Fix $0 \leq p \leq n$. Consider the double differential forms $K^{(p)}_q(v)$ of bidegree $(p, q-1)$ in z and $(n-p, n-q)$ in ζ, t given by

$$\begin{aligned} K^{(p)}_q(v) \;&=\; \frac{(-1)^{q+(n-p)(q-1)}}{(2\pi i)^n n!} \binom{n}{p} \binom{n-1}{q-1} \\ &\times\;\; D_{p,n-p}(\partial z, \partial \zeta) \wedge D_{1,q-1,n-q}\left(v, \bar{\partial}_z v, (\bar{\partial}_\zeta + d_t) v\right), \end{aligned} \tag{4.1}$$

for $1 \leq q \leq n$, and $K^{(p)}_0 \equiv K^{(p)}_{n+1} \equiv 0$.

The double forms (4.1) were first introduced by Koppelman [Kop67]. Here we rehearse some elementary properties of these forms.

Lemma 4.1. *For each smooth function f on $U \times [0, 1]$, we have the equality $K^{(p)}_q(fv) = f^n K^{(p)}_q(v)$.*

Proof. Indeed, if ∂ is one of the differentials $\bar{\partial}_z$, $\bar{\partial}_\zeta$ and d_t, then the Leibniz formula yields $\partial(fv) = (\partial f)v + f\partial v$. As the vector $(\partial f)v$ is proportional to v, it gives no contribution to the last determinant on the right-hand side of (4.1). This proves the lemma. □

In particular, if v satisfies $\langle v, \zeta - z\rangle \neq 0$ pointwise on the set $U \times [0,1]$, then

$$K_q^{(p)}\left(\frac{v}{\langle v, \zeta - z\rangle}\right) = \frac{1}{\langle v, \zeta - z\rangle^n}\, K_q^{(p)}(v)$$

where $\langle \cdot, \cdot\rangle$ denotes the standard bilinear form $\mathbb{C}^n \otimes \mathbb{C}^n \to \mathbb{C}$. Thus, when considering a vector-valued function v with the property that $\langle v, \zeta - z\rangle \neq 0$ on the set $U \times [0,1]$, after multiplication by a nonzero function we may actually assume that $\langle v, \zeta - z\rangle = 1$.

Lemma 4.2. *Suppose v satisfies $\langle v, \zeta - z\rangle = 1$ on $U \times [0,1]$. Then, the equality holds*

$$\left(\bar\partial_\zeta + d_t\right) K_{q+1}^{(p)}(v) = (-1)^{p+q}\, \bar\partial_z K_q^{(p)}(v). \tag{4.2}$$

Proof. See for instance Lemma 1.2 in [AD83] and elsewhere. □

Note that if $v_j = v_j(z,\zeta)$, $j = 0,1$, are smooth functions on u with values in $\mathbb{C}^n$, both satisfying $\langle v_j, \zeta - z\rangle = 1$ on U, then the linear homotopy $v_t = (1-t)v_0 + tv_1$ between them still satisfies $\langle v_t, \zeta - z\rangle = 1$ on the set $U \times [0,1]$. The lemma below allows nonlinear homotopies, too.

Lemma 4.3. *Let v satisfy $\langle v, \zeta - z\rangle = 1$ on $U \times [0,1]$. Write v_0 and v_1 for the values of v at $t = 0$ and $t = 1$, respectively. Then*

$$K_{q+1}^{(p)}(v_1) - K_{q+1}^{(p)}(v_0) = \bar\partial_z I_{q+1}^{(p)}(v) - (-1)^{p+q}\bar\partial_\zeta I_{q+2}^{(p)}(v), \tag{4.3}$$

on the set U, where $I_i^{(p)}(v) = (-1)^{p+(i-1)} \displaystyle\int_0^1 (\partial/\partial t)\rfloor K_{i-1}^{(p)}(v)\, dt.$

Proof. It suffices to integrate equality (4.2) over $t \in [0,1]$ and take into account that

$$\bar\partial_\zeta \int_0^1 (\partial/\partial t)\rfloor K_{q+1}^{(p)}(v)\, dt = -\int_0^1 (\partial/\partial t)\rfloor \bar\partial_\zeta K_{q+1}^{(p)}(v)\, dt$$

because $\bar\partial_\zeta$ and d_t anticommute. □

There is a universal solution to the equation $\langle v, \zeta - z\rangle = 1$ outside of the diagonal in $\mathbb{C}^n_z \times \mathbb{C}^n_\zeta$, given by

$$v_1(z,\zeta) = \frac{\overline{\zeta - z}}{|\zeta - z|^2}$$

for $z \neq \zeta$. Under this choice of v, the double forms $K_q^{(p)}(v)$ fit together to give a fundamental solution of convolution type to the Dolbeault complex on $\mathbb{C}^n$.

Lemma 4.4. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$ with a piecewise smooth boundary and $f \in C^1(\Lambda^{p,q} T^*_{\mathbb{C}} \overline{\mathcal{D}})$. Then,*

$$-\int_{\partial\mathcal{D}} f \wedge K_{q+1}^{(p)}(v_1) + \int_{\mathcal{D}} \bar\partial f \wedge K_{q+1}^{(p)}(v_1) + \bar\partial \int_{\mathcal{D}} f \wedge K_q^{(p)}(v_1) = \chi_{\mathcal{D}} f, \tag{4.4}$$

where $\chi_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$.

Proof. Cf. the original paper [Kop67]. For a thorough treatment we also refer the reader to [AD83]. □

More precisely, let $\mathcal{D}$ be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary. This domain is called linearly convex at a boundary point $\zeta \in \partial\mathcal{D}$ if there exists a complex hyperplane $H_\zeta = \{z \in \mathbb{C}^n : \langle v, z - \zeta\rangle = 0\}$ through ζ which does not meet $\mathcal{D}$.

Pick an open set $\mathcal{S}$ on the boundary of $\mathcal{D}$, such that $\mathcal{D}$ is linearly convex at each point of $\partial\mathcal{D} \setminus \mathcal{S}$. We thus get a distribution H_ζ of hyperplanes in $T_{\mathbb{C}}\overline{\mathcal{D}}$ parametrised by the points $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$.

Assume that $v(\zeta)$ extends to a smooth function in the closure of $\mathcal{D}$, such that no hyperplane H_ζ with $\zeta \in \overline{\mathcal{D}}$ passes through a fixed point $a \in \mathbb{C}^n$. In other words, $\langle v(\zeta), a - \zeta\rangle \neq 0$ holds for all $\zeta \in \overline{\mathcal{D}}$.

Set

$$v_0(z, \zeta) = \frac{v(\zeta)}{\langle v(\zeta), \zeta - z\rangle},$$

thus obtaining a smooth function of $(z, \zeta) \in \mathbb{C}^n \times \overline{\mathcal{D}}$ away from the null set of the denominator $\langle v(\zeta), \zeta - z\rangle$. By assumption, v_0 is smooth on the set $\mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$, and so we readily find

$$I_2((1-t)v_0 + tv_1) = \frac{(-1)^n}{(2\pi\imath)^n} d\zeta \wedge \sum_{k=0}^{n-2} D_{1,1,k,n-2-k}\Big(v_0, v_1, \bar\partial_\zeta v_0, \bar\partial_\zeta v_1\Big)$$

for all $(z, \zeta) \in \mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$.

Theorem 4.5. *Under the above assumptions, if $f \in C(\overline{\mathcal{D}})$ is holomorphic in $\mathcal{D}$, then*

$$\begin{aligned} f(z) &= -\int_{\partial\mathcal{S}} f(\zeta)\, I_2\Big((1-t)v_0 + tv_1\Big) \\ &\quad - \lim_{N\to\infty} \int_{\mathcal{S}} f(\zeta)\Big(K_1(v_1) - \Big(1 - \Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0)\Big) \end{aligned} \tag{4.5}$$

for all $z \in \mathcal{D}$ satisfying $\displaystyle\sup_{\zeta\in\partial\mathcal{D}\setminus\mathcal{S}} \Big|\frac{\langle v(\zeta), z-a\rangle}{\langle v(\zeta), \zeta - a\rangle}\Big| < 1.$

Proof. On applying the Bochner-Martinelli formula (cf. (4.4) for $p = q = 0$) we obtain

$$f(z) = -\int_{\partial\mathcal{D}} f(\zeta)\, K_1(v_1)$$

for $z \in \mathcal{D}$. Write the integral on the right-hand side as the sum of two integrals, the first of the two being over $\mathcal{S}$ and the second being over $\partial\mathcal{D} \setminus \mathcal{S}$. For $z \in \mathcal{D}$ and $\zeta \in \partial\mathcal{D} \setminus \mathcal{S}$, we use (4.3) for $p = q = 0$, to get $K(v_1) = K(v_0) - \bar\partial_\zeta I_2((1-t)v_0 + tv_1)$. This yields

$$\begin{aligned} f(z) &= -\int_{\mathcal{S}} f(\zeta)\, K_1(v_1) - \int_{\partial\mathcal{D}\setminus\mathcal{S}} f(\zeta)\Big(K_1(v_0) - \bar\partial_\zeta I_2((1-t)v_0 + tv_1)\Big) \\ &= -\int_{\partial\mathcal{S}} f(\zeta)\, I_2((1-t)v_0 + tv_1)\Big) - \int_{\mathcal{S}} f(\zeta)\, K_1(v_1) - \int_{\partial\mathcal{D}\setminus\mathcal{S}} f(\zeta)\, K_1(v_0) \end{aligned} \tag{4.6}$$

for each $z \in \mathcal{D}$, which is due to Stokes' formula. It remains to transform the last integral.

By Lemma 4.1,

$$K_1(v_0) = \Big(\frac{1}{\langle v, \zeta - z\rangle}\Big)^n K_1(v)$$

and furthermore

$$\begin{aligned}\frac{1}{\langle v, \zeta - z\rangle} &= \frac{1}{\langle v, \zeta - a\rangle}\,\frac{1}{1-\dfrac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}}\\ &= \lim_{N\to\infty}\Big(1-\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)\,\frac{1}{\langle v, \zeta - z\rangle},\end{aligned}$$

the limit exists because $\Big|\dfrac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big| < 1$ holds for all $\zeta \in \partial\mathcal{D}\setminus\mathcal{S}$. Hence it follows that

$$K_1(v_0) = \lim_{N\to\infty}\Big(1-\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0), \tag{4.7}$$

each member of the sequence being smooth on the closure of $\mathcal{D}$, for no hyperplane $\langle v(\zeta), z-\zeta\rangle = 0$ passes through a whenever $\zeta \in \overline{\mathcal{D}}$.

Our next goal is to show that each member of the sequence in (4.7) is a $\bar\partial$-closed differential form in $\mathcal{D}$. Since the differential forms are smooth on $\overline{\mathcal{D}}$, it suffices to verify this only for those $\zeta \in \mathcal{D}$ which satisfy $\langle v(\zeta), \zeta - z\rangle \neq 0$. When differentiating the form

$$F_N = \Big(1-\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n K_1(v_0),$$

we take into account that $\bar\partial_\zeta K_1(v_0) = 0$, which is a consequence of Lemma 4.2. It follows that

$$\begin{aligned}\bar\partial F_N &= \bar\partial_\zeta\Big(1-\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^n \wedge K_1(v_0)\\ &= n\Big(1-\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^{N+1}\Big)^{n-1}(-1)(N+1)\Big(\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\Big)^N \bar\partial_\zeta\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\wedge K_1(v_0)\end{aligned}$$

and

$$\begin{aligned}\bar\partial_\zeta\frac{\langle v, z-a\rangle}{\langle v, \zeta - a\rangle}\wedge K_1(v_0) &= \frac{\langle\bar\partial_\zeta v, z-a\rangle\langle v, \zeta-a\rangle - \langle v, z-a\rangle\langle\bar\partial_\zeta v, \zeta-a\rangle}{\langle v, \zeta-a\rangle^2}\wedge K_1(v_0)\\ &= (-1)^{n-1}\frac{(n-1)!}{(2\pi\imath)^n}\,\frac{\langle v, z-a\rangle\langle v, \zeta-a\rangle - \langle v, z-a\rangle\langle v, \zeta-a\rangle}{\langle v, \zeta-a\rangle^2\langle v, \zeta-z\rangle^n}\,d\zeta\wedge\bigwedge_{j=1}^n\bar\partial_\zeta v_j\\ &= 0,\end{aligned}$$

as desired.

Combining (4.7) with Stokes' formula yields

$$\begin{aligned}\int_{\partial\mathcal{D}\setminus\mathcal{S}} f(\zeta)\,K_1(v_0) &= \lim_{N\to\infty}\int_{\partial\mathcal{D}\setminus\mathcal{S}} f(\zeta)\,F_N\\ &= \lim_{N\to\infty}\Big(\int_{\partial\mathcal{D}} f(\zeta)\,F_N - \int_{\mathcal{S}} f(\zeta)\,F_N\Big)\\ &= -\lim_{N\to\infty}\int_{\mathcal{S}} f(\zeta)\,F_N,\end{aligned}$$

for every fF_N is $\bar{\partial}$-closed in $\mathcal{D}$ and continuous up to the boundary. On substituting this into (4.6) we arrive at (4.5). □

In particular, let $\mathbb{B}$ be the unit ball around the origin in $\mathbb{C}^n$ and $\mathcal{S}$ a smooth surface in $\mathbb{B} \setminus \{0\}$ which divides the ball into two domains. Denote by $\mathcal{D}$ the domain which does not contain the origin. We simply choose $v(\zeta) = \overline{\zeta}$ and $a = 0$, obtaining

$$\sup_{\zeta \in \partial\mathcal{D} \setminus \mathcal{S}} \Big| \frac{\langle \overline{\zeta}, z \rangle}{\langle \overline{\zeta}, \zeta \rangle} \Big| \leq |z|$$

which is less than 1 for all $z \in \mathcal{D}$. Then, given any function $f \in C(\overline{\mathcal{D}})$ holomorphic in $\mathcal{D}$, the formula holds

$$\begin{aligned} f(z) &= \int_{\partial\mathcal{S} \times [0,1]} f(\zeta)\, K_1\Big((1-t)v_0 + tv_1\Big) \\ &\quad - \lim_{N \to \infty} \int_{\mathcal{S}} f(\zeta) \Big(K_1(v_1) - \Big(1 - \Big(\frac{\langle \overline{\zeta}, z \rangle}{|\zeta|^2}\Big)^{N+1}\Big)^n K_1(v_0) \Big) \end{aligned}$$

for all $z \in \mathcal{D}$.

5. The Cauchy problem for the Dolbeault cohomology

We now return to the general setting of Theorem 4.5. The same proof still goes when we drop the assumptions $q = 0$, thus studying $\bar{\partial}$-closed differential forms of bidegree (p, q) in $\mathcal{D}$.

Assume that $\mathcal{D}$ is a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary and $\mathcal{S} \subset \partial\mathcal{D}$ an open set, such that $\mathcal{D}$ is linearly convex at each point of $\partial\mathcal{D} \setminus \mathcal{S}$. Just as in Section 4, we define

$$v_0(z, \zeta) = \frac{v(\zeta)}{\langle v(\zeta), \zeta - z \rangle},$$

which is a smooth function of $(z, \zeta) \in \mathbb{C}^n \times \overline{\mathcal{D}}$ away from the null set of the denominator $\langle v(\zeta), \zeta - z \rangle$. By assumption, v_0 is smooth on the set $\mathcal{D} \times (\partial\mathcal{D} \setminus \mathcal{S})$ whence

$$\begin{aligned} I_q^{(p)}((1-t)v_0 + tv_1) &= \frac{(-1)^{n+(n-p+1)q}}{(2\pi i)^n n!} \binom{n}{p} \\ &\times\ D_{p,n-p}(\partial z, \partial \zeta) \wedge \sum_{k=0}^{n-q} \binom{n-2-k}{q-2} D_{1,1,q-2,k,n-q-k}\Big(v_0, v_1, \bar{\partial}_z v_1, \bar{\partial}_\zeta v_0, \bar{\partial}_\zeta v_1\Big) \end{aligned}$$

for $2 \leq q \leq n$. One also defines $I_1^{(p)} \equiv I_{n+1}^{(p)} \equiv 0$.

Using the double differential form $I_{q+1}^{(p)}((1-t)v_0 + tv_1)$, we may introduce a $\bar{\partial}$-homotopy operator

$$h_q^{(p)} f\,(z) = -\int_{\partial\mathcal{D} \setminus \mathcal{S}} f \wedge I_{q+1}^{(p)}\Big((1-t)v_0 + tv_1\Big) + \int_{\mathcal{D}} f \wedge K_q^{(p)}(v_1), \quad z \in \mathcal{D},$$

on differential forms f of bidegree (p, q) in $\mathcal{D}$ continuous up to the part $\partial\mathcal{D} \setminus \mathcal{S}$ of the boundary. The interest of the operator $h_q^{(p)}$ lies in the fact that we obtain $\bar{\partial}\, h_q^{(p)} f = f$ in $\mathcal{D}$, provided f is $\bar{\partial}$-closed in $\mathcal{D}$ and vanishes (or is merely $\bar{\partial}_b$-exact) on $\mathcal{S}$.

Theorem 5.1. *If f is a $\bar{\partial}$-closed differential form of bidegree (p,q) in $\mathcal{D}$ continuous up to the boundary, then*

$$\begin{aligned} f(z) &= -\int_{\partial\mathcal{S}} f(\zeta)\wedge I^{(p)}_{q+2}\Big((1-t)v_0+tv_1\Big) - \int_{\mathcal{S}} f(\zeta)\wedge K^{(p)}_{q+1}(v_1) \\ &\quad + \bar{\partial}\, h^{(p)}_q f\,(z) \end{aligned} \tag{5.1}$$

for all $z\in\mathcal{D}$.

Proof. This follows by the same way as in Theorem 4.5, the only difference being in the fact that we apply Lemmas 4.4 and 4.3 with $q=0$ replaced by arbitrary $q\geq 1$. If $q>0$, then the double form $K^{(p)}_{q+1}(v_0)$ vanishes for $\zeta\in\partial\mathcal{D}\setminus\mathcal{S}$ because v_0 is holomorphic in z on the set $\langle v(\zeta),\zeta-z\rangle\neq 0$. Therefore, we need not approximate it uniformly in $\zeta\in\partial\mathcal{D}\setminus\mathcal{S}$ by $\bar{\partial}$-closed differential forms on the closure of $\mathcal{D}$, which simplifies the proof. □

Since formula (5.1) does not contain any limit passage, it demonstrates rather strikingly that the Cauchy problem for the Dolbeault cohomology in $\mathcal{D}$ with data on $\mathcal{S}$ is stable, if posed in appropriate function spaces. In particular, this includes a uniqueness result.

Corollary 5.2. *Let f be a differential form of bidegree (p,q) and of class C^1 on the closure of $\mathcal{D}$. If moreover f is $\bar{\partial}$-closed in $\mathcal{D}$ and $\bar{\partial}_b$-exact on $\mathcal{S}$, then f is $\bar{\partial}$-exact in $\mathcal{D}$.*

Proof. Assume that $f=\bar{\partial}_b u$ on $\mathcal{S}$ where u is the restriction to $\mathcal{S}$ of a smooth $(p,q-1)$-form in a neighbourhood of $\overline{\mathcal{S}}$. Let us transform the right-hand side of (5.1). On the boundary of $\mathcal{S}$ which belongs to $\partial\mathcal{D}\setminus\mathcal{S}$ we can invoke decomposition (4.3) to obtain

$$\begin{aligned} -\int_{\partial\mathcal{S}} f\wedge I^{(p)}_{q+2}(v_t) &= \int_{\partial\mathcal{S}} u\wedge(-1)^{p+q-1}\bar{\partial}_\zeta I^{(p)}_{q+2}(v_t) \\ &= \int_{\partial\mathcal{S}} u\wedge\Big(K^{(p)}_{q+1}(v_1)-K^{(p)}_{q+1}(v_0)-\bar{\partial}_z I^{(p)}_{q+1}(v_t)\Big), \end{aligned}$$

where we write $v_t=(1-t)v_0+tv_1$ for short. On the other hand, integrating by parts and using Lemma 4.2 we get

$$\begin{aligned} -\int_{\mathcal{S}} f\wedge K^{(p)}_{q+1}(v_1) &= -\int_{\mathcal{S}} \bar{\partial}u\wedge K^{(p)}_{q+1}(v_1) \\ &= -\int_{\partial\mathcal{S}} u\wedge K^{(p)}_{q+1}(v_1) - \bar{\partial}\int_{\mathcal{S}} u\wedge K^{(p)}_{q}(v_1) \end{aligned}$$

for all $z\in\mathcal{D}$. Adding these two equalities yields

$$\begin{aligned} &-\int_{\partial\mathcal{S}} f(\zeta)\wedge I^{(p)}_{q+2}(v_t) - \int_{\mathcal{S}} f(\zeta)\wedge K^{(p)}_{q+1}(v_1) \\ &= -\int_{\partial\mathcal{S}} u\wedge K^{(p)}_{q+1}(v_0) + \bar{\partial}\left(-\int_{\partial\mathcal{S}} u\wedge I^{(p)}_{q+1}(v_t) - \int_{\mathcal{S}} u\wedge K^{(p)}_{q}(v_1)\right) \end{aligned}$$

for $z\in\mathcal{D}$.

Note that the double form $K^{(p)}_{q+1}(v_0)$ vanishes identically away from the set of singularities of v_0, if $q>0$. Indeed, the determinant (4.1) contains at least one column $\bar{\partial}_z v_0$, if $q-1>0$, and $\bar{\partial}_z v_0 \equiv 0$ because v_0 is holomorphic in z. It follows from Theorem 5.1 that

$$f = \bar{\partial} \left(- \int_{\partial \mathcal{S}} u \wedge I^{(p)}_{q+1}(v_t) - \int_{\mathcal{S}} u \wedge K^{(p)}_{q}(v_1) + h^{(p)}_{q} f \right)$$

in $\mathcal{D}$, proving the corollary. □

6. The Riemann-Hilbert boundary value problem

Consider a system of first order partial differential equations with constant coefficients of the form

$$A_1 \partial_1 f + \ldots + A_n \partial_n f = 0 \tag{6.1}$$

in $\mathbb{R}^n$, where $A_1, \ldots, A_n$ are $(k \times k)$-matrices of complex numbers, ∂_j the partial derivative in the j th coordinate x^j, for $j = 1, \ldots, n$, and f is an unknown function with values in $\mathbb{C}^k$. We will write A for the partial differential operator on the left-hand side of (6.1). The system (6.1) is called a generalised Cauchy-Riemann system if each solution f to (6.1) has only harmonic components f^j (see [AT16] and the references given there). Every generalised Cauchy-Riemann system is elliptic, i.e., its symbol

$$\sigma(A)(\xi) := \sum_{j=1}^{n} A_j (\imath \xi_j)$$

is invertible for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Obviously, there is no restriction of generality in assuming that $A_1 = E_k$, the identity matrix of type $k \times k$. The matrices $A_2, \ldots, A_n$ of system (6.1) with $A_1 = E_k$ are immediately specified as representations of generators of the Clifford algebra $\mathcal{C}_{n-1}$ over the field $\mathbb{C}$ in the algebra of all linear mappings of $\mathbb{C}^k$.

Another characteristic property of generalised Cauchy-Riemann equations is the so-called rotational invariance. Hence it follows that one rewrite them, by rotating the coordinate system x if necessary, in an equivalent form $Au = 0$ with A satisfying $A^* A = -E_k \Delta$, where A^* is the formal adjoint operator for A and Δ the Laplace operator in $\mathbb{R}^n$.

Let $\mathcal{D}$ be a bounded domain with smooth boundary in $\mathbb{R}^n$. Given a function f_0 at $\partial \mathcal{D}$ with values in $\mathbb{C}^{k/2}$, we look for a solution f to (6.1) in $\mathcal{D}$ which moreover satisfies

$$Bf \;=\; f_0 \quad \text{on} \quad \partial \mathcal{D}, \tag{6.2}$$

where B is a $(k/2 \times k)$-matrix of continuous functions on $\partial \mathcal{D}$ whose rank is maximal, i.e., $k/2$. Problem (6.1), (6.2) is usually referred to as the Hilbert boundary value problem.

Remark 6.1. Since each generalised Cauchy-Riemann operator A has a fundamental solution Φ of convolution type, the inhomogeneous system $Af = g$ in $\mathcal{D}$ is reduced to the homogeneous one by substituting $f + \Phi(\chi_{\mathcal{D}} g)$ for f, where $\chi_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$.

The study of the Hilbert boundary value problem for general elliptic systems of first order partial differential equations in a bounded domain $\mathcal{D} \subset \mathbb{R}^n$ goes back at least as far as [Ava82]. The problem was reduced to a system of singular integral equations the boundary of $\mathcal{D}$ and both a condition for the Fredholm property and an index theorem were given. In [MMT11] the Hilbert boundary value problem is studied for generalised Maxwell equations. These equations have more complicated structure than (6.1).

For the classical Cauchy-Riemann equations we get $k = 1$, and so $k/2$ fails to be a whole number. To dispense with the assumption on the evenness of k one can rewrite equations (6.1) in the obvious way over the field of real numbers. Then condition (6.2) takes the form $\Re\,(B'f) = f_0$ on $\partial\mathcal{D}$, where B' is a nonsingular $(k \times k)$-matrix of continuous functions on $\partial\mathcal{D}$ and f_0 a function on the boundary with values in $\mathbb{R}^k$.

The rows of the matrix $B(x)$ are linearly independent at each point $x \in \partial\mathcal{D}$. On applying the Gram-Schmidt orthogonalisation one can orthonormalise them in $\mathbb{C}^k$. The properties of continuity and smoothness of the matrix elements do not change. Hence, we can assume without restriction of generality that the rows of $B(x)$ form pointwise an orthonormal system.

The task is now to find those conditions on the boundary coefficients and geometry of the domain $\mathcal{D}$ under which the Hilbert boundary value problem has the Fredholm property. When considered in the Sobolev spaces, a boundary value problem for an elliptic system is Fredholm if and only if it satisfies the Shapiro-Lopatinskij condition. The paper [Ste93] studies in detail the adjoint boundary value problem to apply the general theory of elliptic boundary value problems to the Hilbert problem in the case, where the Shapiro-Lopatinskij condition is fulfilled. This is precisely the border line over which elliptic theory no longer works, and so the boundary Fourier method in the Hilbert boundary value problem is well motivated.

The boundary Fourier method is based on an integral identity specifying the complementary part of Bu in the Cauchy data of u on the boundary of $\mathcal{D}$ relative to the generalised Cauchy-Riemann operator $A(\partial) = A_1\partial_1 + \ldots + A_n\partial_n$. By the above, there is no loss of generality in assuming that the rows of the matrix $B(x)$ are pointwise orthonormal. Under rather broad (still necessary) topological conditions on B there is a $(k/2 \times k)$-matrix C of smooth functions on $\partial\mathcal{D}$, such that the block matrix

$$T(x) = \begin{pmatrix} B(x) \\ C(x) \end{pmatrix}$$

is unitary for all $x \in \partial\mathcal{D}$.

Lemma 6.2. *There are unique matrices B^{adj} and C^{adj} of continuous functions on $\partial\mathcal{D}$ with the property that*

$$\int_{\partial\mathcal{D}} \big((Bf, C^{\mathrm{adj}}g)_x - (Cf, B^{\mathrm{adj}}g)_x\big)\, ds = \int_{\mathcal{D}} ((Af, g)_x - (f, A^*g)_x)\, dx \qquad (6.3)$$

for all $f \in H^1(\mathcal{D}, \mathbb{C}^k)$ and $g \in H^1(\mathcal{D}, \mathbb{C}^k)$, where ds is the surface measure on the boundary.

Proof. Since $T(x)$ is a unitary matrix for all $x \in \partial\mathcal{D}$, we get $T^*T = E_k$, which is equivalent to $B^*B + C^*C = E_k$.

Given any $f \in H^1(\mathcal{D}, \mathbb{C}^k)$ and $g \in H^1(\mathcal{D}, \mathbb{C}^k)$, the Green formula of [Tar95, 9.2.2] shows that

$$\int_{\partial\mathcal{D}} (\sigma f, g)_x\, ds = \int_{\mathcal{D}} ((Af, g)_x - (f, A^*g)_x)\, dx$$

where $\sigma(x) := \sigma(A)(-\imath\nu(x))$ for $x \in \partial\mathcal{D}$ and $\nu(x)$ is the unit outward normal vector to the boundary at x. On substituting $u = (B^*B + C^*C)u$ into this formula yields

(6.3) with

$$\begin{array}{rcl} C^{\mathrm{adj}} & = & B\sigma^*, \\ B^{\mathrm{adj}} & = & -\,C\sigma^*, \end{array} \tag{6.4}$$

as desired. □

From (6.4) it follows immediately that the ranks of both C^{adj} and B^{adj} are equal to $k/2$.

Formula (6.3) is said to be a Green formula related to the boundary value problem $\{A, B\}$. The formula is not uniquely determined by the pair $\{A, B\}$, for the complementary part C of B in the Cauchy data $\{B, C\}$ can be chosen in many ways. On choosing C we fix a duality on the manifold with boundary $\overline{\mathcal{D}}$ associated with $\{A, B\}$. The problem

$$\left\{ \begin{array}{rcccl} A^*g & = & h & \mathrm{in} & \mathcal{D}, \\ B^{\mathrm{adj}}g & = & g_0 & \mathrm{on} & \partial\mathcal{D} \end{array} \right. \tag{6.5}$$

is called adjoint to $\{A, B\}$ with respect to the Green formula. Clearly, (6.5) is of Hilbert type, too.

From the Green formula it follows that for the solvability of problem (6.1), (6.2) it is necessary that

$$\int_{\partial\mathcal{D}} (f_0, C^{\mathrm{adj}}g)_x \, ds = 0 \tag{6.6}$$

for all $g \in H^1(\mathcal{D}, \mathbb{C}^k)$ which satisfy the homogeneous problem corresponding to (6.5), i.e., $A^*g = 0$ in $\mathcal{D}$ and $B^{\mathrm{adj}}g = 0$ on $\partial\mathcal{D}$. The moment conditions (6.6) specify a closed space of boundary data f_0 which contains the range of the Hilbert boundary value problem. However, this space need not have finite codimension, for the space of solutions of the homogeneous adjoint problem may be infinite dimensional.

For a smooth function f in $\mathcal{D}$, we set $f_\varepsilon(y) := f(y - \varepsilon\nu(y))$, thus obtaining a family of smooth functions on $\partial\mathcal{D}$ parametrised by a small parameter $\varepsilon > 0$. We say that f admits distribution limit values on $\partial\mathcal{D}$, if

$$\lim_{\varepsilon\to 0+} \int_{\partial\mathcal{D}} f_\varepsilon g \, ds =: \langle f_0, g \rangle \tag{6.7}$$

exists for all $g \in C^\infty(\partial\mathcal{D})$. In this case the limit defines a distribution f_0 on the boundary and the convergence is not only in the weak sense but also in the strong topology on $\mathcal{D}'(\partial\mathcal{D})$. A harmonic function f in $\mathcal{D}$ admits distribution limit values on $\partial\mathcal{D}$ if and only if f is of finite order of growth near $\partial\mathcal{D}$, i.e., there is an integer N and $C > 0$, such that $|f(x)| \le C/(\mathrm{dist}(x, \partial\mathcal{D}))^N$ for all $x \in \mathcal{D}$, see Theorem 1.1 in [Str84].

Let now f be a smooth function in $\mathcal{D}$ with values in $\mathbb{C}^k$ satisfying the generalised Cauchy-Riemann equations $Au = 0$ in $\mathcal{D}$. If there exists an integer N and $C > 0$, such that $|f(x)| \le C/(\mathrm{dist}(x, \partial\mathcal{D}))^N$ for all $x \in \mathcal{D}$, then the same is true for the components of f. By the above, each component admits distribution limit values on $\partial\mathcal{D}$. Hence, f admits limit values on the boundary which form is a continuous linear functional on $C^\infty(\partial\mathcal{D}, \mathbb{C}^k)$. Moreover, both Bu and Cu admit limit values on $\partial\mathcal{D}$ which are distributions with values in $\mathbb{C}^k$. This is precisely the sense in which we interpret them in the following formula analogous to the Cauchy integral formula.

Let $e(x)$ be the standard fundamental solution of convolution type for Δ, i.e., $e(x) = (2\pi)^{-1} \log |x|$, if $n = 2$, and

$$e(x) = \frac{1}{\sigma_n} \frac{1}{2-n} \frac{1}{|x|^{n-2}},$$

if $n \geq 3$, where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$. The matrix $\Phi = -A^* e$ is a (two-sided) fundamental solution of convolution type of the operator A, i.e., $\Phi A = A\Phi = I$ on compactly supported distributions in $\mathbb{R}^n$ whose values belong to $\mathbb{C}^k$.

Lemma 6.3. *For each solution f to equations (6.1) in $\mathcal{D}$ of finite order of growth near $\partial\mathcal{D}$, it follows that*

$$-\int_{\partial\mathcal{D}} \big((Bf, C^{\mathrm{adj}} \Phi(x-\cdot)^*)_y - (Cf, B^{\mathrm{adj}} \Phi(x-\cdot)^*)_y \big)\, ds = \begin{cases} f(x), & \text{if } x \in \mathcal{D}, \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \overline{\mathcal{D}}. \end{cases} \tag{6.8}$$

Note that $(\Phi(x-y))^* = (Ae)(x-y)$ for all x and y away from the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$, as is easy to check.

Proof. See Theorem 9.4.1 of [Tar95]. □

This reasoning, when looked at from a more general point of view, leads to new investigations of Fredholm boundary value problems in Hardy-Smirnov spaces, see [Tar95, 11.2.2].

The operator-theoretic foundations of the method of Fischer-Riesz equations within the framework of the Cauchy problem for solutions of elliptic equations are elaborated in [Tar95, 11.1]. It goes back at least as far as [PF50]. Here we adapt this method for studying the Hilbert boundary value problem for generalised Cauchy-Riemann equations.

Any solution of generalised Cauchy-Riemann equations in $\mathcal{D}$ is a k-column of harmonic functions in this domain. Therefore, the k-fold product of the Hardy-Smirnov space $E^2(\mathcal{D})$ (for which we write $E^2(\mathcal{D}, \mathbb{C}^k)$) fits well to constitute the domain of problem (6.1), (6.2). When endowed with the $L^2(\partial\mathcal{D}, \mathbb{C}^k)$-norm, this space is Hilbert.

Denote by H_1 the closed subspace of $E^2(\mathcal{D}, \mathbb{C}^k)$ consisting of those f which satisfy $Af = 0$ in $\mathcal{D}$. When endowed with the induced unitary structure, H_1 is a Hilbert space. Besides, set $H_2 = L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$ and $H = H_2 \times L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$. Consider the mapping $M : H_1 \to H$ given by $Mf = (Bf, Cf)$, which corresponds to the Cauchy problem for solutions of $Au = 0$ in $\mathcal{D}$ with Cauchy data $Bf = f_0$ and $Cf = f_1$ on $\partial\mathcal{D}$. The operator M is continuous and has closed range. Write $M^* : H \to H_1$ for the operator adjoint to $M : H_1 \to H$ in the sense of Hilbert spaces. The null-space $\ker M^*$ of the operator M^* is separable in the topology induced from H. Let $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$ stand for the space of all solutions to the formal adjoint system $A^* g = 0$ on neighbourhoods of $\overline{\mathcal{D}}$. Since A^* is elliptic, these are real analytic functions with values in $\mathbb{C}^k$.

Lemma 6.4. *Assume that $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$. Then the couple $(C^{\mathrm{adj}} g, -B^{\mathrm{adj}} g)$ belongs to $\ker M^*$.*

Proof. One has to show that $(Mf,(C^{\mathrm{adj}}g,-B^{\mathrm{adj}}g))_H=0$ for all $f\in H_1$. By the Green formula, we get

$$\begin{array}{rcl}(Mf,(C^{\mathrm{adj}}g,-B^{\mathrm{adj}}g))_H & = & \displaystyle\int_{\partial\mathcal{D}}\big((Bf,C^{\mathrm{adj}}g)_x-(Cf,B^{\mathrm{adj}}g)_x\big)\,ds\\ & = & 0,\end{array}$$

as desired. □

The subspace of $\ker M^*$ consisting of all elements of the form $(C^{\mathrm{adj}}g,-B^{\mathrm{adj}}g)$, where $g\in\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, is separable. Hence, there are many ways to choose a sequence $\{g_i\}_{i=1,2,\ldots}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that the system $\{(C^{\mathrm{adj}}g_i,-B^{\mathrm{adj}}g_i)\}$ is complete in this subspace. In Example 6.6 we show some explicit sequences $\{g_i\}$ with this property. In fact, one establishes that the system $\{(C^{\mathrm{adj}}g_i,-B^{\mathrm{adj}}g_i)\}_{i=1,2,\ldots}$ is complete in $\ker M^*$.

Write P for the orthogonal projection of H onto the first factor H_2. The composition $PM=B$ acting from H_1 to H_2 just amounts to the operator of boundary value problem (6.1), (6.2) in the updated setting. More precisely, given any $f_0\in L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$, find $f\in E^2(\mathcal{D},\mathbb{C}^k)$ satisfying $Af=0$ in $\mathcal{D}$ and $Bf=f_0$ weakly on the boundary of $\mathcal{D}$. The following lemma expresses the most important property of the system $\{g_i\}$.

Lemma 6.5. *The system $\{B^{\mathrm{adj}}g_i\}_{i=1,2,\ldots}$ is complete in $L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$ if and only if PM is injective.*

Proof. See [AT16]. □

After removing the elements which are linear combinations of the previous ones from the system $\{B^{\mathrm{adj}}g_i\}_{i=1,2,\ldots}$, we get a sequence $\{g_{i_n}\}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that the system $\{B^{\mathrm{adj}}g_{i_n}\}$ is linearly independent. Applying then the Gram-Schmidt orthogonalisation to the system $\{B^{\mathrm{adj}}g_{i_n}\}$ in $L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$, we obtain a new system $\{e_n\}_{n=1,2,\ldots}$ in $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$, such that $\{B^{\mathrm{adj}}e_n\}$ is an orthonormal system in the space $L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$. Moreover, $\{B^{\mathrm{adj}}e_n\}$ is an orthonormal basis in $L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$, provided that PM is injective. Note that the elements e_n of the new system have explicit expressions through the elements $\{g_{i_1},\ldots,g_{i_n}\}$ of the old system in the form of Gram's determinants.

Example 6.6. Since $\mathcal{D}$ is a bounded domain with smooth boundary, its complement has only finitely many connected components. Choose a finite number of points $\{x_i\}$ away from the closure of $\mathcal{D}$, such that each connected component of $\mathbb{R}^n\setminus\overline{\mathcal{D}}$ contains at least one of the points x_i. Then the columns of the matrix $\partial^\alpha\Phi(x_i-\cdot)^*$ belong obviously to $\mathcal{S}_{A^*}(\overline{\mathcal{D}})$ and the system $\{B^{\mathrm{adj}}\partial^\alpha\Phi(x_i-\cdot)^*\}$ is complete in the subspace of $L^2(\mathcal{D},\mathbb{C}^{k/2})$ formed by elements of the type $B^{\mathrm{adj}}g$ with $g\in\mathcal{S}_{A^*}(\overline{\mathcal{D}})$.

The proof of this fact actually repeats the reasoning of Example 11.4.14 in [Tar95]. Apparently the system of Example 6.6 is most convenient for numerical simulations.

Given any $f_1\in L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$, we denote by $k_n(f_1)$ the Fourier coefficients of f_1 with respect to the system $\{B^{\mathrm{adj}}e_n\}$, i.e.,

$$k_n(f_1)=\int_{\partial\mathcal{D}}(f_1,B^{\mathrm{adj}}e_n)_y\,ds$$

for $n = 1, 2, \ldots$.

Lemma 6.7. *If $f \in E^2(\partial\mathcal{D}, \mathbb{C}^k)$ satisfies $Af = 0$ in $\mathcal{D}$, then*

$$k_n(Cf) = \int_{\partial\mathcal{D}} (Bf, C^{\mathrm{adj}} e_n)_y \, ds,$$

where $n = 1, 2, \ldots$.

Proof. Using Lemma 6.4 we obtain

$$\begin{aligned} k_n(Cf) &= \int_{\partial\mathcal{D}} (Cf, B^{\mathrm{adj}} e_n)_y \, ds + (Mf, (C^{\mathrm{adj}} e_n, -B^{\mathrm{adj}} e_n))_H \\ &= \int_{\partial\mathcal{D}} (Bf, C^{\mathrm{adj}} e_n)_y \, ds, \end{aligned}$$

as desired. □

Thus, in order to find the Fourier coefficients of the data Cf on the boundary with respect to the system $\{B^{\mathrm{adj}} e_n\}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$, it suffices to know only the data Bf of problem (6.1), (6.2).

Theorem 6.8. *Let $f_0 \in L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$. In order that there be a $f \in E^2(\mathcal{D}, \mathbb{C}^k)$ such that $Au = 0$ in $\mathcal{D}$ and $Bu = u_0$ on $\partial\mathcal{D}$, it is necessary and sufficient that*

1) $\displaystyle\sum_{n=1}^{\infty} |c_n|^2 < \infty$, *where* $c_n = \displaystyle\int_{\partial\mathcal{D}} (f_0, C^{\mathrm{adj}} e_n)_y \, ds$, *and*

2) $\displaystyle\int_{\partial\mathcal{D}} (f_0, C^{\mathrm{adj}} g)_y \, ds = 0$ *for all $g \in \mathcal{S}_{A^*}(\overline{\mathcal{D}})$ satisfying $B^{\mathrm{adj}} g = 0$ on the boundary.*

Proof. See [AT16]. □

The convergence of the series in *1)* guarantees the stability of boundary value problem (6.1), (6.2). Under this condition, the range of the mapping PM is described in terms of continuous linear functionals on the space H, cf. *2)*, which is impossible in the general case. On the other hand, if the homogeneous adjoint boundary value problem (6.5) has no smooth solutions in $\overline{\mathcal{D}}$ different from zero, then condition *1)* is necessary and sufficient for the existence of a solution $f \in E^2(\mathcal{D}, \mathbb{C}^k)$ to problem (6.1), (6.2).

Note that the proof of Theorem 6.8 works without the assumption that the operator PM in H is injective. Our next objective will be to construct an approximate solution to boundary value problem (6.1), (6.2). To this end it is natural to assume that the corresponding homogeneous boundary value problem has only zero solution in the space $E^2(\mathcal{D})^k$, i.e., the mapping PM is injective. In this case the orthonormal system $\{B^{\mathrm{adj}} e_n\}$ is actually complete in the space $L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$. The orthonormal bases in $L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$ of this form are said to be special, cf. [Tar95, 11.3].

For $x \in \mathcal{D} \setminus \partial\mathcal{D}$, we denote by $k_n(B^{\mathrm{adj}} \varPhi(x - \cdot)^*)$ the k-row whose entries are the Fourier coefficients of the columns of the $((k/2) \times k)$-matrix $B^{\mathrm{adj}} \varPhi(x - \cdot)^*$ with respect to the orthonormal basis $\{B^{\mathrm{adj}} e_n\}_{n=1,2,\ldots}$ in $L^2(\partial\mathcal{D}, \mathbb{C}^{k/2})$. More precisely, we set

$$k_n(B^{\mathrm{adj}} \varPhi(x - \cdot)^*) = \int_{\partial\mathcal{D}} (B^{\mathrm{adj}} \varPhi(x - \cdot)^*, B^{\mathrm{adj}} e_n)_y \, ds$$

for $n = 1, 2, \ldots$. The coefficients $k_n(B^{\mathrm{adj}}\Phi(x-\cdot)^*)$ are analytic functions in $\mathcal{D}\setminus\partial\mathcal{D}$ with values in $(\mathbb{C}^k)^*$.

We now introduce the (Schwartz) kernels Φ_N defined for $x \in \mathcal{D}$ and y in a neighbourhood of $\overline{\mathcal{D}}$ by

$$\Phi_N(x,y) = \sum_{n=1}^{N} k_n(B^{\mathrm{adj}}\Phi(x-\cdot)^*)^* \, e_n(y)^*,$$

where $N = 1, 2, \ldots$. Obviously, the kernels Φ_N are analytic in $x \in \mathcal{D}$ and y in a neighbourhood of $\overline{\mathcal{D}}$, and $A^*(y,\partial)\Phi_N(\cdot,y)^* = 0$ on this set. The sequence $\{\Phi_N\}$ provides a very refined approximation of the fundamental solution Φ on the boundary of $\mathcal{D}$.

Lemma 6.9. *The sequence $\{B^{\mathrm{adj}}\,(\Phi(x-\cdot) - \Phi_N(x,\cdot))^*\}_{N=1,2,\ldots}$ converges to zero in the norm of $L^2(\partial\mathcal{D}, \mathbb{C}^{(k/2)\times k})$ uniformly in x on the compact subsets of the domain $\mathcal{D}$.*

In this way we recover immediately the concept of a suppressing function within the framework of the Hilbert problem for generalised Cauchy-Riemann equations. More precisely, on determining Q_N from the equality $\Phi - \Phi_N = e^{Q_N}\Phi$ we get formally

$$Q_N(x,y) = \log\big(I - \Phi_N(x,y)\Phi(x-y)^{-1}\big)$$

for all $x \in \mathcal{D}$ and y in a neighbourhood of $\overline{\mathcal{D}}$. Hence, whenever $\Phi_N(x,y)$ approximates $\Phi(x-y)$ the logarithm on the right- hand side tends to $-\infty$. For such $(x,y) \in \mathcal{D}\times\partial\mathcal{D}$ the factor e^{Q_N} suppresses Φ, as $N \to \infty$. This allows one to reconstruct solutions f of the class $E^2(\mathcal{D},\mathbb{C}^k)$ to $Af = 0$ in $\mathcal{D}$ through their data Bf.

Theorem 6.10. *Every function $f \in E^2(\mathcal{D},\mathbb{C}^k)$ satisfying $Au = 0$ in $\mathcal{D}$ can be represented by the integral formula*

$$f(x) = \lim_{N\to\infty}\Big(-\int_{\partial\mathcal{D}} (Bf, C^{\mathrm{adj}}R_N(x,\cdot)^*)_y\, ds\Big)$$

for all $x \in \mathcal{D}$, where $R_N = \Phi - \Phi_N$.

Proof. Fix a point $x \in \mathcal{D}$. Since $R_N(x,\cdot)^*$ and $\Phi(x-\cdot)^*$ differ by a k-row of smooth solutions of the system $A^*g = 0$ in a neighbourhood of $\overline{\mathcal{D}}$, one can write by the Green formula

$$f(x) = -\int_{\partial\mathcal{D}} \big((Bf, C^{\mathrm{adj}}R_N(x,\cdot)^*)_y - (Cf, B^{\mathrm{adj}}R_N(x,\cdot)^*)_y\big)\, ds \qquad (6.9)$$

for any $N = 1, 2, \ldots$. From $f \in E^2(\mathcal{D},\mathbb{C}^k)$ we deduce that $Cf \in L^2(\partial\mathcal{D},\mathbb{C}^{k/2})$. Hence it follows by Lemma 6.9 that

$$\lim_{N\to\infty}\int_{\partial\mathcal{D}} (Cf, B^{\mathrm{adj}}R_N(x,\cdot)^*)_y\, ds = 0.$$

Thus, letting $N \to \infty$ in (6.9) establishes the formula. □

As mentioned, for many problems of mathematical physics formulas for approximate solution like that of Theorem 6.10 were earlier obtained by Kupradze and his colleagues, see [Kup67]. Note that V. I. Smirnov supervised a PhD thesis of V. Kupradze in the 1930s, cf. also [KK58].

7. A nonstandard Cauchy problem for parabolic equations

The problem we discuss with in this sections originates from [LRS80] and it consists in finding an explicit formula for the temperature inside a domain by using partial lateral and initial data. This problem is treated in [Ike09]. A more general class of related formulas was obtained in [MMT16].

Let $\mathcal{D}$ be a bounded domain with smooth boundary in $\mathbb{R}^n$ and $\mathcal{S}$ a nonempty open piece of the boundary surface $\partial\mathcal{D}$. Consider the Cauchy problem for the heat equation in the cylinder $\mathcal{C}_T = \mathcal{D} \times (0,T)$ with data on the strip $\mathcal{S} \times (0,T)$ of the lateral surface of the cylinder, where $(0,T)$ is a finite interval of the time axis. More precisely, given functions f in $\mathcal{C}_T$ and u_0, u_1 on $\mathcal{S} \times (0,T)$, find a function u in $\mathcal{C}_T$ which satisfies

$$\left\{ \begin{array}{rclll} u'_t & = & \varDelta u + f & \text{in} & \mathcal{C}_T, \\ u & = & u_0 & \text{on} & \mathcal{S} \times (0,T), \\ u'_\nu & = & u_1 & \text{on} & \mathcal{S} \times (0,T), \end{array} \right. \tag{7.1}$$

where ν is the unit outward normal vector for $\partial\mathcal{D}$.

The uniqueness of a solution to the problem can be easily proved in anisotropic Sobolev or Hölder spaces by the same method as for elliptic equations, see for instance [LO74]. Since in any slice $t = t_0$ (7.1) is a Cauchy problem for an elliptic equation, the problem is ill-posed in standard function spaces unless $\mathcal{S} = \partial\mathcal{D}$, see [LRS80].

To construct an explicit formula for solutions to problem (7.1) of Golusin-Krylov type, we continue the data f and u_0, u_1 to the whole semicylinder $t > 0$, i.e., we assume $T = \infty$ in the sequel. On applying the Laplace transform in the variable $t \in (0, \infty)$

$$\hat{u}(x,\tau) := \mathcal{L}u\,(x,\tau) = \int_0^\infty e^{-\imath\tau t} u(x,t)\,dt$$

to all equations of (7.1) we get the family of Cauchy problems

$$\left\{ \begin{array}{rclll} (\varDelta - \imath\tau)\hat{u}(\cdot,\tau) & = & -u(\cdot,0) - \hat{f}(\cdot,\tau) & \text{in} & \mathcal{D}, \\ \hat{u}(\cdot,\tau) & = & \hat{u}_0(\cdot,\tau) & \text{on} & \mathcal{S}, \\ \hat{u}'_\nu(\cdot,\tau) & = & \hat{u}_1(\cdot,\tau) & \text{on} & \mathcal{S} \end{array} \right. \tag{7.2}$$

in the domain $\mathcal{D}$, parametrised by the complex parameter τ running over a horizontal line in the lower half-plane. For any fixed τ, one readily specifies (7.2) within the framework of ill-posed Cauchy problem for the Helmholtz equation in $\mathcal{D}$ with data on $\mathcal{S}$.

Denote by $\hat{G}(x;\tau)$ the fundamental solution of convolution type for the Helmholtz operator $\varDelta - \imath\tau$ in the class of temperate distributions on $\mathbb{R}^n$. The Goluzin-Krylov techniques of the suppressing function enable one to construct an approximate solution of problem (7.2). To wit, one finds a sequence of kernels $\hat{C}_N(x,y;\tau)$, for $N = 1, 2, \ldots$, which differ from $\hat{G}(x-y;\tau)$ by a smooth solution of the Helmholtz equation in $y \in \mathcal{D}$ and whose Cauchy data in y tend to zero at $\partial\mathcal{D} \setminus S$, see [Tar95]. Then, for fixed τ, an approximate solution of (7.2) is obtained from the so-called Carleman formula

$$\hat{u}(x,\tau) = \lim_{N\to\infty} \Big(-\int_{\mathcal{S}} \Big(\hat{C}_N(x,\cdot;\tau)\partial_\nu \hat{u} - \partial_\nu \hat{C}_N(x,\cdot;\tau)\hat{u} \Big)\,ds + \int_{\mathcal{D}} \hat{C}_N(x,\cdot;\tau)(\varDelta - \imath\tau)\hat{u}\,dy \Big)$$

for all $x \in \mathcal{D}$, where ds is the surface measure on $\partial\mathcal{D}$. Then, assuming that the inverse Laplace transform is possible under the limit passage in the last formula, we arrive at the formula

$$\begin{aligned} u(x,t) = \lim_{N\to\infty} \Big(& -\int_{\mathcal{D}} (\mathcal{L}^{-1}\hat{C}_N)(x,\cdot;t)u(\cdot,0)dy \\ & -\int_{\mathcal{S}}\int_0^t \Big((\mathcal{L}^{-1}\hat{C}_N)\,(x,\cdot;t-t')\partial_\nu u - \partial_\nu(\mathcal{L}^{-1}\hat{C}_N)(x,\cdot;t-t')u \Big)\, dsdt' \\ & +\int_{\mathcal{D}}\int_0^t (\mathcal{L}^{-1}\hat{C}_N)(x,\cdot;t-t')(\Delta - \partial_{t'})u\, dydt' \Big) \end{aligned}$$

whenever $(x,t) \in \mathcal{C}_T$. For (x,y) away from the diagonal in $\mathcal{D}\times\overline{\mathcal{D}}$ and for $t>0$, set

$$C_N(x,y;t) = \mathcal{L}^{-1}\hat{C}_N(x,y;\tau) = \frac{1}{2\pi}\int_{\Im\tau=\gamma} e^{\imath t\tau}\hat{C}_N(x,y;\tau)\,d\tau,$$

where γ is a sufficiently small negative number. A direct calculation shows that

$$\begin{aligned} (\Delta_y - \partial_t)C_N(x,y;t) &= \frac{1}{2\pi}\int_{\Im\tau=\gamma} e^{\imath t\tau}(\Delta_y - \imath\tau)\hat{C}_N(x,y;\tau)\,d\tau \\ &= 0 \end{aligned}$$

for (x,y) and t in the domain of C_N. Moreover, $C_N(x,y;t)$ tends to zero in certain sense in y away from $\mathcal{S}$ on the boundary of $\mathcal{D}$, as $N\to\infty$.

Hence, the sequence of kernels $C_N(x,y;t)$ generalises immediately the concept of Carleman function in the Cauchy problem for elliptic equations to problem (7.1), see [Tar95].

Lemma 7.1. *If $C_N(x,y;t)$ is a Carleman function of problem (7.1), then the formula*

$$u_N = \mathcal{P}_{i,N}(u(\cdot,0)) + \mathcal{P}_{s,N}(u_1) + \mathcal{P}_{d,N}(u_0) + \mathcal{P}_{v,N}(f)$$

gives an approximate solution of the problem in the cylinder $\mathcal{C}_T$, where

$$\mathcal{P}_{i,N}(u(\cdot,0)) = -\int_{\mathcal{D}} C_N(x,\cdot;t)u(\cdot,0)dy, \qquad \mathcal{P}_{s,N}(u_1) = -\int_{\mathcal{S}}\int_0^t C_N(x,\cdot;t-t')u_1 dsdt',$$

$$\mathcal{P}_{d,N}(u_0) = \int_{\mathcal{S}}\int_0^t \partial_\nu C_N(x,\cdot;t-t')u_0 dsdt', \quad \mathcal{P}_{v,N}(f) = -\int_{\mathcal{D}}\int_0^t C_N(x,\cdot;t-t')f dydt'.$$

We now show how it works for the so-called cap type domains. To this end assume that $\mathcal{D}$ is a bounded domain in the upper half-space $\{x_3>0\}$ of $\mathbb{R}^3$ whose boundary consists of a smooth surface $\mathcal{S}$ lying in the half-space $\{x_3>0\}$, and a closed piece of the plane $\{x_3=0\}$. Let σ be a positive number. Consider the entire function

$$K(w) = \exp(\sigma w^2)$$

of complex variable $w\in\mathbb{C}$. The restriction of K to any vertical line $w = u_0 + \imath v$ just amounts to $K(u+\imath v) = K(u)\exp(2\imath\sigma uv - \sigma v^2)$, which is a rapidly decreasing function of v.

Given two different points $x=(x',x_3)$ and $y=(y',y_3)$ in $\mathbb{R}^3$, set $r' = |y'-x'|$ and introduce the integral

$$\Phi(x,y) = \frac{-1}{2\pi^2}\frac{1}{K(x_3)}\int_0^\infty \Im\Big(\frac{K(w)}{w-x_3}\Big)\frac{\cos\lambda\vartheta}{\sqrt{r'^2+\vartheta^2}}\,d\vartheta, \tag{7.3}$$

where $w = y_3 + \imath\sqrt{r'^2+\vartheta^2}$ and λ is a complex parameter. On separating the imaginary part we get

$$\Phi(x,y) = \int_0^\infty k(x,y;\vartheta)\cos\lambda\vartheta\, d\vartheta$$

with $k(x,y;\vartheta)$ given by

$$\frac{-1}{2\pi^2}\, \frac{e^{\sigma(y_3^2-x_3^2)}e^{-\sigma(r'^2+\vartheta^2)}}{\vartheta^2+r^2}\Big((y_3-x_3)\frac{\sin 2\sigma y_3\sqrt{r'^2+\vartheta^2}}{\sqrt{r'^2+\vartheta^2}} - \cos 2\sigma y_3\sqrt{r'^2+\vartheta^2}\Big).$$

The convergence of the improper integral on the right-hand side of (7.3) is thus guaranteed by the factor $e^{-\sigma\vartheta^2}$.

The following two lemmas are crucial for the Carleman formula.

Lemma 7.2. *As defined by equality (7.3), the function $\Phi(x,y)$ is represented in the form*

$$\Phi(x,y) = \frac{-1}{4\pi}\frac{e^{-\lambda r}}{r} + R(x,y),$$

where $R(x,y)$ is a twice continuously differentiable function of the variable $y \in \mathbb{R}^3$ including the point $y = x$.

Lemma 7.3. *As defined in (7.3), the function $\Phi(x,y)$ satisfies the Helmholtz equation $\Delta\Phi - \lambda^2\Phi = 0$ in $y \in \mathbb{R}^3 \setminus \{x\}$.*

Following (7.3) we introduce

$$\hat{C}_\sigma(x,y;\tau) = \frac{-1}{2\pi^2}\frac{1}{K(x_3)}\int_0^\infty \Im\Big(\frac{K(w)}{w-x_3}\Big)\frac{\cos(\sqrt{\imath\tau}\vartheta)}{\sqrt{r'^2+\vartheta^2}}\, d\vartheta,$$

where $w = y_3 + \imath\sqrt{r'^2+\vartheta^2}$ and τ is a complex parameter. An easy calculation shows that

$$\hat{C}_\sigma(x,y;\tau) = \int_0^\infty k_\sigma(x,y;\vartheta)\cos(\sqrt{\imath\tau}\vartheta)\, d\vartheta$$

where $k_\sigma(x,y;\vartheta)$ is given by

$$\frac{-1}{2\pi^2}\, \frac{e^{\sigma(y_3^2-x_3^2)}e^{-\sigma(r'^2+\vartheta^2)}}{\vartheta^2+r^2}\Big((y_3-x_3)\frac{\sin 2\sigma y_3\sqrt{r'^2+\vartheta^2}}{\sqrt{r'^2+\vartheta^2}} - \cos 2\sigma y_3\sqrt{r'^2+\vartheta^2}\Big).$$

Hence it follows that

$$\begin{aligned} k_\sigma(x,y;\vartheta) &= \frac{1}{2\pi^2}\,\frac{e^{-\sigma x_3^2}e^{-\sigma(r'^2+\vartheta^2)}}{\vartheta^2+r^2}, \\ \partial_{y_3}k_\sigma(x,y;\vartheta) &= \frac{1}{\pi^2}\,\frac{e^{-\sigma x_3^2}e^{-\sigma(r'^2+\vartheta^2)}}{(\vartheta^2+r^2)^2}x_3\left(1+\sigma(\vartheta^2+r^2)\right) \end{aligned}$$

on the plane $y_3 = 0$. On applying Lemma 7.3 we conclude that $\hat{C}_\sigma(x,y;\tau)$ is a Carleman function of Cauchy problem (7.2) in the domain $\mathcal{D}$ with data on $\mathcal{S}$, parametrised by τ. It remains to evaluate the inverse Laplace transform

$$C_\sigma(x,y;t) = \mathcal{L}^{-1}\hat{C}_\sigma(x,y;\tau) = \int_0^\infty k_\sigma(x,y;\vartheta)\,\mathcal{L}^{-1}\cos(\sqrt{\imath\tau}\vartheta)\, d\vartheta$$

of $\hat{C}_\sigma(x,y;\tau)$, which reduces to evaluating the inverse Laplace transform of the function $\cos(\sqrt{\imath\tau}\vartheta)$ within the theory of [GS53].

Theorem 7.4. *Let $\mathcal{D}$ be a cap type domain in $\mathbb{R}^3$. Then, given any u in the anisotropic Sobolev space $H^{2,1}(\mathcal{C}_T)$, it follows that*

$$\begin{aligned} u(x,t) = \lim_{\sigma\to\infty} \Big(& - \int_{\mathcal{D}} C_\sigma(x,\cdot;t)u(\cdot,0)dy \\ & - \int_{\mathcal{S}}\int_0^t (C_\sigma(x,\cdot;t-t')\partial_\nu u - \partial_\nu C_\sigma(x,\cdot;t-t')u)\, ds dt' \\ & + \int_{\mathcal{D}}\int_0^t C_\sigma(x,\cdot;t-t')(\Delta - \partial_{t'})u\, dy dt' \Big) \end{aligned}$$

for all $(x,t) \in \mathcal{C}_T$.

For $\lambda = 0$ in formula (7.3), Lemmas 7.2 and 7.3 are still valid for the particular choice $K(w) = \exp(\sigma w)$, which leads to a simple suppressing function in the Cauchy problem for the Laplace equation in the cap type domains in $\mathbb{R}^3$, see [Yar04]), [Ike09].

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